# MATHEMATICAL BEHAVIORAL FINANCE

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The purpose of the talk is to present a new research area - Mathematical Behavioral Finance. Its characteristic feature is the systematic application of behavioral approaches combined with the mathematical modeling of financial markets. The focus of work is on the fundamental questions and problems pertaining to Finance and Financial Economics, especially those related to equilibrium asset pricing and portfolio selection. The models under study reflect the psychology of market participants and go beyond the traditional paradigm of fully rational utility maximization. They do not rely upon restrictive hypotheses (perfect foresight) and avoid using unobservable agents' characteristics such as individual utilities and beliefs. The theory developed may be regarded as a plausible alternative to the classical general equilibrium theory (Walras, Arrow, Debreu, Radner, and others) responding to the challenges of today's economic and financial reality.

Joint work with

**Rabah Amir**, Economics Department, University of Iowa

**Thorsten Hens**, Department of Banking and Finance, University of Zurich

**Klaus R. Schenk-Hoppé**, School of Mathematics and Business School, University of Leeds

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**The Focus of Work:**

The general goal is to develop a new theory of market dynamics and equilibrium – a plausible alternative to the classical General Equilibrium theory (Walras, Arrow, Debreu, Radner and others).

The characteristic feature of the theory is the systematic application of **behavioral approaches** combined with the **evolutionary modeling** of financial markets.

The theory addresses from new positions the fundamental questions and problems pertaining to Finance and Financial Economics, especially those related to **equilibrium asset pricing** and **portfolio selection**, and is aimed at quantitative applications.

**Methodology:** Economic Theory + Mathematical Financial Modeling

## Walrasian Equilibrium

Conventional models of equilibrium and dynamics of asset markets are based on the principles of **Walrasian General Equilibrium theory**. This theory typically assumes that market participants are **fully rational** and act so as to **maximize utilities** of consumption subject to budget constraints.

Walras, Arrow, Debreu.

Hicks, Lindahl, Hildenbrand, Grandmont – **temporary equilibrium**.

Radner: equilibrium in (incomplete) **asset markets**.

Text: Magill and Quinzii.

”Although academic models often assume that all investors are rational, this assumption is clearly an expository device **not to be taken seriously**.”

**Mark Rubinstein** (Financial Analysts Journal, 05/06 2001, p. 15)

## The Fundamental Drawbacks of Conventional GET

- the hypothesis of “perfect foresight”
- the indeterminacy of temporary equilibrium
- coordination of plans of market participants
- the use of unobservable agent’s characteristics (individual utilities and beliefs)

## Behavioral Equilibrium

We develop an alternative equilibrium concept – **behavioral equilibrium**, admitting that market actors may have different patterns of behavior determined by their individual psychology, which are *not necessarily describable in terms of utility maximization.*

Their **strategies** may involve, for example, mimicking, satisficing, rules of thumb based on experience, spiteful behavior, etc.

The **objectives** of market participants might be of an **evolutionary** nature: **survival** (especially in crisis environments), **domination** in a market segment, capital **growth**, etc. – this kind of behavioral objectives will be in the main focus of this talk.

The strategies and objectives might be **interactive** – taking into account the behavior and the performance of the other economic agents.

**Behavioral economics** – studies at the interface of psychology and economics: Tversky, Kahneman, Smith, Shleifer (1990s); the 2002 Nobel Prize in Economics: Kahneman and Smith.

**Behavioral finance:** e.g. Shiller, Thaler (2000s).

# THE BASIC MODEL

**Randomness.**

$S$ space of "states of the world" (a measurable space);

$s_t \in S$ $(t = 1, 2, ...)$ state of the world at date $t$;

$s_1, s_2, ...$ an exogenous stochastic process.

**Assets.** There are $K$ *assets* .

**Dividends.** At each date $t$, assets $k = 1, ..., K$ pay *dividends* $D_{t,k}(s^t) \geq 0$, $k = 1, ..., K$, depending on the history

$$s^t := (s_1, ..., s_t)$$

of the states of the world up to date $t$.

**Assumptions:**

$$\sum_{k=1}^{K} D_{t,k}(s^t) > 0; \quad ED_{t,k}(s^t) > 0,\ k = 1, ..., K,\ t = 1, 2, ...,$$

where $E$ is the expectation with respect to the underlying probability $P$.

