Final report: Derived Categories of T-varieties

- Nathan Broomhead visited the HIM for the period 06/01/2014 - 06/03/2014
- Andreas Hochenegger visited the HIM for the period 06/01/2014 - 25/04/2014
- Hendrik Süß visited the HIM for the period 09/02/2014 - 10/03/2014

Research activity

The project that we proposed was to understand the derived categories of T-varieties, ie. normal varieties X with the effective action of some torus T (see [1] for a survey). For such an X, there is a rational map $X \dashrightarrow X/T$ to its Chow quotient, and using this map, X can be described by some combinatorial object on X/T. The combinatorics describe the fibres of this quotient map, which has a toric variety Z as its general fibre.

The original aim of the project was to see if the property of being a T-variety is a derived invariant. In other words, for a (smooth projective) T-variety X and a variety Y such that $\mathcal{D}^b(X) \cong \mathcal{D}^b(Y)$ we asked whether it follows automatically that Y is also a T-variety. This question has a positive answer, and the proof turned out to be far simpler than we had previously envisioned; using an application of [6, Thm. 4.18], it is rather immediate that there is also an induced action of T on Y, which makes Y a T-variety. We decided that this observation was unfortunately not worth a paper.

Given an equivalence between T-varieties, our next step was then to ask whether this equivalence also induces an equivalence of the derived categories of the Chow quotients or the (general) fibres. Even though this seems a reasonable expectation, we were able to construct some examples where the fibres and the Chow quotients are only birational but not derived equivalent. It remains an interesting question to understand which of the Fourier-Mukai kernels giving equivalences $\mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ induce derived equivalences of the Chow quotients or the fibres. However the Chow quotient map lacks many desirable properties like flatness, and a general result in this direction seemed out of reach.

We decided therefore to change our focus in this project, and instead to look at the derived categories of a different generalisation of toric varieties, namely of Mori dream spaces, which include an important class of T-varieties, namely those of complexity one.

Given a complete normal variety, the Cox ring is defined as a k-vector space spanned by the global section of all divisor classes:

$$\mathrm{Cox}(X) = \bigoplus_{[D] \in \mathrm{Cl}(X)} H^0(X, \mathcal{O}(D)).$$

A natural multiplication can be defined, as well. If $\mathrm{Cox}(X)$ is finitely generated as a k-algebra X is called a Mori dream space. Such a variety comes with a natural embedding into a toric variety and with a quotient representation $X = \hat{X} /\!\!/ H$ of an affine variety by a torus H. The birational geometry of X is governed by the geometric invariant theory (GIT) of this quotient. Here, T-varieties give rise to a wide class of interesting examples, since for them the Cox ring and the described quotient construction are known very explicitly [3].

In certain smooth situations Ballard, Favero and Katzarkov [2] describe how the derived category of a GIT quotient $\hat{X} /\!\!/_{\chi} H$ varies with the choice of character χ. As an application, they obtain an exceptional sequence for any projective toric variety X. By choosing a path in the GIT fan from the chamber corresponding to X to the boundary, they obtain a sequence of birational transformations until one arrives at a Mori fibration over a toric varity Y of lower dimension. $\mathcal{D}^b(X)$ can be written as a semi-orthogonal decomposition whose parts come from the wall-crossings and $\mathcal{D}^b(Y)$. Both contributions come from lower dimensional varieties, so this allows an inductive procedure. This can be seen as a generalisation of the results of Kawamata ([4], [5]). One could hope for a similar description in the case of a general Mori dream space (or more precisely to the associated stack). There is however a significant obstacle: Cox rings of non-toric varieties are always singular, which prevents us from directly applying the theory of [2]. Moreover, the case of Picard rank one (which is due to Orlov), shows that the description of the derived category of X should involve contributions from singularity categories associated to the singularities of the Cox ring.

In all of the explicit examples that we have calculated so far we are able to circumvent these problems, using techniques such as root constructions, and produce semiorthogonal decompositions. However, the machinery has become complex and there are still complications in producing general statements. We expect that there is at least a reasonable class of Mori Dream stacks, where our techniques will yield interesting semi-orthogonal decompositions. For this reason we don't yet have a completed preprint. However work on the project is still ongoing. N.B. and A.H. met to discuss the project including regular meetings when A.H. visited Hannover (1/15-2/15) and when N.B. visited Köln (6/15-7/15). N.B. and H.S. have met several time to discuss, including two days in Edinburgh (5/14) and a meeting in Manchester in September 2015.

Other activities

Together with the group "Derived categories of hyperkähler varieties" we organised a workshop on derived categories which ran in the Hausdorf Institute between 10-13th February 2014. We participated in many workshops and seminars and mini-courses run by other groups. We also had many interesting discussions with members of other groups.

References

[1] Klaus Altmann, Nathan Owen Ilten, Lars Petersen, Hendrik Süß and Robert Vollmert, *The geometry of T-varieties,* In: Piotr Pragacz (Editor) *Contributions to Algebraic Geometry*, EMS Series of Congress Reports, 17–70, 2012.

[2] Matthew Ballard, David Favero and Ludmil Katzarkov, *Variation of Geometric Invariant Theory Quotients and Derived Categories,* `arXiv:1203.6643`.

[3] Jürgen Hausen and Hendrik Süß, *The Cox ring of an algebraic variety with torus action,* Adv. Math. 225(2):977–1012, 2010.

[4] Yujiro Kawamata, *Derived categories of toric varieties,* Michigan Math. J. 54(3):517–535, 2006.

[5] Yujiro Kawamata, *Derived categories of toric varieties II,* Michigan Math. J. 62(2):353–363, 2013.

[6] Raphaël Rouquier, *Automorphismes, graduations et catégories triangulées,* J. Inst. Math. Jussieu 10(3):713–751, 2011.