Fractional Kelly Strategies in Continuous Time: Recent Developments

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Abstract

The Kelly criterion and fractional Kelly strategies hold an important place in investment management theory and practice. Both the Kelly criterion and fractional Kelly strategies, e.g. invest a fraction f of one's wealth in the Kelly portfolio and a proportion $1 - f$ in the risk-free asset, are optimal in the continuous time setting of the Merton [33] model. However, fractional Kelly strategies are no longer optimal when the basic assumptions of the Merton model, such as the lognormality of asset prices, are removed. In this chapter, we present an overview of some recent developments related to Kelly investment strategies in an incomplete market environment where asset prices are not lognormally distributed. We show how the definition of fractional Kelly strategies can be extended to guarantee optimality. The key idea is to get the definition of fractional Kelly strategies to coincide with the fund separation theorem related to the problem at hand. In these instances, fractional Kelly investment strategies appear as the natural solution for investors seeking to maximize the terminal power utility of their wealth.

1 Introduction

The Kelly criterion and fractional Kelly strategies hold an important place in investment management theory and practice. The Kelly criterion maximizes the log-return on invested wealth and is therefore related to the seminal work of Bernoulli [9]. Early contributions to the theory and application of the Kelly criterion to gambling and investment include Kelly [20], Latané [23], Breiman [8], Thorp [43] or Markowitz [32]. The main reference is undeniably [30]. Readers interested in an historical account of the Kelly criterion and of its use at the gambling table and in the investment industry will refer to Poudstone [39]. From a practical investment management perspective, several of the most successful investors, including Keynes, Buffett

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and Gross have used Kelly-style strategies in their funds (see Ziemba [48], Thorp [44] and Ziemba and Ziemba [47] for details).

The Kelly criterion has a number of good as well as bad properties, as discussed by MacLean, Thorp and Ziemba [27]. Its 'good' properties extend beyond practical asset management and into asset pricing theory, as the Kelly portfolio is the numéraire portfolio associated with the physical probability measure. This observation forms the basis of the 'benchmark approach to finance' proposed by Platen [37] and Heath and Platen [38] (see also Long [26] and Becherer [3]). The 'bad' properties of the criterion are also well studied and understood. Samuelson, in particular, was a long time critique of the Kelly criterion (see [40], [42] and [41]). A main drawback of the Kelly criterion is that it is inherently a very risky investment.

To address this shortcoming, MacLean, Ziemba and Blazenko [31] propose the following *fractional* Kelly strategy: invest a fraction f of one's wealth in the Kelly portfolio and a proportion $1 - f$ in the risk-free asset. MacLean, Sanegre, Zhao and Ziemba [29] MacLean, Ziemba and Li [28] pursued further research in this direction. There are two key advantages to this definition: first, a fractional Kelly strategy is significantly less risky than the full Kelly portfolio, while maintaining a significant part of the upside. Second, fractional Kelly strategies are optimal in the continuous time setting of the Merton [33] model. In fact, fractional Kelly strategies correspond to the optimal investment of a power utility investor seeking to maximize the terminal utility of his/her wealth. Unfortunately, fractional Kelly strategies are no longer optimal when the basic assumptions of the Merton model, such as the lognormality of asset prices, are removed (see MacLean, Ziemba and Li [28]). In recent years, a number of attempts have been made to remedy this situation and extend the definition of fractional Kelly strategies to guarantee their optimality.

In this chapter, we present an overview of some recent developments related to Kelly investment strategies in an incomplete market environment where asset prices are not lognormally distributed. In section 2, we introduce the Kelly portfolio and fractional Kelly strategies in the context of the Merton model. Next, we consider in Section 3 an Intertemporal Capital Asset Pricing Model (ICAPM) where the drift of the asset price dynamics are affine functions of some affine factors. In this ICAPM, traditionally defined fractional Kelly strategies are no longer optimal. We must therefore extend the definition of the fractional Kelly strategies along the lines of a fund separation theorem in order to guarantee optimality. In section 4, we present an extension to a benchmarked investor, that is an investor with the objective of outperforming a given investment benchmark, before considering the impact of partial observation on the underlying valuation factors in Section 6. Finally, we look at optimal investment strategies in a jump-diffusion setting, where asset prices follow jump-diffusion processes and the underlying factors are modelled as diffusion processes.

2 The Kelly Criterion Portfolio and Fractional Kelly Strategies in the Merton World

2.1 The Kelly Criterion Portfolio in the Merton Model

We start by introducing some of the notation that we will need in the remainder of the chapter. Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ be the underlying probability space. On this space is defined an $\mathbb{R}^m$-valued $(\mathcal{F}_t)$-Brownian motion $W(t)$ with components $W_k(t)$, $k = 1, \ldots, N$. $S_i(t)$ denotes the price at time t of the ith security, with $i = 0, \ldots, m$. Let S_0 denote the wealth invested in a money market account. The dynamics of the money market account is given by:

$$\frac{dS_0(t)}{S_0(t)} = rdt, \qquad S_0(0) = s_0 \tag{2.1}$$

where $r \in \mathbb{R}^+$ is the risk-free rate. The dynamics of the m risky securities and n factors can be expressed as:

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t), \qquad S_i(0) = s_i, \quad i = 1, \ldots, m \tag{2.2}$$

where the market parameters $\mu = (\mu_1, \ldots, \mu_m)'$ represents the rate of return vector and the volatility $\Sigma := [\sigma_{ij}]\,,\ i = 1, \ldots, m,\ j = 1, \ldots, m$ is a $m \times m$ matrix. More synthetically,

$$dS(t) = D(S(t))\mu dt + D(S(t))\Sigma dW(t) \tag{2.3}$$

where $D(S(t))$ denotes the diagonal matrix with $S_1(t), \ldots, S_M(t)$ on the diagonal.

We make the further assumption that:

Assumption 2.1. The matrix Σ is positive definite.

This assumption rules out simple arbitrage opportunities.

Let $\mathcal{G}_t := \sigma((S(s), X(s)), 0 \leq s \leq t)$ be the sigma-field generated by the security, liability and factor processes up to time t. An *investment strategy* or *control process* is an $\mathbb{R}^m$-valued process with the interpretation that $h_i(t)$ is the fraction of current portfolio value invested in the ith asset,

$i = 1, \ldots, m$. The fraction invested in the money market account is then $h_0(t) = 1 - \sum_{i=1}^{m} h_i(t)$.

Definition 2.2. An $\mathbb{R}^m$-valued control process $h(t)$ is in class $\mathcal{A}(T)$ if the following conditions are satisfied:

1. $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t \geq 0}$ and is left continuous with right limit (càdlàg);

2. $P\left(\int_0^T |h(s)|^2 \, ds < +\infty\right) = 1, \quad \forall T > 0;$

3. the Doléans exponential χ_t^h, given by

$$\chi_t^h \quad := \quad \exp\left\{\gamma \int_0^t h(s)' \Sigma dW_s - \frac{1}{2}\gamma^2 \int_0^t h(s)' \Sigma \Sigma' h(s) ds\right\} \quad (2.4)$$

is an exponential martingale, i.e. $\mathbf{E}\left[\chi_T^h\right] = 1$

Definition 2.3. We say that a control process $h(t)$ is *admissible* if $h(t) \in \mathcal{A}(T)$.

Taking the budget equation into consideration, the wealth, $V(t)$, of the asset in response to an investment strategy $h \in \mathcal{H}$ follows the dynamics

$$\frac{dV(t)}{V(t)} = r dt + h'(t)\left(\mu - r\mathbf{1}\right) dt + h'(t) \Sigma dW_t \quad (2.5)$$

with initial endowment $V(0) = v$ and where $\mathbf{1} \in \mathbb{R}^m$ is the m-element unit column vector. Thus,

$$\ln V(t) = \ln v + \int_0^t r + h'(s)\left(\mu - r\mathbf{1}\right) - \frac{1}{2} h'(s) \Sigma \Sigma' h(s) ds + \int_0^t h'(s) \Sigma dW_s \quad (2.6)$$

The objective of a Kelly investor with a fixed time horizon T is to maximize

$$J(t, h; T, \gamma) \quad = \quad \mathbf{E}\left[U(V_T)\right] = \mathbf{E}\left[\ln V_T\right]$$

A pointwise maximization of the criterion J yields the Kelly portfolio:

$$h^* = \left(\Sigma\Sigma'\right)^{-1}\left(\mu - r\mathbf{1}\right)$$

Substituting back in (2.6), we find that the wealth of a Kelly investor is

$$\begin{aligned} V^*(T) \quad = \quad & v \exp\left\{\left[r + \frac{1}{2}\left(\mu - r\mathbf{1}\right)'\left(\Sigma\Sigma'\right)^{-1}\left(\mu - r\mathbf{1}\right)\right](T - t) \right. \\ & \left. + \left(\mu - r\mathbf{1}\right)'\left(\Sigma\Sigma'\right)^{-1}\Sigma(W_T - W_t)\right\} \end{aligned} \quad (2.7)$$

and the instantaneous growth rate of the strategy follows the dynamics

$$dR_t = \left[r + \frac{1}{2}(\mu - r\mathbf{1})'(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1})\right]dt + (\mu - r\mathbf{1})'(\Sigma\Sigma')^{-1}\Sigma dW_t \tag{2.8}$$

2.2 Fractional Kelly Strategies

To mitigate the risks inherent in a the Kelly investment strategy, MacLean, Ziemba and Blazenko [31] propose the following *fractional* Kelly strategy: invest a fraction f of one's wealth in the Kelly portfolio and a proportion $1-f$ in the risk-free asset. MacLean, Sanegre, Zhao and Ziemba [29] propose a methodology for computing the optimal fractional Kelly weights at discrete time intervals. In a continuous time setting where asset prices follow a geometric Brownian motion, they show that a fractional Kelly strategy is optimal with respect to Value at Risk and a Conditional Value at Risk criteria. MacLean, Ziemba and Li [28] further prove that fractional Kelly strategies are efficient when asset prices are lognormally distributed.

This last result is actually a corollary to Merton's Fund Separation theorem. In the Merton problem with no consumption and a power utility function, the objective of an investor is to maximize the expected utility of terminal wealth over a fixed time horizon T:

$$J(t,h;T,\gamma) = \mathbf{E}\left[U(V_T)\right] = \mathbf{E}\left[\frac{V_T^\gamma}{\gamma}\right] = \mathbf{E}\left[\frac{e^{\gamma \ln V_T}}{\gamma}\right]$$

with risk aversion coefficient $\gamma \in (-\infty, 0) \cup (0, 1)$. We define the value function Φ corresponding to the maximization of the auxiliary criterion function $J(t,h;T,\gamma)$ as

$$\Phi(t) = \sup_{h \in \mathcal{A}} J(t,h;T,\gamma) \tag{2.9}$$

By Itô's lemma,

$$e^{\gamma \ln V(t)} = v^\gamma \exp\left\{\gamma \int_0^t g(h(s);\gamma)ds\right\} \chi_t^h \tag{2.10}$$

where

$$g(h;\gamma) = -\frac{1}{2}(1-\gamma)h'\Sigma\Sigma' h + h'(\mu - r\mathbf{1}) + r$$

and χ_t^h is defined in (2.4)

We can solve the stochastic control problem associated with (2.9) by a change of measure argument (see exercise 8.18 in [35] or [21] in the context of risk sensitive control). Let $\mathbb{P}_h$ be the measure on $(\Omega, \mathcal{F}_T)$ defined via the Radon-Nikodým derivative[1]

$$\frac{d\mathbb{P}_h}{d\mathbb{P}} := \chi_T^h \tag{2.11}$$

For $h \in \mathcal{A}(T)$,

$$W_t^h = W_t - \gamma \int_0^t \Sigma' h(s) ds$$

is a standard Brownian motion under the measure $\mathbb{P}_h$. Moreover, the control criterion under this new measure is

$$I(t, h; T, \gamma) = \frac{v^\gamma}{\gamma} \mathbf{E}_{t,x}^h \left[\exp\left\{ \gamma \int_t^T g(h(s); \gamma) ds \right\} \right] \tag{2.12}$$

where $\mathbf{E}_t^h[\cdot]$ denotes the expectation taken with respect to the measure $\mathbb{P}_h$ at an initial time t.