**Asset supply.** Total mass (the number of "physical units") of asset $k$ available at each date $t$ is $V_k > 0$.

**Investors and their portfolios.** There are $N$ *investors (traders)* $i \in \{1, ..., N\}$.

Investor $i$ at date $t = 0, 1, 2, ...$ selects a *portfolio*

$$x_t^i = (x_{t,1}^i, ..., x_{t,K}^i) \in \mathbb{R}_+^K,$$

where $x_{t,k}^i$ is the number of units of asset $k$ in the portfolio $x_t^i$. The portfolio $x_t^i$ for $t \geq 1$ depends, generally, on the current and previous states of the world:

$$x_t^i = x_t^i(s^t), \; s^t = (s_1, ..., s_t).$$

**Asset prices.** We denote by $p_t \in \mathbb{R}_+^K$ the vector of market prices of the assets. For each $k = 1, ..., K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, ..., p_{t,K})$ stands for the price of one unit of asset $k$ at date $t$. The prices might depend on the current and previous states of the world:

$$p_{t,k} = p_{t,k}(s^t),\ s^t = (s_1, ..., s_t).$$

The scalar product

$$\langle p_t, x_t^i \rangle := \sum_{k=1}^{K} p_{t,k} x_{t,k}^i$$

expresses the market value of the investor $i$'s portfolio $x_t^i$ at date $t$.

**The state of the market** at date $t$:

$$(p_t, x_t^1, ..., x_t^N),$$

where $p_t$ is the vector of asset prices and $x_t^1, ..., x_t^N$ are the portfolios of the investors.

**Investors' budgets.** At date $t = 0$ investors have initial endowments $w_0^i > 0$ ($i = 1, 2, ..., N$). Trader $i$'s budget at date $t \geq 1$ is

$$B_t^i(p_t, x_{t-1}^i) := \langle D_t + p_t, x_{t-1}^i \rangle,$$

where

$$D_t(s^t) := (D_{t,1}(s^t), ..., D_{t,K}(s^t)).$$

Two components:

the *dividends* $\langle D_t(s^t), x_{t-1}^i \rangle$ paid by the yesterday's portfolio $x_{t-1}^i$;

the *market value* $\langle p_t, x_{t-1}^i \rangle$ of the portfolio $x_{t-1}^i$ in the today's prices $p_t$.

**Investment rate.** A fraction $\alpha$ of the budget is invested into assets. We will assume that the *investment rate* $\alpha \in (0, 1)$ is fixed, the same for all the traders.

**Investment proportions.** For each $t \geq 0$, each trader $i = 1, 2, ..., N$ selects a vector of *investment proportions*

$$\lambda_t^i = (\lambda_{t,1}^i, ..., \lambda_{t,K}^i) \in \Delta^K$$

in the unit simplex $\Delta^K$, according to which the budget is distributed between assets.

**Game-theoretic slidework.** We regard the investors $i = 1, 2, ..., N$ as **players** in an $N$-person **stochastic dynamic game**. The vectors of investment proportions $\lambda_t^i$ are the players' *actions* or *decisions*.

Players' decisions might depend on the history $s^t := (s_1, ..., s_t)$ of states of the world and the *market history*

$$H^{t-1} := (p^{t-1}, x^{t-1}, \lambda^{t-1}),$$

where

$$p^{t-1} := (p_0, ..., p_{t-1}),$$

$$x^{t-1} := (x_0, x_1, ..., x_{t-1}),\ x_l = (x_l^1, ..., x_l^N),$$

$$\lambda^{t-1} := (\lambda_0, \lambda_1, ..., \lambda_{t-1}),\ \lambda_l = (\lambda_l^1, ..., \lambda_l^N).$$

**Investment strategies.** A vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions with values in $\Delta^K$

$$\Lambda_t^i(s^t, H^{t-1}),\ t = 1, 2, ...,$$

form an **investment strategy** (**portfolio rule**) $\Lambda^i$ of investor $i$.

**Basic strategies:** those for which $\Lambda_t^i$ depends only on $s^t$, and not on the market history $H^{t-1} = (p^{t-1}, x^{t-1}, \lambda^{t-1})$. We will call such portfolio rules **basic**.

**Investor $i$'s demand function.** Given a vector of investment proportions $\lambda_t^i = (\lambda_{t,1}^i, ..., \lambda_{t,K}^i)$ of investor $i$, the $i$'s demand function is

$$X_{t,k}^i(p_t, x_{t-1}^i) = \frac{\alpha \lambda_{t,k}^i B_t^i(p_t, x_{t-1}^i)}{p_{t,k}}.$$

where $\alpha$ is the investment rate.

