Variational Methods and Special Holonomy

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1 On the project

Metrics of special holonomy are at the crossroads of Riemannian, symplectic and complex geometry and play an important rôle in modern developments of differential geometry and theoretical physics. Further advances in the understanding of these metrics are clearly desirable. Our research project approached these metrics from a variational point of view which more generally leads to the investigation of various deformation and moduli spaces of special geometric structures. Within this framework we worked on several subprojects.

1.1 Energy functionals and soliton equations for G_2-forms (H. Weiß and F. Witt)

We extended short-time existence and stability of the Dirichlet energy flow as introduced in an earlier paper of ours [6] to a broader class of energy functionals. Furthermore, we derived some monotonely decreasing quantities for the Dirichlet energy flow and investigated an equation of soliton type. In particular, we showed that nearly parallel G_2-structures satisfy this soliton equation and studied their infinitesimal soliton deformations. This paper was finalised at HIM.

1.2 A spinorial energy functional: critical points and gradient flow (B. Ammann, H. Weiß and F. Witt)

On the universal bundle of unit spinors we studied a natural energy functional whose critical points, if $\dim M \geq 3$, are precisely the pairs (g, ϕ) consisting of a Ricci-flat Riemannian metric g together with a parallel g-spinor ϕ. In particular, the underlying metric is of special holonomy. We investigated the basic properties of this functional and studied its negative gradient flow. In particular, we proved short-time existence and uniqueness for this flow. Furthermore, we investigated the moduli space of critical points and showed smoothness under additional assumptions. This paper was initiated at HIM and written jointly with Bernd Ammann (Universität Regensburg).

1.3 Morse homology for the Yang–Mills gradient flow (J. Swoboda)

We used the Yang-Mills gradient flow on the space of connections over a closed Riemann surface to construct a Morse chain complex. The chain groups are generated by Yang-Mills

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connections. The boundary operator is defined by counting the elements of appropriately defined moduli spaces of Yang-Mills gradient flow lines that converge asymptotically to Yang-Mills connections. This paper was finalised at HIM.

1.4 A symplectically non-squeezable small set and the regular coisotropic capacity (J. Swoboda und F. Ziltener)

We proved that for $n \geq 2$ there exists a compact subset X of the closed ball in $\mathbb{R}^{2n}$ of radius $\sqrt{2}$, such that X has Hausdorff dimension n and does not symplectically embed into the standard open symplectic cylinder. The second main result gives a lower bound on the d-th regular coisotropic capacity, which is sharp up to a factor of 3. For an open subset of a geometrically bounded, aspherical symplectic manifold, this capacity is a lower bound on its displacement energy. The proofs of the results involved a certain Lagrangian submanifold of linear space, which was considered by M. Audin [1] and L. Polterovich [5]. Essential parts of this paper were written together with Fabian Ziltener (KIAS Seoul) at HIM.

1.5 SQuaRE project “Nonlinear analysis and special geometric structures” (R. Mazzeo, J. Swoboda, H. Weiß and F. Witt)

Apart from taking individual projects further we quickly focused on a new project which involved all three of us. Inspired by very useful conversations with Justin Sawon on the moduli space of Higgs bundles and related issues as well as by our jointly organised conference on hyperkähler geometry, we decided to attack Hausel's conjecture [2, 3] on the L^2-cohomology of Hitchin's Higgs bundle moduli space [4]. Together with Rafe Mazzeo (Stanford University) we formulated a SQuaRE project “Nonlinear analysis and special geometric structures" (http://www.aimath.org/research/squares.html) which was accepted in December 2011.

2 Our stay at HIM

During our stay at HIM we interacted quite a lot with other participants such as H.-J. Hein, R. Haslhofer, R. Glover, J. Sawon and J. Nordström through various topics such as geometric flows, generalised geometry (à la Hitchin), hyperkähler geometry and G_2-manifolds. As our guests we were pleased to have with us Fabian Ziltener (KIAS Seoul) and Michael Struwe (ETH Zürich) who also delivered a small lecture series on *Conformal metrics of prescribed Gauss curvature on closed surfaces of higher genus*. Apart from giving talks at the MPI Seminar, the workshop on geometric flows and in the trimester seminar (co-)organised by J. Swoboda, we organised (jointly with J. Sawon) the workshop on hyperkähler geometry. As a whole we found very good working conditions and our stay at HIM very stimulating.

3 Publications

- B. Ammann, H. Weiß and F. Witt, *A spinorial energy functional: critical points and gradient flow*, submitted.
- J. Swoboda, *Morse homology for the Yang-Mills gradient flow*, J. Math. Pures Appl. **98**, 160–210, 2012.
- J. Swoboda and F. Ziltener, *A symplectically non-squeezable small set and the regular coisotropic capacity*, to appear in: J. Symplectic Geometry

- H. Weiß and F. Witt, *Energy functionals and soliton equations for* G_2*-forms.* Ann. Global Anal. Geom. **42** (2012) no. 4, 585–610.

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