Under the measure $\mathbb{P}_h$, the control problem can be solved through a pointwise maximisation of the auxiliary criterion function $I(v, x; h; t, T)$. The optimal control h^* is simply the maximizer of the function $g(x; h; t, T)$ given by

$$h^* = \frac{1}{1-\gamma} (\Sigma\Sigma')^{-1} (\mu - r\mathbf{1})$$

which represents a position of $\frac{1}{1-\gamma}$ in the Kelly criterion portfolio.

Remark 2.4. The change of measures simplifies the problem considerably: under the measure $\mathbb{P}_h$ we can solve the optimization through a pointwise maximization, as we did earlier on in the logarithmic utility case.

Substituting (2.13) into (2.10) and (2.4), we derive the value function $\Phi(t)$, or optimal utility of wealth,

$$\Phi(t) = \frac{v^\gamma}{\gamma} \exp\left\{ \gamma \left[r + \frac{1}{2(1-\gamma)} (\mu - r\mathbf{1}) (\Sigma\Sigma')^{-1} (\mu - r\mathbf{1}) \right] (T - t) \right\}$$

as well as an exact form for the exponential martingale χ_t^* associated with the control h^*:

$$\begin{aligned} \chi_t^* \quad := \quad & \exp\left\{ \frac{\gamma}{1-\gamma} (\mu - r\mathbf{1})' \Sigma^{-1} W(t) \right. \\ & \left. -\frac{1}{2} \left(\frac{\gamma}{1-\gamma} \right)^2 (\mu - r\mathbf{1})' (\Sigma\Sigma')^{-1} (\mu - r\mathbf{1}) t \right\} \end{aligned} \tag{2.13}$$

[1] Loève [25] provides a thorough treatment of the Radon-Nikodým theorem

Therefore, fractional Kelly strategies appear as a consequence of a classical Fund Separation Theorem:

Theorem 2.5 (Fund Separation Theorem)**.** *Any portfolio can be expressed as a linear combination of investments in the Kelly (log-utility) portfolio*

$$h^K(t) \quad = \quad (\Sigma\Sigma')^{-1}(\mu - r\mathbf{1}) \tag{2.14}$$

and the risk-free rate. Moreover, if an investor has a risk sensitivity γ, the proportion of the Kelly portfolio will equal $\frac{1}{1-\gamma}$.

A key limitation of fractional Kelly strategies is that they are only optimal within the Merton model, that is when asset prices are lognormally distributed (see also MacLean, Ziemba and Li [28]). This situation suggests that the definition of Fractional Kelly strategies could be broadened in order to guarantee optimality. We can take a first step in this direction by revisiting the ICAPM (see Merton[34]).

3 Incomplete Markets and Intertemporal Capital Asset Pricing Model

3.1 The Model

Merton [34] proposed an Intertemporal Capital Asset Pricing Model (ICAPM) in which the drift rate of the asset prices depend on a number of Normally-distributed factors. Bielecki and Pliska (see in particular [5] and [6]), Kuroda and Nagai [21] as well as Davis and Lleo (see [11],[12] and [15]) further developed this idea in the context of risk-sensitive control.

Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ be the underlying probability space. On this space is defined an $\mathbb{R}^N$-valued $(\mathcal{F}_t)$-Brownian motion $W(t)$ with components $W_k(t)$, $k = 1, \ldots, N$. $S_i(t)$ denotes the price at time t of the ith security, with $i = 0, \ldots, m$, and $X_j(t)$ denotes the level at time t of the jth factor, with $j = 1, \ldots, n$. We also assume that the factors are observable.

Let S_0 denote the wealth invested in a money market account. The dynamics of the money market account is given by:

$$\frac{dS_0(t)}{S_0(t)} = \big(a_0 + A_0'X(t)\big)\,dt, \qquad S_0(0) = s_0 \tag{3.1}$$

where $a_0 \in \mathbb{R}$ is a scalar constant, $A_0 \in \mathbb{R}^n$ is a n-element column vector and throughout the paper x' denotes the transpose of the matrix or vector x.. We further assume that the expected rates of return of the assets depend on n valuation factors $X_1(t), \ldots, X_n(t)$ which follow the dynamics given in equation (3.3) below Let $N = n+m$. The dynamics of the m risky securities and n factors are

$$dS(t) = D\left(S(t)\right)(a + AX(t))dt + D\left(S(t)\right)\Sigma dW(t), \qquad S(0) = s \tag{3.2}$$

and

$$dX(t) = (b + BX(t))dt + \Lambda dW(t), \qquad X(0) = x \tag{3.3}$$

where $X(t)$ is the $\mathbb{R}^n$-valued factor process with components $X_j(t)$ and the market parameters $a,\ A,\ b,\ B,\ \Sigma := [\sigma_{ij}]\,,\ i = 1,\ldots,m,\ j = 1,\ldots,N$, $\Lambda := [\Lambda_{ij}]\,,\ i = 1,\ldots,n,\ j = 1,\ldots,N$ are matrices of appropriate dimensions.

Throughout the rest of the chapter, we assume the following:

Assumption 3.1. The matrices $\Sigma\Sigma'$ and $\Lambda\Lambda'$ are positive definite.

The wealth $V(t)$ of the portfolio in response to an investment strategy $h \in \mathcal{A}(T)$ is now factor-dependent with dynamics

$$\frac{dV(t)}{V(t)} = \left(a_0 + A_0'X(t)\right)dt + h'(t)\left(\hat{a} + \hat{A}X(t)\right)dt + h'(t)\Sigma dW_t \tag{3.4}$$

where $\hat{a} := a - a_0\mathbf{1}$, $\hat{A} := A - \mathbf{1}A_0'$, and the initial endowment $V(0) = v$.

The investor seeks to maximize

$$\Phi(t,x) = \sup_{h\in\mathcal{A}} J(t,x,h;T,\gamma) \tag{3.5}$$

where the expected utility of terminal wealth $J(t,x,h;T,\gamma)$ is factor-dependent:

$$J(t,x,h;T,\gamma) = \mathbf{E}\left[U(V_T)\right] = \mathbf{E}\left[\frac{V_T^\gamma}{\gamma}\right] = \mathbf{E}\left[\frac{e^{\gamma\ln V_T}}{\gamma}\right]$$

By Itô's lemma,

$$e^{\gamma\ln V(t)} = v^\gamma \exp\left\{\gamma\int_0^t g(X_s,h(s);\gamma)ds\right\}\chi_t^h \tag{3.6}$$

where

$$g(x,h;\gamma) = -\frac{1}{2}(1-\gamma)\,h'\Sigma\Sigma'h + h'(\hat{a} + \hat{A}x) + a_0 + A_0'x \tag{3.7}$$

and the exponential martingale χ_t^h is still given by (2.4).

Applying the change of measure argument, we obtain the control criterion under the measure $\mathbb{P}_h$

$$I(t,x,h;T,\gamma) = \frac{v^\gamma}{\gamma}\mathbf{E}^h_{t,x}\left[\exp\left\{\gamma\int_t^T g(X_s,h(s);\gamma)ds\right\}\right] \tag{3.8}$$

where $\mathbf{E}^h_{t,x}[\cdot]$ denotes the expectation taken with respect to the measure $\mathbb{P}_h$ and with initial conditions (t,x). The dynamics of the state variable $X(t)$ under the new measure is

$$dX(t) = \left(b + BX(t) + \gamma\Lambda\Sigma' h(t)\right)dt + \Lambda dW^h_t, \qquad t \in [0,T] \tag{3.9}$$

The value function Φ for the auxiliary criterion function $I(t,x;h;T,\gamma)$ is defined as

$$\Phi(t,x) = \sup_{h\in\mathcal{A}(T)} I(t,x;h;T,\gamma) \tag{3.10}$$

After solving the stochastic control problem (see [13] for an outline of the argument connecting ICAPM and risk-sensitive asset management, as well as [5] and [21] for details), we obtain the optimal investment policy $h^*(t)$

$$h^*(t) = \frac{1}{1-\gamma}\left(\Sigma\Sigma'\right)^{-1}\left[\hat{a} + \hat{A}X(t) + \gamma\Sigma\Lambda' D\tilde{\Phi}(t,X(t))\right] \tag{3.11}$$

as well as a solution for the logarithmically transformed value function $\tilde{\Phi}$

$$\frac{1}{\gamma}\ln\Phi(t,x) := \tilde{\Phi}(t,x) = \frac{1}{2}x'Q(t)x + x'q(t) + k(t) \tag{3.12}$$

where Q is a $n \times n$ symmetric non-negative matrix solving a Riccati equation, q is a n-element column vector solving a linear ODE and k is a scalar respectively satisfying (see [21] for details). As a result,

$$h^*(t) = \frac{1}{1-\gamma}\left(\Sigma\Sigma'\right)^{-1}\left[\hat{a} + \hat{A}X(t) + \gamma\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right)\right] \tag{3.13}$$

In the ICAPM, the classical Kelly strategy splitting the wealth of an investor in an allocation to the Kelly portfolio and an allocation to the money market account is no longer optimal. However, a new view of Fractional Kelly investing emerges as a consequence of the following ICAPM Fund Separation theorem:

Theorem 3.2 (ICAPM Fund Separation Theorem)**.** *Any portfolio can be expressed as a linear combination of investments into two funds with respective risky asset allocations:*

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h^I(t) &= -(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right) \end{aligned} \tag{3.14}$$

and respective allocation to the money market account given by

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h_0^I(t) &= 1 + \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right) \end{aligned}$$

Moreover, if an investor has a risk aversion γ, then the respective weights of each mutual fund in the investor's portfolio equal $\frac{1}{1-\gamma}$ and $-\frac{\gamma}{1-\gamma}$, respectively.

In the factor-based ICAPM,

$$\left(\Sigma\Sigma'\right)^{-1}\left[\hat{a} + \hat{A}X(t)\right] \tag{3.15}$$

represents the Kelly (log utility) portfolio and

$$\left(\Sigma\Sigma'\right)^{-1}\Sigma\Lambda'\left(Q(t)X(t) + q(t)\right) \tag{3.16}$$

is the 'intertemporal hedging porfolio' identified by Merton. The appropriate definition of Kelly strategies is not an investment in the Kelly portfolio supplemented by cash, but an investment in the Kelly portfolio and the intertemporal hedging portfolio. This new definition raises some questions as to the practicality of interemporal hedging portfolio as an investment option.