**Short-run (temporary) equilibrium:** for each $t$, aggregate demand for every asset is equal to supply:

$$\sum_{i=1}^N X_{t,k}^i(p_t, x_{t-1}^i) = V_k,\ k = 1, ..., K.$$

**Equilibrium market dynamics**

**Prices:**

$$p_{t,k} V_k = \sum_{i=1}^{N} \alpha \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle,\ k = 1, ..., K.$$

**Portfolios:**

$$x_{t,k}^i = \frac{\alpha \lambda_{t,k}^i \langle D_t(s^t) + p_t, x_{t-1}^i \rangle}{p_{t,k}},\ k = 1, ..., K,\ i = 1, 2, ..., N.$$

The vectors of *investment proportions* $\lambda_t^i = (\lambda_{t,k}^i)$ are recursively determined by the *investment strategies*

$$\lambda_t^i(s^t) := \Lambda_t^i(s^t, H^{t-1}),\ i = 1, 2, ..., N.$$

Under mild "admissibility" assumptions on the strategy profile, the pricing equation has a unique solution $p_t$, $p_{t,k} > 0$.

## Random dynamical system

Put $w_t^i = \langle D_t + p_t, x_{t-1}^i \rangle$ (investor $i$'s **wealth**). Denote by $r_t = (r_t^1, ..., r_t^N)$ the random vector of the **market shares**

$$r_t^i = \frac{w_t^i}{w_t^1 + ... + w_t^N}$$

of $N$ investors. We will examine the **dynamics of the vectors of market shares** $r_t$. It is giverned by the **random dynamical system:**

$$r_{t+1}^i = \sum_{k=1}^{K} [\alpha \langle \lambda_{t+1,k}, r_{t+1} \rangle + (1-\alpha) R_{t+1,k}] \frac{\lambda_{t,k}^i r_t^i}{\langle \lambda_{t,k}, r_t \rangle},$$

$i = 1, ..., N$, $t \geq 0$, where

$$R_{t,k} = R_{t,k}(s^t) := \frac{D_{t,k}(s^t) V_k}{\sum_{m=1}^{K} D_{t,m}(s^t) V_m},\ k = 1, ..., K,\ t \geq 1,$$

are the **relative dividends**.

Nonlinear, defined implicitly in terms of rational functions (ratios of polynomials) with $N$ variables.

# COMMENTS ON THE MODEL

**Marshallian temporary equilibrium**.

We use the Marshallian "moving equilibrium method," to model the dynamics of the asset market as a sequence of consecutive temporary equilibria.

To employ this method one needs to distinguish between at least two sets of economic variables changing with different speeds.

Then the set of variables changing slower (in our case, the set of vectors of investment proportions) can be temporarily fixed, while the other (in our case, the asset prices) can be assumed to rapidly reach the unique state of partial equilibrium.

Samuelson (1947), describing the Marshallian approach, writes:

> I, myself, find it convenient to visualize equilibrium processes of quite different speed, some very slow compared to others. Within each long run there is a shorter run, and within each shorter run there is a still shorter run, and so forth in an infinite regression. For analytic purposes it is often convenient to treat slow processes as data and concentrate upon the processes of interest. For example, in a short run study of the level of investment, income, and employment, it is often convenient to assume that the stock of capital is perfectly or sensibly fixed.

Samuelson thinks about a *hierarchy* of various equilibrium processes with different speeds. In our model, it is sufficient to deal with only two levels of such a hierarchy.

# SURVIVAL STRATEGIES

**Market shares of the investors.** Investor $i$'s *wealth* at time $t$ is

$$w_t^i = \langle D_t(s^t) + p_t, x_{t-1}^i \rangle$$

(dividends + portfolio value). Investor $i$'s *market share* is

$$r_t^i = \frac{w_t^i}{w_t^1 + ... + w_t^N}.$$

**Survival strategies.** Given a strategy profile $(\Lambda^1, ..., \Lambda^N)$, we say that the portfolio rule $\Lambda^1$ (or the investor 1 using it) *survives* with probability one if

$$\inf_{t \geq 0} r_t^1 > 0 \text{ (a.s.)},$$

(the market share of investor 1 is bounded away from zero a.s. by a strictly positive random variable).

