DERIVED CATEGORIES OF HYPERKÄHLER VARIETIES FINAL REPORT, SEPTEMBER 2015

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Project. We participated in the Junior Hausdorff Trimester Program on algebraic geometry from January to April 2014. Our project title was "Derived categories of hyperkähler varieties."

Hyperkähler or *irreducible holomorphic symplectic* varieties are one of three basic classes of varieties with vanishing first Chern class, and are the most mysterious of the three owing to the small number of examples. The best-understood examples are the "varieties of $\mathrm{K3}^{[n]}$-type," which include moduli spaces of sheaves on K3 surfaces. In recent years, several important results on varieties in this family have been obtained using *derived categories of coherent sheaves* both directly, e.g. the work of Bayer–Macrì and Bayer–Hassett–Tschinkel on the minimal model program for varieties of $\mathrm{K3}^{[n]}$-type, and indirectly, e.g. Markman's work on the monodromy group, which takes Verbitsky's Torelli theorem for general hyperkählers and gives a much more effective statement for varieties of $\mathrm{K3}^{[n]}$-type.

One of our dreams is to describe every hyperkähler variety as a moduli space of objects in a triangulated category that behaves like the derived category of a K3 surface. This might allow us to extend the results just mentioned to other families of hyperkähler varieties, or even to find new families or explain why there are so few. Since 2011, N.A. and Tom Bridgeland (Sheffield) have tried several unsuccessful approaches to constructing such a category. While in Bonn, W.D. brought in some new ideas, related to [8] below; this new approach has yet to bear fruit, but we continue to work on it.

Papers. The main product of our time together in Bonn was the following pair of joint papers:

[1] N. Addington, W. Donovan, and C. Meachan. Mukai flops and $\mathbb{P}$-twists. Preprint, 1507.02595.

Mukai flops are an important class of birational maps between hyperkähler varieties. Associated to a Mukai flop $X \dashrightarrow X'$ are on the one

hand a sequence of equivalences $D^b(X) \to D^b(X')$, due to Kawamata and Namikawa, and on the other hand a sequence of autoequivalences of $D^b(X)$, due to Huybrechts and Thomas. In this paper we work out a complete picture of the relationship between the two. We make use of these results in a particularly rich example in [2]. W.D. has proved related results for other classes of flops in work with Ed Segal and Michael Wemyss, including [8].

[2] N. Addington, W. Donovan, and C. Meachan. On derived categories of moduli spaces of torsion sheaves on K3 surfaces. Preprint, 1507.02597.

In 2011, N.A. constructed a new autoequivalence of the derived category of the Hilbert scheme of n points on a K3 surface, and conjectured that the same construction would work for any moduli space of sheaves on a K3 surface. In this paper we prove the conjecture for many moduli spaces of torsion sheaves, and find a geometrically meaningful interpretation of the autoequivalences both in the new examples and the earlier one. This paper makes use of the results of [1].

We also finished or started the following papers with other collaborators:

[3] N. Addington, W. Donovan, and E. Segal. The Pfaffian–Grassmannian equivalence revisited. *Alg. Geom.*, 2(3):332–364, 2015. 1401.3661.

This paper is unrelated to the hyperkähler project, but we put the finishing touches on it shortly after arriving in Bonn.

[4] A. Krug and C. Meachan. Spherical functors on the Kummer surface. *Nagoya Math. J.*, to appear. 1402.1651.

This note completes an earlier result of C.M. on autoequivalences of derived categories of higher-dimensional Kummer varieties, by showing that the same construction on Kummer *surfaces* yields an autoequivalence which factors into a product of familiar terms. Krug was based at Universität Bonn at the time.

[5] N. Addington and M. Lehn. On the symplectic eightfold associated to a Pfaffian cubic fourfold. *J. Reine Angew. Math.*, to appear. 1404.5657.

This note completes a result of Lehn, Lehn, Sorger, and van Straten, who constructed a new hyperkähler 8-fold but left open the question of whether it was of $\mathrm{K3}^{[4]}$-type or represented a whole new family of hyperkählers, by showing that it is of $\mathrm{K3}^{[4]}$-type. Pleasingly, derived categories appear in the proof only, not in the statement of the results,

so we are really "using derived categories to do honest geometry." We started working on this paper the previous fall, but we were able to work much faster once N.A. got to Bonn as Lehn is based in Mainz.

[6] C. Meachan and Z. Zhang. Birational geometry of singular moduli spaces of O'Grady type. Preprint, 1404.6783.

This paper follows Bayer–Macrì's work on the minimal model program for varieties of $K3^{[n]}$-type, mentioned above, proving similar results for another class of hyperkähler varieties.

[7] N. Addington. On two rationality conjectures for cubic fourfolds. *Math. Res. Lett.*, to appear. 1405.4902.

The variety of lines on any cubic 4-fold is a variety of $K3^{[2]}$-type; this note describes exactly when it is birational to a Hilbert scheme of 2 points or another moduli space of sheaves on a K3 surface, conditions thought to be relevant to the question of whether the cubic is rational. Derived categories do not appear in the statements or the proofs, but they are present philosophically. This paper benefitted from N.A.'s conversations in Bonn with JHTP participants François Charles and Giovanni Mongardi.

[8] W. Donovan and M. Wemyss. Twists and braids for general 3-fold flops. Preprint, 1504.05320.

Not about hyperkählers, but like [1] it deals with equivalences and autoequivalences associated to a certain class of flops; and ideas about noncommutative deformations from this paper and its predecessor revived the project with N.A. and Tom Bridgeland mentioned earlier.

Other activities. Together with Nathan Broohmead's group ("The derived category of a T-variety"), we organized a workshop on derived categories which ran from February 10 to 13. N.A. and W.D. gave a mini-course on autoequivalences at the request of some other JHTP participants. We all attended many workshop and seminar talks by the other groups and their guests, and had many interesting mathematical conversations with them.

We found our time in Bonn very productive, and the working conditions and collegial atmosphere very pleasant. N.A.'s family stayed in Bonn for the duration of the program and have fond memories of their time there.

It is a pleasure to thank the Hausdorff Institute for its hospitality.