When the asset price noise and the factor noise are uncorrelated, i.e. $\Sigma\Lambda' = 0$, the intertemporal hedging portfolio vanishes and is replaced by an allocation to the money market asset. In this case, fractional Kelly strategies are optimal and we have the following corollary:

Corollary 3.3 (ICAPM Fund Separation Theorem with Uncorrelated Noise)**.** *Any portfolio can be expressed as a linear combination of investments in the Kelly (log-utility) portfolio*

$$h^K(t) = (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \tag{3.17}$$

and the money market asset. Moreover, if an investor has a risk sensitivity γ, the proportion of the Kelly portfolio will equal $\frac{1}{1-\gamma}$ and $-\frac{\gamma}{1-\gamma}$.

3.2 Example: The Relevance of Valuation Factors

In this section, we compare the investment strategies in the ICAPM and the Merton model in a simple setting with $m = 1$ risky asset and $n = 1$ factor.

3.2.1 Setting

The single factor is a short term interest rate behaving according to the Vasicek model (see [45]):

$$dX(t) = (b_0 - b_1 X(t))dt + \Lambda dW(t), \qquad X(0) = x \tag{3.18}$$

or equivalently

$$dX(t) = b_1 \left(\frac{b_0}{b_1} - X(t) \right) dt + \Lambda dW(t),$$

where Λ is a 2-element row vector and $W(t) = (W_1(t), W_2(t))'$ is a two-dimensional Brownian motion where $W_1(t)$ is independent from $W_2(t)$.

The money market account pays the short term rate, and therfore

$$\frac{dS_0(t)}{S_0(t)} = X(t)dt, \qquad S_0(0) = s_0 \tag{3.19}$$

The dynamics of the stock is

$$\frac{dS(t)}{S(t)} = (a + AX(t))dt + \Sigma dW(t), \qquad S(0) = s, \tag{3.20}$$

where Σ is a 2-element row vector. Typically, we would view $W_1(t)$ as the noise associated with the short term interest rate so that $\Lambda = (\lambda, 0)$. The Brownian motion $W_2(t)$ would then capture the idiosyncratic noise in the share. In fact, we could define a new Brownian motion $W_S(t)$ as

$$W_S(t) = \rho W_1(t) + \sqrt{1 - \rho^2} W_2(t)$$

where ρ is the correlation coefficient of $W_S(t)$ and $W_1(t)$, and express the dynamics of the stock as

$$\frac{dS(t)}{S(t)} = (a + AX(t))dt + \sigma_S dW_S(t), \qquad S(0) = s, \tag{3.21}$$

Based on Theorem 3.2, the investor will allocate a fraction $\frac{1}{1-\gamma}$ of his/her wealth to the Kelly portfolio with risky asset allocation

$$h^K(t) \quad = \quad (\Sigma\Sigma')^{-1} \left(a + \hat{A}X(t) \right)$$

and take a *short* position amounting to a fraction $\frac{1}{1-\gamma}$ of his/her wealth in the intertemporal hedging portfolio

$$h^I(t) \quad = \quad -(\Sigma\Sigma')^{-1} \Sigma \Lambda' \left(Q(t)X(t) + q(t) \right)$$

We compare the ICAPM with a naive implementation of the Merton model. The Merton model ignores the valuation factor $X(t)$. One possibility would be to set the risk-free rate r to the long-term average money market rate $\frac{b_0}{b_1}$ so that

$$\frac{dS_0(t)}{S_0(t)} = rdt = \frac{b_0}{b_1}dt, \qquad S_0(0) = s_0 \tag{3.22}$$

Similarly, the drift of the stock could be set to a long-term average $\mu := a + A\frac{b_0}{b_1}$, so that

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma_S dW_S(t), \qquad S(0) = s, \tag{3.23}$$

Based on Theorem 2.14

$$h^K(t) = (\Sigma\Sigma')^{-1}(\mu - r\mathbf{1})$$

In equilibrium, that is when $X(t) = \mathbb{E}[X(t)] =: \bar{X}(t)$, the composition of the Kelly portfolio in the ICAMP and in the classical Merton model will coincide

$$\bar{h}^K(t) = (\Sigma\Sigma')^{-1}\left(a + \hat{A}\bar{X}(t)\right) = (\Sigma\Sigma')^{-1}(\mu - r\mathbf{1})$$

so that the only difference in terms of allocation lies in the intertemporal hedging portfolio.

3.2.2 Numerical Example

We consider an example with $b = 0.04$, $B = -1$ and $\Lambda = (0.08, 0)$, $a = 0.056$, $A = 1.1$ and $\Sigma = (0.20)$, meaning that the noise related to the asset is perfectly positively correlated with the noise from the factor. A perfect correlation between asset and factor noise is unrealistic. However, this condition will help us observe more clearly the behaviour of the intertemporal hedging portfolio. Indeed, the role of the intertemporal hedging portfolio is to use the covariance structure of the asset and factors in order to adjust the risk of the portfolio. If the assets and factor noise are uncorrelated, the intertemporal hedging portfolio has no use and the investor will invest his money in the money market account, as shown in Corollary 3.3.

In equilibrium, the (full) Kelly portfolio is invested at 150% in the stock, regardless of whether we are considering the classical Merton model or the ICAPM. Figure 1 displays the allocation of the intertemporal hedging portfolio to the risky stock for various levels of risk aversion. The proportion of

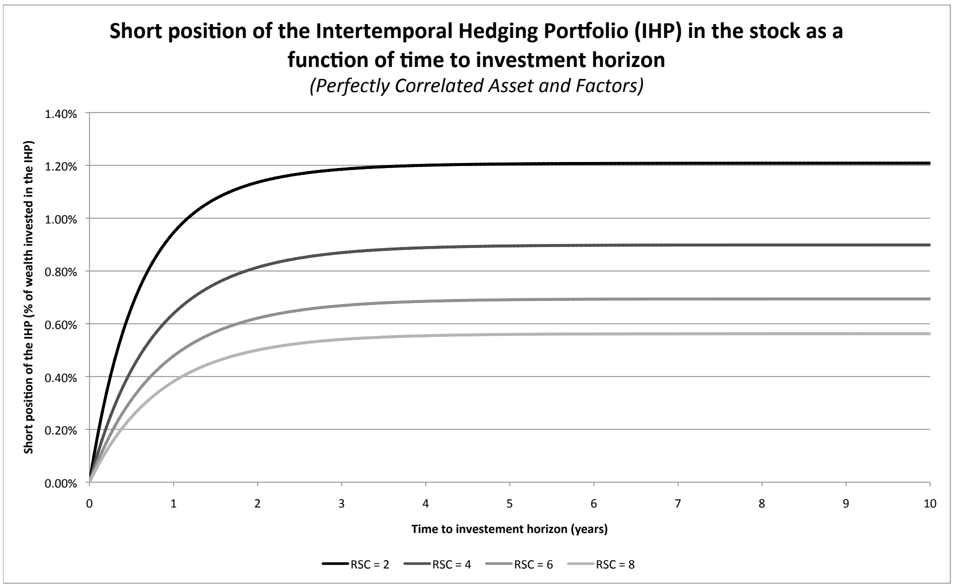


Figure 1: Short position of the Intertemporal Hedging Portfolio (IHP) in the stock as a function of time to investment horizon for various levels of risk aversion (γ = -RSC) in the case of perfectly correlated asset and factors.

the intertemporal hedging portfolio short in the stock is modest: it reaches 1.20% for a risk aversion $\gamma = -2$ and stands at 0.56% for a risk aversion $\gamma = -8$. With 10 years left on the investment horizon, the short position is at its highest. It then declines at an accelerating rate as the horizon draws near, to finally reach zero at the end of the horizon. This is in sharp contrast with the myopic Kelly portfolio which remains fully invested regardless of the investment horizon. Finally, the short position is inversely related to the aversion: the higher the risk aversion, the smaller the short position.

As a result of the difference in magnitude between the Kelly portfolio and the intertemporal hedging portfolio, the proportion of the investor's total wealth invested in the risky stock remains relatively constant through the investment horizon, as shown in Figure 2.

As the value $X(t)$ of the factor varies away from equilibrium conditions, the investment of the Kelly portfolio in the ICAPM will deviate from the

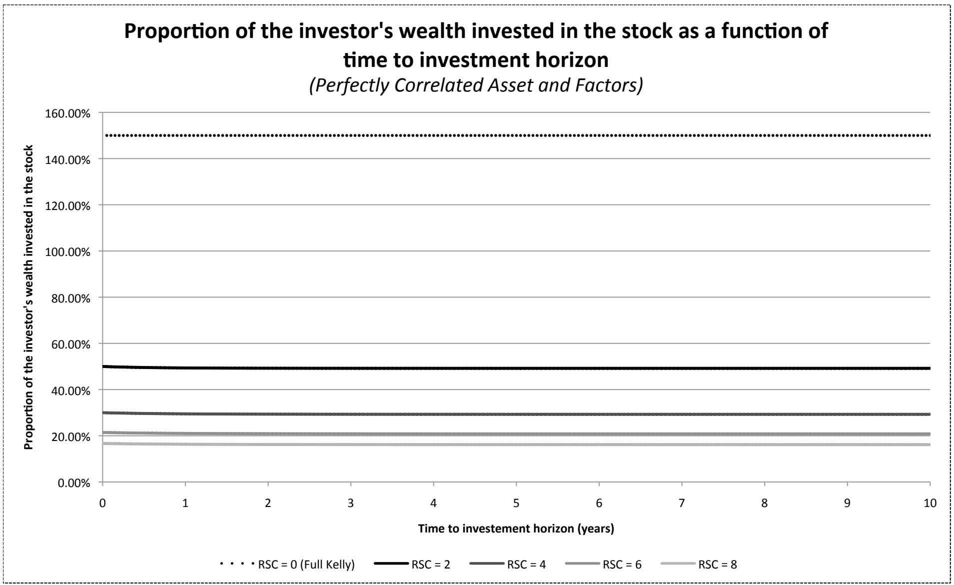


Figure 2: Proportion of the investor's wealth in the stock as a function of time to investment horizon for various levels of risk aversion (γ = -RSC) in the case of perfectly correlated asset and factors.

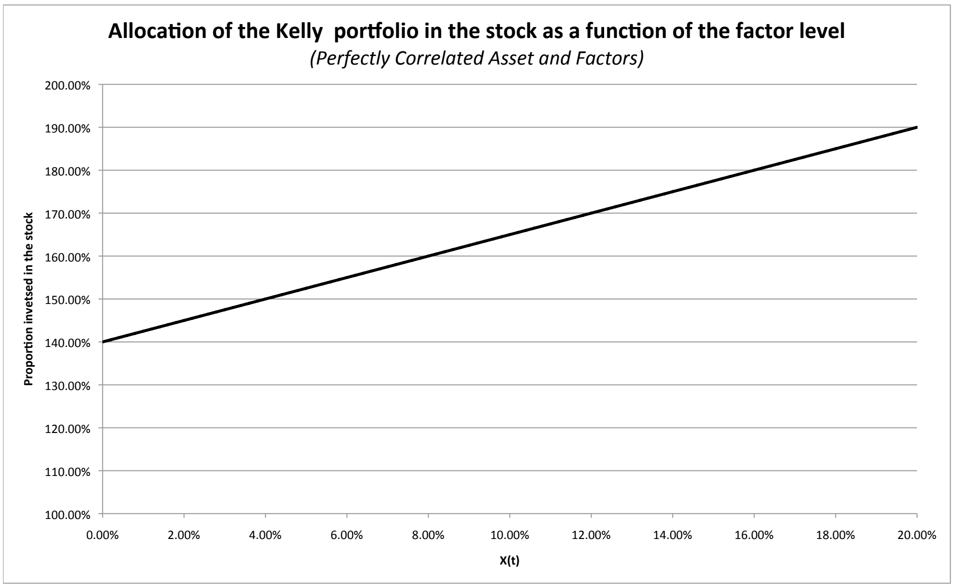


Figure 3: Proportion of the Kelly portfolio invested in the stock as a function of the value of the underlying factor $X(t)$ in the case of perfectly correlated asset and factors.