**Definition.** A portfolio rule is called a **survival strategy** if the investor using it survives with probability one (*irrespective of what portfolio rules are used by the other investors!*).

**Our central goal is to identify survival strategies.**

# THE MAIN RESULTS

**Relative dividends.** Define the *relative dividends* of the assets $k = 1, ..., K$ by

$$R_{t,k} = R_{t,k}(s^t) := \frac{D_{t,k}(s^t)V_k}{\sum_{m=1}^{K} D_{t,m}(s^t)V_m},\ k = 1, ..., K,\ t \geq 1,$$

and put $R_t(s^t) = (R_{t,1}(s^t), ..., R_{t,K}(s^t))$.

**Definition of the survival strategy $\Lambda^*$.** Put

$$\alpha_l = \alpha^{l-1}(1-\alpha).$$

Define

$$\lambda_t^*(s^t) = (\lambda_{t,1}^*(s^t), ..., \lambda_{t,K}^*(s^t)),$$

where

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} \alpha_l R_{t+l,k}\ .$$

Here, $E_t(\cdot) = E(\cdot|s^t)$ stands for the conditional expectation given $s^t$; $E_0(\cdot)$ is the unconditional expectation $E(\cdot)$.

Assume $\lambda^*_{t,k} > 0$ (a.s.)

**The central results** are as follows.

**Theorem 1.** *The portfolio rule $\Lambda^*$ is a survival strategy.*

We emphasize that the strategy $\Lambda^*$ is basic, and it survives in competition with any (not necessarily basic) strategies!

In the class of *basic* strategies, the survival strategy $\Lambda^*$ is **asymptotically unique**:

**Theorem 2.** *If $\Lambda = (\lambda_t)$ is a basic survival strategy, then*

$$\sum_{t=0}^{\infty} ||\lambda^*_t - \lambda_t||^2 < \infty \text{ (a.s.).}$$

**The meaning of $\Lambda^*$.** The portfolio rule $\Lambda^*$ defined by

$$\lambda^*_{t,k} = E_t \sum_{l=1}^{\infty} \alpha_l R_{t+l,k} \ ,$$

combines three general investment principles known in Financial Economics.

(a) $\Lambda^*$ prescribes the allocation of wealth among assets in the proportions of their **fundamental values** – the expectations of the flows of the discounted future dividends.

(b) The strategy $\Lambda^*$, defined in terms of the **relative (weighted) dividends,** is analogous to the CAPM strategy involving investment in the **market portfolio**.

(c) The portfolio rule $\Lambda^*$ is closely related (and in some special cases reduces) to the **Kelly portfolio rule** prescribing to maximize the expected logarithm of the portfolio return – see below.

**The i.i.d. case.** If $s_t \in S$ are **independent and identically distributed** (i.i.d.) and

$$R_{t,k}(s^t) = R_k(s_t),$$

then

$$\lambda^*_{t,k} = \lambda^*_k = ER_k(s_t),$$

does not depend on $t$, and so $\Lambda^*$ is a *fixed-mix (constant proportions) strategy.* It is independent of the investment rate $\alpha$ !

In the case of Arrow securities ("horse race model"), the expectations $ER_k(s_t)$ are equal to the **probabilities** of the states of the world ("betting your beliefs"). This is the **Kelly portfolio rule** maximizing the expected log returns.

## Global evolutionary stability of $\Lambda^*$

Consider the i.i.d. case in more detail. It is important for quantitative applications and admits a deeper analysis of the model. Let us concentrate on fixed-mix strategies. In the class of such strategies, $\Lambda^*$ is **globally evolutionarily stable**:

**Theorem 3.** *If among the $N$ investors, there is a group using $\Lambda^*$, then those who use $\Lambda^*$ survive, while all the others are driven out of the market (their market shares tend to zero a.s.).*

## GAME-THEORETIC ASPECTS

**A synthesis of evolutionary and dynamic games.** The notion of a survival strategy is the **solution concept** we adopt in the analysis of the market game.

This is a solution concept of a *purely evolutionary nature*.

No utility maximization or Nash equilibrium is involved.

On the other hand, the strategic slidework we consider is the one characteristic for **stochastic dynamic games** (Shapley 1953).

**Survival strategy and ESS.** The notion of a survival portfolio rule, stable with respect to the market selection process, is akin to the notions of **evolutionary stable strategies (ESS)** introduced by Maynard Smith and Price (1973) and Schaffer (1988, 1989).