150% level of the Merton model. Figure 3 shows that in the ICAPM, the allocation of the Kelly portfolio increases linearly with the factor level. The equation of the line is $h^K = 1.40 + 2.5 \times x$, implying that the Kelly portfolio is highly leveraged.

The proportion of the intertemporal hedging portfolio invested in the stock increases with the factor level, as a result of the perfect correlation between the factor noise, although it still remains modest even with a 10-year investment horizon (see Figure 4). Finally, the overall asset allocation is significantly influenced by both the increase in leverage in the Kelly portfolio resulting from a rise in the factor level and the dilution of the Kelly portfolio associated with an increase in the risk aversion (see Figure 5).

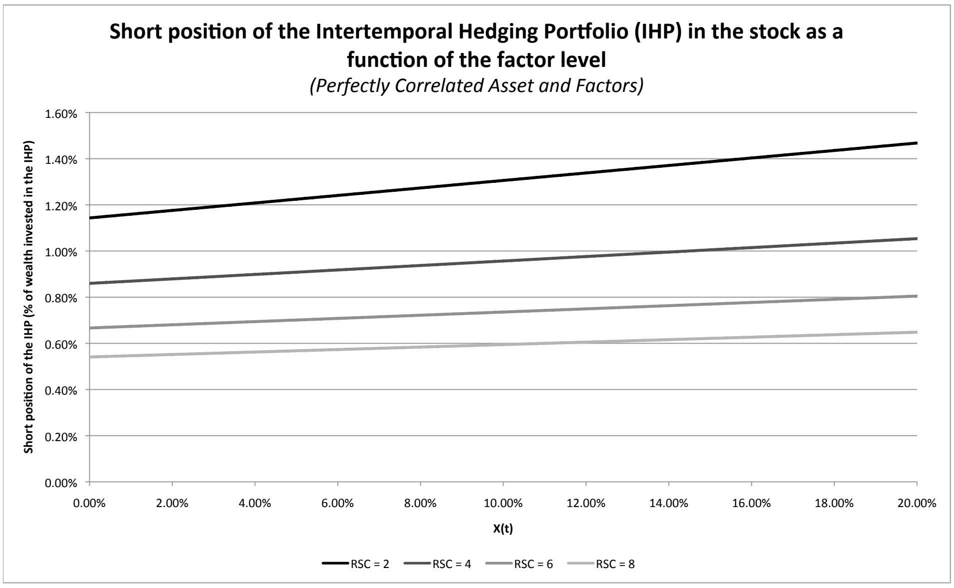


Figure 4: hort position of the Intertemporal Hedging Portfolio (IHP) in the stock as a function of the value of the underlying factor $X(t)$ for various levels of risk aversion (γ = -RSC) in the case of perfectly correlated asset and factors.

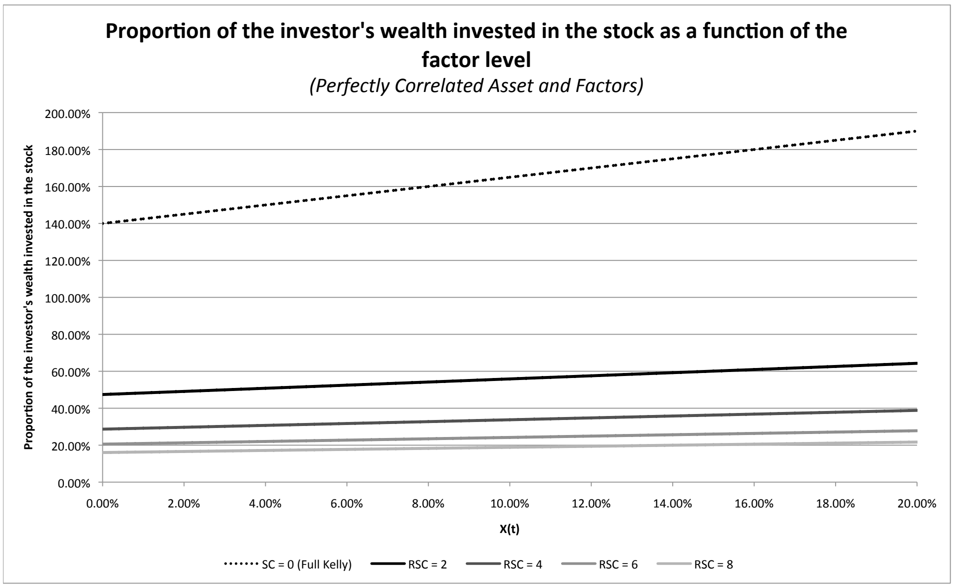


Figure 5: Proportion of the investor's wealth in the stock as a function of the value of the underlying factor $X(t)$ for various levels of risk aversion (γ = -RSC) in the case of perfectly correlated asset and factors.

4 Benchmarked Investment Management

4.1 The Model

Davis and Lleo [11] and [12] study a benchmarked asset management using risk-sensitive control. Merton's ICAPM with no consumption and a power utility function is closely related to risk-sensitive asset management (see [11] and [12]) and therefore to Linear Exponential-of-Quadratic Gaussian (LEQG) stochastic control. The results obtained in [11] and [12] readily extend to a benchmarked version of Merton's ICAPM, up to some minor changes in the value function.

The objective of a benchmarked investor is to outperform a given investment benchmark, such as the S&P 500 or the Salomon Smith Barney World Government Bond Index. Davis and Lleo [11] model the evolution of the benchmark level using the SDE

$$\frac{dL(t)}{L(t)} = (c + C'X(t))dt + \varsigma' dW(t), \qquad L(0) = l \tag{4.1}$$

where c is a scalar constant, C is a n-element column vector and ς is a N-element column vector. The objective of the investor is to maximize the expected utility of terminal outperformance $J(t, x, h; T, \gamma)$:

$$J(t, x, h; T, \gamma) = \mathbf{E}\left[U(F_T)\right] = \mathbf{E}\left[\frac{F_T^{\gamma}}{\gamma}\right] = \mathbf{E}\left[\frac{e^{\gamma \ln F_T}}{\gamma}\right]$$

where $F(t, x; h)$ is defined as the (log) excess return of the investor's portfolio over the return of the benchmark, i.e.

$$F(t, x, h) := \ln \frac{V(t, x, h)}{L(t, x, h)}$$

By Itô's lemma, the log of the excess return in response to a strategy h is

$$\begin{aligned}
F(t, x; h) &= \ln \frac{v}{l} + \int_0^t d \ln V(s) - \int_0^t d \ln L(s) \\
&= \ln \frac{v}{l} + \int_0^t \left(a_0 + A_0' X(s) + h(s)' \left(\hat{a} + \hat{A} X(s)\right)\right) ds \\
&\quad - \frac{1}{2} \int_0^t h(s)' \Sigma \Sigma' h(s) ds + \int_0^t h(s)' \Sigma dW(s) \\
&\quad - \int_0^t (c + C'X(s)) ds + \frac{1}{2} \int_0^t \varsigma' \varsigma ds \\
&\quad - \int_0^t \varsigma' dW(s)
\end{aligned} \tag{4.2}$$

where

$$F(0,x;h) = f_0 := \ln \frac{v}{l}$$

Following an appropriate change of measure along the line described in the previous sections, the criterion function under the new measure can be expressed as

$$I(t,,x,h;T,\gamma) = \frac{f_0^\gamma}{\gamma}\mathbf{E}_{t,x}^h\left[\exp\left\{\gamma\int_t^T g(X_s,h(s);\gamma)ds\right\}\right]$$

where

$$\begin{aligned} g(x,h;\gamma) &= \frac{1}{2}(1-\gamma)\,h'\Sigma\Sigma'h - a_0 - A_0'x - h'(\hat{a}+\hat{A}x) \\ &\quad +\gamma h'\Sigma\varsigma + (c+C'x) + \frac{1}{2}(1-\gamma)\,\varsigma'\varsigma \end{aligned}$$

The value function Φ for the auxiliary criterion function $I(t,x;h;T,\gamma)$ is defined as

$$\Phi(t,x) = \sup_{h\in\mathcal{A}(T)} I(t,x;h;T,\gamma) \tag{4.3}$$

Solving the control problem, we find that the optimal investment policy $h^*(t)$ is given by

$$h^* = \frac{1}{1-\gamma}(\Sigma\Sigma')^{-1}\left(\hat{a}+\hat{A}x+\gamma\Sigma\Lambda'D\Phi-\gamma\Sigma\varsigma\right) \tag{4.4}$$

and the logarithmically-transformed value function $\tilde{\Phi}(t,x) = \frac{1}{\gamma}\ln\Phi(t,x)$ is still given by

$$\Phi(t,x) = x'Q(t)x + x'q(t) + k(t)$$

where $Q(t)$ solves a n-dimensional matrix Riccati equation and $q(t)$ solves a n-dimensional linear ordinary differential equation.

The following fund separation theorem defines benchmarked fractional Kelly strategies:

Theorem 4.1 (Benchmarked Mutual Fund Theorem)**.** *Given a time t and a state vector $X(t)$, any portfolio can be expressed as a linear combination of investments into two "mutual funds" with respective risky asset allocations:*

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a}+\hat{A}X(t)\right) \\ h^C(t) &= (\Sigma\Sigma')^{-1}\left[\Sigma\varsigma - \Sigma\Lambda'\left(q(t)+Q(t)X(t)\right)\right] \end{aligned} \tag{4.5}$$

and respective allocation to the money market account given by:

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h_0^C(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left[\Sigma\varsigma - \Sigma\Lambda'\left(q(t) + Q(t)X(t)\right)\right] \end{aligned}$$

Moreover, if an investor has a risk aversion γ, the respective weights of each mutual fund in the investor's portfolio are equal to $\frac{1}{1-\gamma}$ and $-\frac{\gamma}{1-\gamma}$.

This result splits the benchmarked fractional Kelly strategy in three "funds":

1. An allocation of $\frac{1}{1-\gamma}$ to the Kelly portfolio. As $\gamma \to 0$, this investors's allocation to the Kelly portfolio converges to 100% of his/her wealth, implying that a Kelly investor will ignore the benchmark and focus solely on growth maximization;

2. An allocation to the intertemporal hedging portfolio. Note that the intertemporal hedging portfolio in the benchmarked case is subtly different from its counterpart in the asset-only case. Indeed, the coefficient q solve a slightly different linear ODE which includes terms related to the benchmark dynamics;

3. An allocation to a benchmark-tracking portfolio. The allocation of this portfolio, $(\Sigma\Sigma')^{-1}\Sigma\varsigma$, is in fact a projection of the benchmark risk on the subspace spanned by asset risk, that is an unbiased estimator of a linear relationship between asset risks and benchmark risk $\varsigma = \Sigma' u$;

4.2 Example: Replicable Benchmark

We start with a similar setting as in Section 3.2, namely a one-factor model where the single factor is a short term interest rate behaving according to the Vasicek model:

$$dX(t) = (b_0 - b_1 X(t))dt + \Lambda dW(t), \qquad X(0) = x \tag{4.6}$$

where Λ is now a 3-element row vector and $W(t) = (W_1(t), W_2(t), W_3(t))'$ is a three-dimensional Brownian motion. $W_1(t)$, $W_2(t)$ and $W_3(t)$ are independent standard Brownian motions on $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$. For numerical applications, we still take $b = 0.04$, $B = -1$ and $\Lambda = (0.08, 0)$.

The money market account pays the short term rate, and therfore

$$\frac{dS_0(t)}{S_0(t)} = X(t)dt, \qquad S_0(0) = s_0 \tag{4.7}$$

The investor can buy either of two stocks S_1 and S_2 with respective time-t price $S_1(t)$ and $S_2(t)$. The dynamics of the price vector $S(t) := (S_1(t), S_2(t))$ is given by equation (3.2), that is

$$dS(t) = D\left(S(t)\right)(a + AX(t))dt + D\left(S(t)\right)\Sigma dW(t), \qquad S(0) = s,$$

In our example, we will take

$$a = \begin{pmatrix} 0.018 \\ 0.064 \end{pmatrix}$$

$$A = \begin{pmatrix} 1.3 \\ 0.9 \end{pmatrix}$$

and

$$\Sigma = \begin{pmatrix} 0.108 & 0.0523 & 0 \\ 0.064 & 0 & 0.196 \end{pmatrix}$$

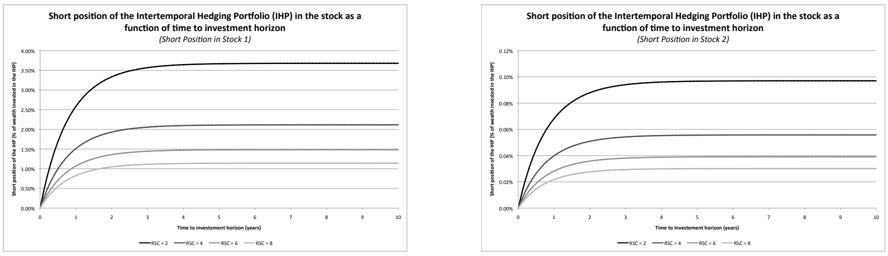
Hence, the drift of stock S_1 is more sensitive to changes in the factor than the drift of stock S_2. To understand the relation between the diffusion of the stock price and the diffusion of the factor, we define two new Brownian motions:

$$W_S^1(t) = \rho_1 W_1(t) + \sqrt{1-\rho_1^2}W_2(t)$$
$$W_S^2(t) = \rho_2 W_1(t) + \sqrt{1-\rho_2^2}W_3(t)$$

The Brownian motions $W_S^1(t)$ and $W_S^2(t)$ represent the respective noise associated with stock S_1 and S_2. They are correlated with the factor noise $W_1(t)$, with correlation coefficient $\rho_1 = 0.9$ and $\rho_2 = 0.2$. Thus, stock S_1 has a lower volatility and a higher correlation with the factor than stock S_2.

The investor's benchmark is an index with a 60% allocation to stock S_1 and a 40% allocation to S_2. The benchmark is therefore easily replicable. Let $w_B = (60\%, 40\%)'$ be the vector of benchmark weights, then the parameters of the benchmark's SDE (4.1) are $c = w_B' a$, $C = w_B' A$ and $\varsigma = w_B' \Sigma$.

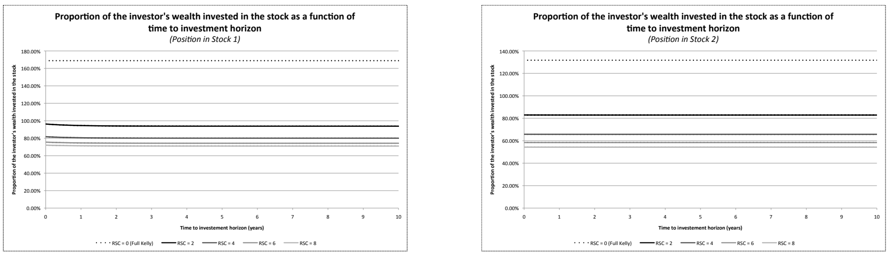
In equilibrium, that is when $X(t) = \bar{X}(t) = 0.04$, the (full) Kelly portfolio is invested at 168.83% in stock S_1 and 131.71% in stock S_2. The Kelly portfolio is myopic: its allocation does not change as the investment horizon changes. It only varies with the factor level. As expected, the allocation to the benchmark replicating portfolio equals $(\Sigma\Sigma')^{-1}\Sigma\varsigma = (60\%, 40\%) = w_B$ irrespective of the time horizon or factor level: we can (and do) fully replicate the benchmark. Figure 6 displays the allocation of the intertemporal hedging portfolio to the two risky stock for various levels of risk aversion. The proportion of the intertemporal hedging portfolio short in either stock is still modest: it reaches 3.68% in stock S_1 and 0.01% in stock S_2 for a



(a) Short position in Stock 1

(b) Short position in Stock 2

Figure 6: Short position of the Intertemporal Hedging Portfolio (IHP) in the stock as a function of time to investment horizon



(a) Allocation to Stock 1

(b) Allocation to Stock 2

Figure 7: Proportion of the investor's wealth in the stock as a function of time to investment horizon for various levels of risk aversion (γ = -RSC)

risk aversion $\gamma = -2$. The evolution of the allocation is similar to what we already observed in the asset only case: the short position is at its highest level when the investment horizon is at 10 years, and it then declines at an accelerating rate to finally reach zero at the end of the horizon. Figure 7 shows that the overall asset allocation is dominated by the Kelly portfolio at low risk-aversion levels and by the benchmark replicating portfolio at high levels of risk aversion. As a result, the proportion of the investor's total wealth invested in the stocks remains relatively constant through the investment horizon.

Keeping the time horizon fixed at ten years, and letting the factor level vary from 0 to 20%, we see that the allocation of the Kelly portfolio to the two stocks evolves in opposite directions (see Figure 8). The asset allocation

h_1^K and h_2^K to the two stocks are an affine function of the factor level x:

$$\begin{aligned} h_1^K &= 0.7960 + 22.3066x \\ h_1^K &= 1.5134 - 4.9071x \end{aligned} \tag{4.8}$$

Not that the sign of the slope and of the intercept reflect the sign of the entries of vectors $\hat{A}$ and $\hat{a}$.In our example, $\hat{A}_1 = 0.3$ while $\hat{A}_2 = -0.1$.

The allocation to the benchmark replicating portfolio remains fixed at 60% in stock S_1 and 40% in stock S_2. The proportion of the intertemporal hedging portfolio short stock S_1 and S_2 increases with the factor level (see Figure 8). The short position for the highly correlated stock S_1 reaches 13.85% for a factor level of 20% and a risk aversion $\gamma = -2$. By comparison, the short position in less correlated stock S_2 is only 0.36% for the same risk factor and risk aversion levels. Overall, the asset allocation is dominated by a combination of the Kelly portfolio and the benchmark replicating portfolio (see Figure 9). Within the range of factor values considered, the allocation to both stocks remains above the benchmark level. This implies that the investor will use leverage, even at fairly high risk aversion levels.

5 ICAPM with Partial Observation

Davis and Lleo [13] propose an extension to the case where the factor process $X(t)$ is not directly observed and the asset allocation strategy h_t must be adapted to the filtration $\mathcal{F}_t^S = \sigma\{S_i(u), 0 \leq u \leq t, i = 0, \dots, m\}$ generated by the asset price processes alone. In the partial observation case, the pair of processes $(X(t), Y(t))$, where $Y_i(t) = \log S_i(t)$, takes the form of the 'signal' and 'observation' processes in a Kalman filter system, and consequently the conditional distribution of $X(t)$ is normal $N(\hat{X}(t), P(t))$ where $\hat{X}(t) = \mathbf{E}[X(t)|\mathcal{F}_t^S]$ satisfies the Kalman filter equation and $P(t)$ is a deterministic matrix-valued function. By using this idea we can obtain an equivalent form of the problem in which $X(t)$ is replaced by $\hat{X}(t)$ and the dynamic equation (3.3) by the Kalman filter. Optimal strategies take the form $h(t, \hat{X}(t))$. This very old idea in stochastic control goes back at least to Wonham [46].

To simplify the presentation of the idea, we will assume that A_0, implying the short-term interest rate is constant. The more general case is outlined in Davis and Lleo [13]. In the partial obsrevation case we need to assume that the initial value X_0 of the factor process is a normal random vector $N(m_0, P_0)$ with known mean m_0 and covariance P_0, and is independent of the Brownian motion W.

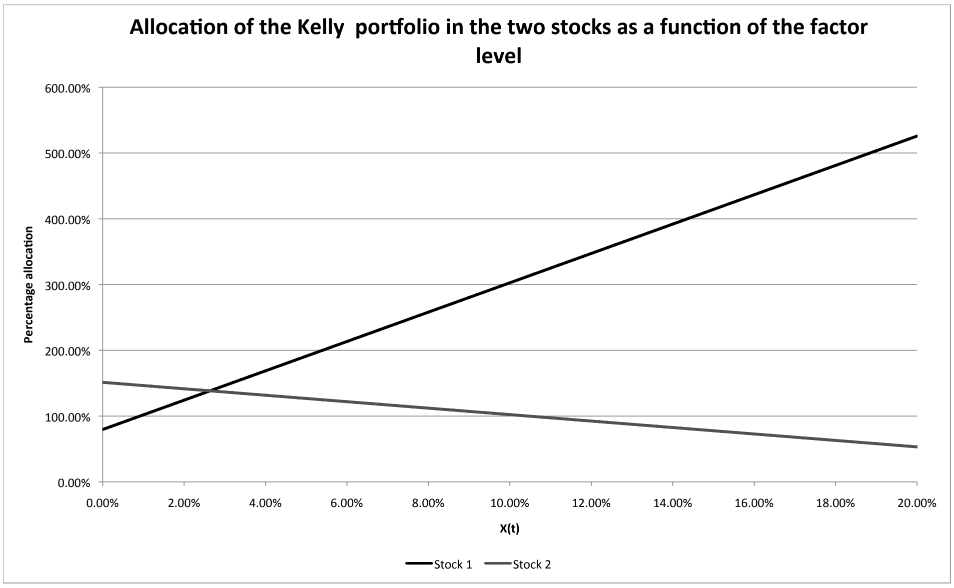
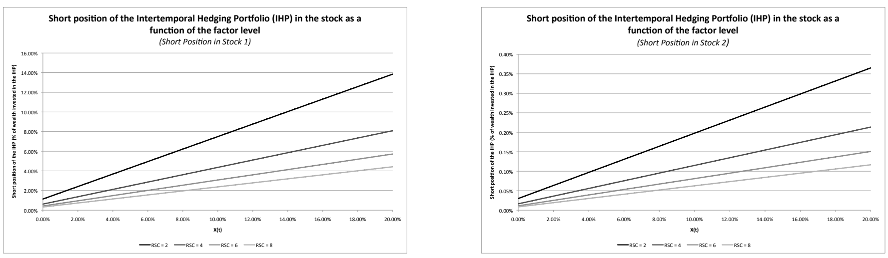


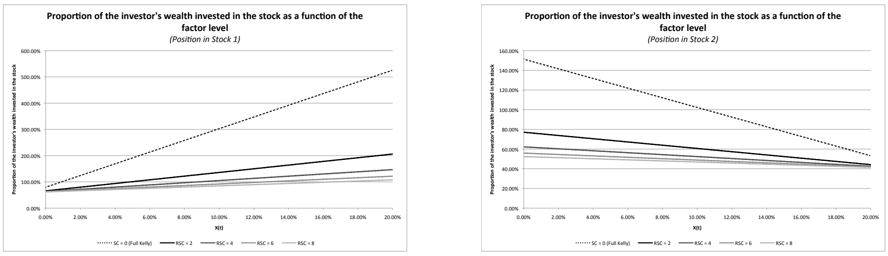
Figure 8: Proportion of the Kelly portfolio invested in the stocks as a function of the value of the underlying factor $X(t)$