However, the mechanism of market selection in our model is radically distinct from the typical schemes of evolutionary game theory, based on a given **static game**, where repeated random matchings of species or agents in large populations result in their survival or extinction in the long run.

Our notion of survival is defined in the original terms of the **dynamic game** describing wealth accumulation of investors, which makes it possible to address directly those questions that are of interest in the quantitative modeling of asset market dynamics.

## IN ORDER TO SURVIVE YOU HAVE TO WIN!

### Equivalence of Survival and Unbeatable Strategies

One might think that the focus on survival substantially restricts the scope of the analysis: ”one should care of survival only if things go wrong”.

It turns out, however, that the class of survival strategies coincides with the class of **unbeatable** strategies having a better relative performance in the long run (in terms of wealth accumulation) than any other strategies competing in the market.

Thus, **in order to survive you must win!**

## Winning (=unbeatable) strategies of capital accumulation

♦ For two sequences of positive random numbers $(w_t)$ and $(w'_t)$, define

$$(w_t) \preccurlyeq (w'_t) \text{ iff } w_t \leq H w'_t \text{ (a.s.)}$$

for some random constant $H$, i.e. $w_t$ *does not grow asymptotically faster than* $w'_t$.

♦ Let $(w^i_t)$ denote the wealth process of investor $i$.

**Proposition.** *A portfolio rule* $\Lambda^1$ *is a survival strategy if and only if the following condition holds. If investor* 1 *uses* $\Lambda^1$*, then*

$$(w^i_t) \preccurlyeq (w^1_t)$$

*for all* i=2,...,N *and any strategies* $\Lambda^2, ..., \Lambda^N$.

Thus $\Lambda^1$ is an **unbeatable (winning) strategy in terms of the growth rate of wealth if and only if it is a survival strategy.**

## Unbeatable strategies: a general definition

Consider an abstract game of $N$ players $i = 1, ..., N$ selecting strategies $\Lambda^i$ from some given sets.

Let $w^i = w^i(\Lambda^1, ..., \Lambda^N)$ be the outcome of the game for player $i$ corresponding to the strategy profile $(\Lambda^1, ..., \Lambda^N)$.

Possible outcomes $w^i$ are elements of a set $\mathcal{W}$.

Suppose that a preference relation

$$w^i \succcurlyeq w^j, w^i, w^j \in \mathcal{W},\ i \neq j,$$

is given, comparing relative performance of players $i$ and $j$.

**Definition.** A strategy $\Lambda$ of player $i$ is **unbeatable** if for any admissible strategy profile $(\Lambda^1, \Lambda^2, ..., \Lambda^N)$ in which $\Lambda^i = \Lambda$, we have

$$w^i(\Lambda^1, \Lambda^2, ..., \Lambda^N) \succcurlyeq w^j(\Lambda^1, \Lambda^2, ..., \Lambda^N) \text{ for all } j \neq i.$$

Thus, if player $i$ uses the strategy $\Lambda$, he/she cannot be outperformed by any of the rivals $j \neq i$, irrespective of what strategies they employ.

**Pre-von Neumann / Pre-Nash game theory.** The notion of a *winning* or *unbeatable* strategy was a central solution concept in the pre-von Neumann and pre-Nash game theory (as a branch of mathematics, pioneered by Bouton, Zermelo, Borel, 1900s - 1920s).

The question of **determinacy** of a game (existence of a winning strategy for one of the players) was among the key topics in game theory until 1950s. Dynamic games of complete information: Gale, Stewart, Martin ("Martin's axiom").

The first mathematical paper in game theory "solving" a game (=finding a winning strategy for one of the players) was:

**Bouton,** C. L. (1901-2) Nim, a game with a complete mathematical theory, **Annals of Mathematics**, 3, 35–39.

**Unbeatable strategies and evolutionary game theory.** The basic solution concepts in evolutionary game theory – **evolutionary stable strategies** (Maynard Smith & Price, Schaffer) – may be regarded as **"conditionally" unbeatable strategies** (the number of mutants is small enough, or they are identical). Unconditional versions: Kojima (2006).

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## Versions of the model

## Short-lived assets

Assets live one period, yield payoffs, and then are identically reborn at the beginning of the next period. A simplified version of the basic model (reduces to it when $\alpha \to 0$).

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