(a) Short position in Stock 1

(b) Short position in Stock 2

Figure 9: Short position of the Intertemporal Hedging Portfolio (IHP) in the stock as a function of the value of the underlying factor $X(t)$



(a) Allocation to Stock 1

(b) Allocation to Stock 2

Figure 10: Proportion of the investor's wealth in the stock as a function of the value of the underlying factor $X(t)$

The observation process is the log price process $Y(t) = \ln S_t$:

$$dY_i(t) = \left[(a + AX(t))_i - \frac{1}{2}\Sigma\Sigma'_{ii}\right] dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t),$$
$$Y_i(0) := y_i(0) = \ln S_i(0) \qquad (5.1)$$

The processes $(X(t), Y(t))$ and the filtering equations are standard:

Proposition 5.1 (Kalman Filter)**.** *The conditional distribution of $X(t)$ given $\mathcal{F}_t^S$ is $N(\hat{X}(t), P(t))$, calculated as follows.*
(i) The innovations process *$U(t) \in \mathbb{R}^m$ defined by*

$$dU(t) = \left(\Sigma\Sigma'\right)^{-1/2} (dY(t) - A\hat{X}(t)dt), \qquad U(0) = 0 \qquad (5.2)$$

is a vector Brownian motion.
(ii) $\hat{X}(t)$ is the unique solution of the SDE

$$d\hat{X}(t) = (b + B\hat{X}(t))dt + \left(\Lambda\Sigma' + P(t)A'\right)\left(\Sigma\Sigma'\right)^{-1/2} dU(t), \qquad \hat{X}(0) = m_0. \qquad (5.3)$$

(iii) $P(t)$ is the unique non-negative definite symmetric solution of the matrix Riccati equation

$$\begin{aligned} \dot{P}(t) &= \Lambda\Xi\Xi'\Lambda' - P(t)A'\left(\Sigma\Sigma'\right)^{-1} AP(t) + \left(B - \Lambda\Sigma'\left(\Sigma\Sigma'\right)^{-1} A\right) P(t) \\ &\quad + P(t)\left(B' - A'\left(\Sigma\Sigma'\right)^{-1}\Sigma\Lambda'\right), \qquad P(0) = P_0 \end{aligned}$$

where $\Xi := I - \Sigma'\left(\Sigma'\Sigma\right)^{-1}\Sigma$.

The Kalman filter has replaced our initial state process $X(t)$ by an estimate $\hat{X}(t)$ with dynamics given in (5.3). To recover the asset price process, we use (5.1) together with (5.2) to obtain the dynamics of $Y(t)$

$$dY_i(t) = \left[a + A\hat{X}(t)\right]_i dt - \frac{1}{2}\Sigma\Sigma'_{ii}dt + \sum_{k=1}^{N} \left[\left(\Sigma\Sigma'\right)^{1/2}\right]_{ik} dU_k(t) \tag{5.4}$$

and then apply Itô to $S_i(t) = \exp Y_i(t)$ to get

$$\begin{aligned} dS(t) &= D\left(S(t)\right)(a + A\hat{X}(t))dt + D\left(S(t)\right)\left(\Sigma\Sigma'\right)^{1/2} dU(t) \\ & S_i(0) = s_i, \quad i = 1, \dots, m \end{aligned} \tag{5.5}$$

Now we can solve the stochastic control problem with partial observation as in Section 3 simply by replacing the original asset price description (3.2) by (5.5), and the factor process description (3.3) by the Kalman filter equation (5.3) (see [13] for the full detail). The optimal investment policy $h^*(t)$

$$h^*(t) = \frac{1}{1-\gamma}\left(\Sigma\Sigma'\right)^{-1}\left[\hat{a} + A\hat{X}(t) + \gamma\left(\Lambda\Sigma' + P(t)A'\right)'\left(Q(t)\hat{X}(t) + q(t)\right)\right] \tag{5.6}$$

where $Q(t)$ satisfies a matrix Riccati equation and $q(t)$ satisfies a vector linear ODE.

The following fund separation theorem defines fractional Kelly strategies subject to partial observation:

Corollary 5.2 (Fund Separation Theorem - Partial Observation)**.** *Any portfolio can be expressed as a linear combination of investments into two funds with respective risky asset allocations:*

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a} + A\hat{X}(t)\right) \\ h^{IPO}(t) &= -(\Sigma\Sigma')^{-1}\left(\Sigma\Lambda' + AP(t)\right)\left(Q(t)\hat{X}(t) + q(t)\right) \end{aligned} \tag{5.7}$$

where $P(t)$ solves the matrix Riccati equation (5.4) and $h^{IPO}(t)$ can be viewed as a full allocation to two risky portfolios, an intertemporal portfolio h^I, and a partial observation portfolio h^{PO}:

$$\begin{aligned} h^I(t) &= -(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(Q(t)\hat{X}(t) + q(t)\right) \\ h^{PO}(t) &= -(\Sigma\Sigma')^{-1}AP(t)\left(Q(t)\hat{X}(t) + q(t)\right) \end{aligned} \tag{5.8}$$

The funds have respective allocation to the money market account given by

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'h^K(t) \\ h_0^{IPO}(t) &= 1 + \mathbf{1}'h^{IPO}(t) \end{aligned}$$

Moreover, if an investor has a risk aversion γ, then the respective weights of each mutual fund in the investor's portfolio equal $\frac{1}{1-\gamma}$ and $-\frac{\gamma}{1-\gamma}$, respectively.

This result splits the benchmarked fractional Kelly strategy in three "funds":

1. An allocation of $\frac{1}{1-\gamma}$ to the Kelly portfolio subject to partial observation, that is with the Kalman estimate $\hat{X}(t)$ instead of the true value $X(t)$;

2. An allocation to the intertemporal hedging portfolio subject to partial observation. Here again the intertemporal hedging portfolio in the partial observation case is subtly different from its counterpart in the full observation case;

3. An allocation of to a partial observation portfolio;

6 Fractional Kelly Strategies in a Jump-Diffusion Setting

Optimal investment and consumption problems in a jump diffusion setting have been an active area of research since Merton [33] introduced the possibility of jumps in asset prices. Investigations have tended to concentrate on two axes: the mathematical resolution of the investment problem, and the economic implications of the jumps. Mathematical research has generally focused on the resolution of the HJB partial integro-differential equation, to the detriment of an analysis of the optima asset allocation. Recent references include Øksendal and Sulem [36], Barles and Imbert [2] and Bouchard and Touzi [7] in the general context of jump-diffusion control, as well as Davis and Lleo [15] [16] [14] in the context of risk-sensitive asset management. A notable exception, Aït-Sahalia, Cacho-Diaz and Hurd [1] proposed a method based on orthogonal decompositions to solve jump-diffusion asset allocation problems. In an infinite-horizon setting and under specific assumptions on the jump structure, they derive an analytical solution and a fund separation.

Financial economics research has used jumps to model to model market events and systemic shocks. Although the emphasis is decidedly on investment strategies, this stream of investigation generally takes place in specific models and tends to rely on numerical analysis rather than a mathematical derivation. Liu, Longstaff and Pan [24] use the event risk model proposed by Duffie Pan and Singleton [17] to study the impact of a jump in asset prices and volatility on investment policies. They find that the joint risk of an increase in volatility and of a jump in asset prices lead to conservative investment strategies with limited leveraged and short positions. Das and Uppal [10] model asset prices as jump-diffusion processes to investigate the impact of systemic risk, embodied in an appropriately calibrated jump component, on asset allocations. They conclude that the impact of a systemic event has less to do with reduced diversification benefits than with the large losses leveraged investors experience.

Somewhere in between, Davis and Lleo [13] consider two classes of jump-diffusion factor models. The first has affine drift, constant diffusion and jumps in asset prices only. The second class of models accommodates factor dependent drift and diffusion as well as jumps in factor levels. For clarity, we will only present the affine model with diffusion factors in this section.

6.1 Setting

Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ be the underlying probability space. On this space is defined an $\mathbb{R}^M$-valued $(\mathcal{F}_t)$-Brownian motion $W(t)$ with components $W_k(t)$, $k = 1, \ldots, M$. Moreover, let $(\mathbf{Z}, \mathcal{B}_{\mathbf{Z}})$ be a Borel space[2]. Let $\mathbf{p}$ be an $(\mathcal{F}_t)$-adapted σ-finite Poisson point process on $\mathbf{Z}$ whose underlying point functions are maps from a countable set $\mathbf{D_p} \subset (0, \infty)$ into $\mathbf{Z}$. Define

$$\mathfrak{Z}_{\mathbf{p}} := \{U \in \mathcal{B}(\mathbf{Z}), \mathbb{E}[N_{\mathbf{p}}(t, U)] < \infty \ \forall t\} \tag{6.1}$$

where $N_{\mathbf{p}}(dt, dz)$ is the Poisson random measure on $(0, \infty) \times \mathbf{Z}$ induced by $\mathbf{p}$.

Our analysis focuses on stationary Poisson point processes of class (QL) with associated Poisson random measure $N_{\mathbf{p}}(dt, dz)$. The class (QL) is defined in [19] (Definition II.3.1 p. 59) as

Definition 6.1. An $(\mathcal{F}_t)$-adapted point process $\mathbf{p}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be *of class (QL)* with respect to $(\mathcal{F}_t)$ if it is σ-finite and there exists $\hat{N}_{\mathbf{p}} = \left(\hat{N}_{\mathbf{p}}(t, U)\right)$ such that

(i.) for $U \in \mathfrak{Z}_p$, $t \mapsto \hat{N}_{\mathbf{p}}(t, U)$ is a continuous $(\mathcal{F}_t)$-adapted increasing process;

(ii.) for each t and a.a. $\omega \in \Omega$, $U \mapsto \hat{N}_{\mathbf{p}}(t, U)$ is a σ-finite measure on $(\mathbf{Z}, \mathcal{B}(\mathbf{Z}))$;

(iii.) for $U \in \mathfrak{Z}_p$, $t \mapsto \tilde{N}_{\mathbf{p}}(t, U) = N_{\mathbf{p}}(t, U) - \hat{N}_{\mathbf{p}}(t, U)$ is an $(\mathcal{F}_t)$-martingale;

The random measure $\left\{\hat{N}_{\mathbf{p}}(t, U)\right\}$ is called the *compensator* of the point process p.

[2] $\mathbf{Z}$ is a Polish space and $\mathcal{B}_{\mathbf{Z}}$ is the Borel σ-field

Because the Poisson point processes we consider are stationary, then their compensators are of the form $\hat{N}_{\mathbf{p}}(t,U) = \nu(U)t$ where ν is the σ-finite characteristic measure of the Poisson point process $\mathbf{p}$.

Finally, for notational convenience, we fix throughout the paper a set $\mathbf{Z}_0 \in \mathcal{B}_{\mathbf{Z}}$ such that $\nu(\mathbf{Z}\backslash\mathbf{Z}_0) < \infty$ and define the Poisson random measure $\bar{N}_{\mathbf{p}}(dt,dz)$ as

$$
\begin{aligned}
&\bar{N}_{\mathbf{p}}(dt,dz) \\
= &\begin{cases} N_{\mathbf{p}}(dt,dz) - \hat{N}_{\mathbf{p}}(dt,dz) = N_{\mathbf{p}}(dt,dz) - \nu(dz)dt =: \tilde{N}_{\mathbf{p}}(dt,dz) & \text{if } z \in \mathbf{Z}_0 \\ N_{\mathbf{p}}(dt,dz) & \text{if } z \in \mathbf{Z}\backslash\mathbf{Z}_0 \end{cases}
\end{aligned}
$$

The dynamics of the n factors $X(t)$ and of the wealth invested in the money market account $S_0(t)$ are the same as in the diffusionc ase:

$$
dX(t) = (b + BX(t))dt + \Lambda dW(t), \qquad X(0) = x \tag{6.2}
$$

where $X(t)$ is the $\mathbb{R}^n$-valued factor process with components $X_j(t)$ and $b \in \mathbb{R}^n$, $B \in \mathbb{R}^{n\times n}$ and $\Lambda\mathbb{R}^{n\times M}$,

$$
\frac{dS_0(t)}{S_0(t)} = \left(a_0 + A_0'X(t)\right)dt, \qquad S_0(0) = s_0 \tag{6.3}
$$

where $a_0 \in \mathbb{R}$ is a scalar constant, $A_0 \in \mathbb{R}^n$ is a n-element column vector and throughout the paper x' denotes the transpose of the matrix or vector x.

The securities prices can exhibit jumps: let $S_i(t)$ denote the price at time t of the ith security, with $i = 1,\ldots,m$. The dynamics of risky security i can be expressed as:

$$
\begin{aligned}
\frac{dS_i(t)}{S_i(t^-)} &= (a + AX(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t) + \int_{\mathbf{Z}} \eta_i(z)\bar{N}_{\mathbf{p}}(dt,dz), \\
&\quad S_i(0) = s_i, \quad i = 1,\ldots,m
\end{aligned} \tag{6.4}
$$

where $a \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$, $\Sigma := [\sigma_{ij}]$, $i = 1,\ldots,m$, $j = 1,\ldots,M$ and $\gamma(z) \in \mathbb{R}^m$ satisfying Assumption 6.2:

Assumption 6.2. $\eta(z) \in \mathbb{R}^m$ satisfies

$$
-1 \le \eta_i^{min} \le \eta_i(z) \le \eta_i^{max} < +\infty, \qquad i = 1,\ldots,m
$$

and

$$
-1 \le \eta_i^{min} < 0 < \eta_i^{max} < +\infty, \qquad i = 1,\ldots,m
$$

for $i = 1, \ldots, m$. Furthermore, define

$$\mathbf{S} := \text{supp}(\nu) \in \mathcal{B}_{\mathbf{Z}}$$

and

$$\tilde{\mathbf{S}} := \text{supp}(\nu \circ \gamma^{-1}) \in \mathcal{B}(\mathbb{R}^m)$$

where supp$(\cdot)$ denotes the measure's support, then we assume that $\prod_{i=1}^{m}[\gamma_i^{min}, \gamma_i^{max}]$ is the smallest closed hypercube containing $\tilde{\mathbf{S}}$.

In addition, the vector-valued function $\gamma(z)$ satisfies:

$$\int_{\mathbf{Z}_0} |\eta(z)|^2 \nu(dz) < \infty \tag{6.5}$$

Note that Assumption 6.2 requires that each asset has, with positive probability, both upward and downward jump. The effect of this assumption is to bound the space of controls. Relation (6.5) is a standard condition. See Definition II.4.1 in Ikeda and Watanabe [19] [3].

Define the set $\mathcal{J}$ as

$$\mathcal{J} := \left\{ h \in \mathbb{R}^m : -1 - h'\psi < 0 \quad \forall \psi \in \tilde{\mathbf{S}} \right\} \tag{6.6}$$

For a given z, the equation $h'\gamma(z) = -1$ describes a hyperplane in $\mathbb{R}^m$. $\mathcal{J}$ is a convex subset of $\mathbb{R}^m$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$.

Let $\mathcal{G}_t := \sigma((S(s), X(s)), 0 \leq s \leq t)$ be the sigma-field generated by the security and factor processes up to time t.

Definition 6.3. An $\mathbb{R}^m$-valued control process $h(t)$ is in class $\mathcal{H}$ if the following conditions are satisfied:

1. $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0, t]) \otimes \mathcal{G}_t\}_{t \geq 0}$ and is càdlàg;
2. $P\left(\int_0^T |h(s)|^2 \, ds < +\infty\right) = 1, \quad \forall T > 0$;
3. $h'(t)\gamma(z) > -1, \quad \forall t > 0, z \in \mathbf{Z}$, a.s. $d\nu$.

Define the set $\mathcal{K}$ as

$$\mathcal{K} := \{h \in \mathcal{H} : h \in \mathcal{J} \quad \text{for } a.a.t\} \tag{6.7}$$

Definition 6.4. A control process $h(t)$ is in class $\mathcal{A}(T)$ if the following conditions are satisfied:

[3] In [19], $\mathbf{F_P}$ and $\mathbf{F_P^{2,loc}}$ are respectively given in equations II(3.2) and II(3.5)

1. $h \in \mathcal{H}$;
2. $\mathbf{E}\chi_T^h = 1$ where χ_t^h, $t \in (0,T]$, is the Doléans exponential defined as

$$
\begin{aligned}
\chi_t^h \quad := \quad & \exp\left\{\gamma \int_0^t h(s)'\Sigma dW_s - \frac{1}{2}\gamma^2 \int_0^t h(s)'\Sigma\Sigma'(h(s)ds \right. \\
& + \int_0^t \int_{\mathbf{Z}} \ln\left(1 - G(z,h(s))\right) \tilde{N}(ds,dz) \\
& \left. + \int_0^t \int_{\mathbf{Z}} \left\{\ln\left(1 - G(z,h(s))\right) + G(z,h(s))\right\} \nu(dz)ds \right\},
\end{aligned}
\tag{6.8}
$$

and

$$
G(z,h) \quad = \quad 1 - \left(1 + h'\eta(z)\right)^{\gamma} \tag{6.9}
$$

Definition 6.5. We say that a control process $h(t)$ is *admissible* if $h(t) \in \mathcal{A}(T)$.

Taking this budget equation into consideration, the wealth, $V(t)$ of the investor in response to an investment strategy $h(t) \in \mathcal{H}$, follows the dynamics

$$
\frac{dV(t)}{V(t^-)} = \left(a_0 + A_0'X(t)\right)dt + h'(t)\left(\hat{a} + \hat{A}X(t)\right)dt + h'(t)\Sigma dW_t + \int_{\mathbf{Z}} h'(t)\eta(z)\bar{N}_{\mathbf{p}}(dt,dz) \tag{6.10}
$$

where $\hat{a} := a - a_0\mathbf{1}$, $\hat{A} := A - \mathbf{1}A_0'$, $\mathbf{1} \in \mathbf{R}^m$ denotes the m-element unit column vector and with initial endowment $V(0) = v$.

6.2 Control Problem

The objective of an investor is to maximize the expected utility of terminal wealth over a fixed time horizon T:

$$
J(t,h;T,\gamma) \quad = \quad \mathbf{E}\left[U(V_T)\right] = \mathbf{E}\left[\frac{V_T^{\gamma}}{\gamma}\right] = \mathbf{E}\left[\frac{e^{\gamma \ln V_T}}{\gamma}\right]
$$

By Itô,

$$
e^{\gamma \ln V(t)} \quad = \quad v^{\gamma} \exp\left\{\gamma \int_0^t g(X_s,h(s))ds\right\} \chi_t^h \tag{6.11}
$$

where

$$
\begin{aligned}
g(x,h) \quad = \quad & -\frac{1}{2}(1-\gamma)h'\Sigma\Sigma'h + a_0 + A_0'x + h'\left(\hat{a} + \hat{A}x\right) \\
& + \int_{\mathbf{Z}} \left\{\frac{1}{\gamma}\left[\left(1 + h'\eta(z)\right)^{\gamma} - 1\right] + h'\eta(z)1_{\mathbf{Z}_0}(z)\right\} \nu(dz)
\end{aligned}
\tag{6.12}
$$

and the Doléans exponential χ_t^h is given by (6.8).

Let $\mathbb{P}_h$ be the measure on $(\Omega, \mathcal{F}_T)$ defined via the Radon-Nikodým derivative

$$\frac{d\mathbb{P}_h}{d\mathbb{P}} := \chi_T^h \tag{6.13}$$

For a change of measure to be possible, we must ensure that the following technical condition holds:

$$G(z, h(s)) < 1$$

This condition is satisfied iff

$$h'(s)\eta(z) > -1 \tag{6.14}$$

a.s. $d\nu$, which was already required for $h(t)$ to be in class $\mathcal{H}$ (Condition 3 in Definition 6.3). Thus $\mathbb{P}_h$ is a probability measure for $h \in \mathcal{A}(T)$. For $h \in \mathcal{A}(T)$,

Moreover,

$$W_t^h = W_t - \gamma \int_0^t \Sigma' h(s) ds$$

is a standard $\mathbb{P}_h$-Brownian motion and we define the $\mathbb{P}_h$-compensated Poisson random measure by

$$\begin{aligned}\int_0^t \int_{\mathbf{Z}_0} \tilde{N}^h(ds, dz) &= \int_0^t \int_{\mathbf{Z}_0} N(ds, dz) - \int_0^t \int_{\mathbf{Z}_0} \{1 - G(z, h(s))\} \nu(dz) ds \\ &= \int_0^t \int_{\mathbf{Z}_0} N(ds, dz) - \int_0^t \int_{\mathbf{Z}_0} \left\{ \left(1 + h'\eta(z)\right)^\gamma \right\} \nu(dz) ds\end{aligned}$$

As a result, $X(s),\ 0 \le s \le t$ satisfies the SDE:

$$dX(s) = f(X(s), h(s))ds + \Lambda dW_s^h \tag{6.15}$$

where

$$f(x, h) := b + Bx + \gamma \Lambda \Sigma' h \tag{6.16}$$

Applying the change of measure argument, we obtain the criterion function under the measure $\mathbb{P}_h$:

$$I(t, x, h; T; \gamma) = \frac{v^\gamma}{\gamma} \mathbf{E}_{t,x}^h \left[\exp\left\{ \gamma \int_t^T g(X_s, h(s)) ds \right\} \right] \tag{6.17}$$

The value function Φ for the auxiliary criterion function $I(t,x,h;T;\gamma)$ is defined as

$$\Phi(t,x) = \inf_{h\in\mathcal{A}} I(t,x,h;T;\gamma) \tag{6.18}$$

The corresponding HJB PIDE is

$$\frac{\partial \Phi}{\partial t}(t,x) + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi(t,x)\right) + H(x,\Phi,D\Phi) \tag{6.19}$$

subject to terminal condition

$$\Phi(T,x) = \frac{v^\gamma}{\gamma} \tag{6.20}$$

where for $r\in\mathbb{R}$, $p\in\mathbb{R}^n$

$$H(x,r,p) = \inf_{h\in\mathcal{J}} \left\{ f(x,h)'p - \gamma g(x,h) r \right\} \tag{6.21}$$

In this setting, the change of measure argument has reduced the problem to solving a stochastic control problem in the factor process, which has no jumps. As a result, we only need to solve a parabolic PDE, not a Partial Integro-Differential Equation (PIDE), which makes it easier to show that the HJB PDE admits a unique classical solution (see [15] for the full detail).

To obtain the optimal control we will introduce the associated risk-sensitive value function $\tilde{\Phi}(t,x) := \frac{1}{\gamma}\ln\Phi(t,x)$ with associated HJB PIDE

$$\frac{\partial \tilde{\Phi}}{\partial t}(t,x) + \sup_{h\in\mathcal{J}} L^h(x,D\tilde{\Phi},D^2\tilde{\Phi}) = 0 \tag{6.22}$$

where

$$L^h(x,p,M) = f(x,h)'p + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' M\right) + \frac{\gamma}{2}p'\Lambda\Lambda' p - g(x,h) \tag{6.23}$$

and subject to the terminal condition

$$\tilde{\Phi}(T,x) = \ln v - \frac{1}{\gamma}\ln\gamma, \qquad x\in\mathbb{R}^n. \tag{6.24}$$

The supremum in (6.22) can be expressed as

$$\begin{aligned}
&\sup_{h\in\mathbb{R}^m} L^h(x,p,M) \\
= \;& (b+Bx)'p + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' M\right) + \frac{\gamma}{2}p'\Lambda\Lambda' p + a_0 + A_0 x \\
&+ \sup_{h\in\mathcal{J}} \left\{ \frac{1}{2}(\gamma-1) h'\Sigma\Sigma' h + \gamma h'\Sigma\Lambda' p + h'\left(\hat{a} + \hat{A}x\right) \right. \\
&\left. + \frac{1}{\gamma}\int_{\mathbf{Z}} \left\{ \left[\left(1+h'\eta(z)\right)^\gamma - 1\right] - \gamma h'\eta(z) 1_{\mathbf{Z}_0}(z) \right\} \right\}
\end{aligned} \tag{6.25}$$

Proposition 6.6. *The supremum in* (6.22) - (6.23) *admits a unique Borel measurable maximizer* $\hat{h}(t,x,p)$ *for* $(t,x,p) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$.

Proof. See Section 3.3 in [15]. □

6.3 Fractional Kelly Strategy

The basic idea in a Fund Separation Theorem is that any admissible portfolio $h(t)$ satisfying Definition 6.4 can be expressed as a linear combination of two (or more) portfolios. For example, one could find two portfolios A and B with respective asset allocations h^A and h^B such that

$$h(t) = w_A h^A(t) + (1 - w_A) h^B(t)$$

This decomposition is not unique. Focusing on optimal controls $h^*(t)$ only, we could take $w_A = \frac{1}{1-\gamma}$ to guarantee a decomposition between the Kelly portfolio h^K and an intertemporal portfolio h^I_γ

$$h^*(t) = \frac{1}{1-\gamma} h^K(t) + \frac{\gamma}{\gamma - 1} h^I_\gamma(t)$$

The intertemporal hedging portfolio h^I_γ depends indirectly on the risk aversion level γ and as such is not universal.

Theorem 6.7 (Fund Separation Theorem in a Jump-Diffusion Setting)**.** *The optimal asset allocation for an investor with a risk aversion* γ *can be expressed as a linear combination of an investment in the Kelly criterion (log-utility) portfolio and in an intertemporal hedging portfolio:*

$$h^*(t) = \frac{1}{1-\gamma} h^K(t) - \frac{\gamma}{1-\gamma} h^I(t)$$

where the risky allocation h^K *to the Kelly criterion (log-utility) portfolio solves the fixed point problem*

$$h = \left(\Sigma\Sigma'\right)^{-1} \left[\left(\hat{a} + \hat{A}x \right) + \int_{\mathbf{Z}} \left\{ \frac{\eta(z)}{1 + h'\eta(z)} - \eta(z) 1_{\mathbf{Z}_0}(z) \right\} \nu(dz) \right] \tag{6.26}$$

and the risky allocation to the intertemporal hedging portfolio h^I_γ *satisfies:*

$$h^I_\gamma(t) = \frac{\gamma - 1}{\gamma} h^*(t) - \frac{1}{\gamma} h^K(t)$$

The respective allocation to the money market account for each of the two funds are given by

$$\begin{aligned} h_0^K(t) &= 1 - h^K(t) \\ h_0^I(t) &= 1 - h_\gamma^I(t) \end{aligned}$$

Proof. Consider the auxiliary function ℓ defined as

$$\begin{aligned} \ell(h;x,p) &= \frac{1}{2}(1-\gamma)\, h'\Sigma\Sigma' h - \gamma h'\Sigma\Lambda' p - h'\left(\hat{a} + \hat{A}x\right) \\ &\quad - \frac{1}{\gamma}\int_{\mathbf{Z}} \left\{\left[\left(1 + h'\eta(z)\right)^\gamma - 1\right] - \gamma h'\eta(z) \mathit{1}_{\mathbf{Z}_0}(z)\right\} \nu(dz) \end{aligned} \tag{6.27}$$

for $h \in \mathcal{J}$, $x \in \mathbb{R}^n$, $p \in \mathbb{R}^n$.

A Taylor expansion of the integral term $\frac{1}{\gamma}\int_{\mathbf{Z}} \{[(1 + h'\eta(t,z))^\gamma - 1]\}\,\nu(dz)$ around $h = 0$ yields

$$\begin{aligned} &\frac{1}{\gamma}\int_{\mathbf{Z}} \left\{\left[\left(1 + h'\eta(z)\right)^\gamma - 1\right]\right\} \nu(dz) \\ = &\int_{\mathbf{Z}} \left\{ h'\eta(z) + \frac{\gamma-1}{2}\left(h'\eta(z)\right)^2 + \frac{(\gamma-1)(\gamma-2)}{3!}\left(h'\eta(z)\right)^3 + \ldots \right. \\ &\left. + \frac{(\gamma-1)(\gamma-2)...(\gamma-1)(\gamma-k+1)}{k!}\left(h'\eta(z)\right)^k + \ldots \right\} \nu(dz) \end{aligned}$$

Taking the limit as $\gamma \to 0$, we obtain:

$$\lim_{\gamma\to 0}\frac{1}{\gamma}\int_{\mathbf{Z}} \left\{\left[\left(1 + h'\eta(z)\right)^\gamma - 1\right]\right\} \nu(dz) = \int_{\mathbf{Z}} \left\{\ln\left[1 + h'\eta(z)\right]\right\} \nu(dz)$$

We now define the function $\ell^K(h;x,p)$ as the limit of ℓ as $\gamma \to 0$:

$$\begin{aligned} \ell^K(h;x,p) &= \lim_{\gamma\to 0} \ell(h;x,p) \\ &= \frac{1}{2} h'\Sigma\Sigma' h - h'\left(\hat{a} + \hat{A}x\right) - \int_{\mathbf{Z}} \left\{\ln\left[1 + h'\eta(z)\right] - h'\eta(t,z) \mathit{1}_{\mathbf{Z}_0}(z)\right\} \nu(dz) \end{aligned} \tag{6.28}$$

for $h \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ and $p \in \mathbb{R}^n$.

The Kelly allocation is the unique minimizer of (6.28). Applying the first order condition, we conclude that h^k satisfies

$$\begin{aligned} &\frac{\partial \ell^K}{\partial h} = 0 \\ \Leftrightarrow\quad & \Sigma\Sigma' h - \left(\hat{a} + \hat{A}x\right) - \int_{\mathbf{Z}} \left\{\frac{\eta(z)}{1 + h'\eta(z)} - \eta(z)\mathit{1}_{\mathbf{Z}_0}(z)\right\} \nu(dz) = 0 \\ \Leftrightarrow\quad & h = \left(\Sigma\Sigma'\right)^{-1}\left[\left(\hat{a} + \hat{A}x\right) + \int_{\mathbf{Z}} \left\{\frac{\eta(z)}{1 + h'\eta(z)} - \eta(z)\mathit{1}_{\mathbf{Z}_0}(z)\right\} \nu(dz)\right] \end{aligned}$$

□

It is difficult to get much more intuition regarding the behaviour of the intertemporal hedging portfolio due to its dependence on the risk-aversion γ, the asset price jumps and the value function $\tilde{\Phi}$. Practically, one can estimate the intertemporal hedging portfolio through numerical methods such as a policy improvement scheme (see Bellman [4], Fleming and Richel [18] and Davis and Lleo [15]) or a finite difference method (see for example Kushner and Dupuis [22]). Davis and Lleo [13] show how to adapt a policy improvement scheme to compute numerically the allocation to the intertemporal hedging portfolio.

Corollary 6.8 (Three Fund Separation Theorem in a Jump-Diffusion Setting)**.** *The optimal asset allocation for an investor with a risk aversion γ can be expressed as a linear combination of an investment in the Kelly criterion (log-utility) portfolio and in an intertemporal hedging portfolio:*

$$h^*(t) = \frac{1}{1-\gamma} h^K(t) - \frac{\gamma}{1-\gamma} h^I(t)$$

The risky allocation h^K to the Kelly criterion (log-utility) portfolio solves the fixed point problem

$$h = \left(\Sigma\Sigma'\right)^{-1} \left[\left(\hat{a} + \hat{A}X\right) + \int_{\mathbf{Z}} \left\{ \frac{\eta(z)}{1 + h'\eta(z)} - \eta(z) \mathit{1}_{\mathbf{Z}_0}(z) \right\} \nu(dz) \right] \tag{6.29}$$

Alternatively, the risky allocation h^K to the Kelly criterion (log-utility) portfolio can be expressed as a decomposition bewteen the standard Kelly allocation h^{KD} and a jump-related portfolio h^{KJ}:

$$h^K = h^{KD} + h^{KJ} \tag{6.30}$$

where the standard Kelly allocation h^{KD} is given by

$$h^{KD} = \left(\hat{a} + \hat{A}X\right) \tag{6.31}$$

and the jump-related portfolio h^{KJ} solves the fixed point problem

$$h = \left(\Sigma\Sigma'\right)^{-1} \int_{\mathbf{Z}} \left\{ \frac{\eta(z)}{1 + h'\eta(z)} - \eta(z) \mathit{1}_{\mathbf{Z}_0}(z) \right\} \nu(dz) + h^{KS} \tag{6.32}$$

The risky allocation to the intertemporal hedging portfolio h^I_γ can be expressed as

$$h^I_\gamma(t) = h^\gamma(t) + \bar{h}^I(t)$$

that is an allocation between

i. the fully risk-averse portfolio $\bar{h}^I \in \mathcal{S}$ staisfying

$$h^I(t) = -\left(\Sigma\Sigma'\right)^{-1}\Sigma\Lambda' D\tilde{\Phi}^I \tag{6.33}$$

and with $\mathcal{S} := \left\{h \in \mathcal{J} : h'\psi \geq 0 \quad \forall \psi \in \tilde{\mathbf{S}}\right\}$

ii. the risk-aversion induced portfolio h^γ.

The respective allocation to the money market account for each of the two funds are given by

$$\begin{aligned} h^K_0(t) &= 1 - h^K(t) \\ h^I_0(t) &= 1 - h^I_\gamma(t) \end{aligned}$$

7 Conclusion

In this chapter, we have presented an overview of some recent developments related to Kelly investment strategies. In particular, we showed how the definition of fractional Kelly strategies can be extended to guarantee optimality when asset prices are no longer lognormally distributed. The key idea is to get the definition of fractional Kelly strategies to correspond with the fund separation theorem related to the problem at hand. In these instances, fractional Kelly investment strategies appear as the natural solution for investors seeking to maximize the terminal power utility of their wealth.

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