REPORT ON OUR PROJECT
“*PRESCRIBED SCALAR CURVATURE ON OPEN MANIFOLDS*”
AT THE HAUSDORFF INSTITUTE (SEPTEMBER – DECEMBER 2011)

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The project was centered on the scalar curvature of Riemannian metrics on open, i.e. noncompact connected, manifolds. In the first part of the project we examined properties of a conformal invariant, the Yamabe constant; this subproject led to a publication and is now finished. In the second part we investigated — and still investigate — to which extent a given function on an open manifold can be realised as the scalar curvature of a Riemannian metric.

1. The Yamabe constant

The *Yamabe constant* $Y(g) \in [-\infty, \infty[$ of a nonempty Riemannian manifold (M, g) of dimension $n \geq 3$ is the infimum of $Q_g(v)$ over all compactly supported smooth functions $v\colon M \to \mathbb{R}_{\geq 0}$ which are not identically 0; here $Q_g(v) \in \mathbb{R}$ is defined by (see [16])

$$Q_g(v) = \frac{\int_M \left(\frac{4(n-1)}{n-2}|\mathrm{d}v|_g^2 + \mathrm{scal}_g v^2\right)\mathrm{d}\mu_g}{\left(\int_M v^{2n/(n-2)}\mathrm{d}\mu_g\right)^{(n-2)/n}} \quad .$$

$Y(g)$ depends only on the conformal class of g. The existence of constant scalar curvature metrics in this conformal class is closely related to $Y(g)$. Therefore $Y(g)$ is a very important and especially on closed manifolds extensively investigated quantity [1],[2],[14],[16],[18]. The noncompact case has been considered for instance in [13, 19, 6].

Bérard Bergery proved in [3] that on a *compact* manifold M, the function $g \mapsto Y(g)$ is continuous with respect to the C^2-topology on the space of Riemannian metrics on M. We generalized this to noncompact manifolds in the following way:

1.1. **Theorem.** [7] *For every nonempty manifold M of dimension ≥ 3, the map $g \mapsto Y(g) \in [-\infty, \infty[$ is continuous with respect to the fine (a.k.a.* strong *or* Whitney*) C^2-topology on the space of Riemannian metrics on M.*

The definition of the fine C^k-topology can be found in [10], for instance. On compact manifolds, it is equal to the compact-open C^k-topology; but in the noncompact case, it is much finer (not even first countable, in particular not metrizable). We also discussed to which extent the Yamabe constant is continuous with respect to coarser topologies on the space of Riemannian metrics. On each open manifold, continuity with respect to any compact-open C^k-topology fails at every metric. Continuity with respect to any uniform C^k-topology fails at least at certain metrics on some manifolds.

This part of the project resulted in the publication [7], which contains also other results that we are not going to discuss in the present report.

2. PRESCRIBED SCALAR CURVATURE

Kazdan–Warner proved in 1975 [12, Theorem 1.4] that on every open manifold M of dimension $n \geq 2$ *which can be embeddded into a compact n-manifold*, every smooth function $s\colon M \to \mathbb{R}$ is the scalar curvature of a smooth Riemannian metric. Every manifold (without embeddability assumption) of dimension ≥ 3 admits a complete Riemannian metric of constant negative scalar curvature [4]. (In contrast, many open manifolds do not admit a *complete* Riemannian metric of positive scalar curvature [5], and many closed manifolds do not admit a scal > 0 metric either [17].)

However, it is still not known [11] whether our following conjecture is true:

Conjecture. *On every open manifold M of dimension ≥ 2 (without further topological assumptions), every smooth function $s\colon M \to \mathbb{R}$ is the scalar curvature of a smooth Riemannian metric.*

We intend to solve this problem in our second project, as a corollary to much sharper results. Namely, we consider the following problem: Given are a smooth function $s\colon M \to \mathbb{R}$, a codimension-zero submanifold-with-boundary A of M which is a closed subset of M, and a smooth metric g on M whose scalar curvature is equal to s on the set A. The task is to decide whether there exists a metric g' on M which coincides with g on A and whose scalar curvature is s on all of M.

We splitted this project into three steps. First we asked whether one can always find a metric g_1 that coincides with g on A and whose scalar curvature is larger than s on $M \setminus A$. This part is finished, the answer is always *yes* if each connected component of $M\setminus A$ is not relatively compact in M. The corresponding article is in preparation [8].

Using this information, we show in the second step that for all $\varepsilon \in C^\infty(M, \mathbb{R}_{>0})$ there is a metric g_2 that coincides with g on A, fulfills $s \leq \mathrm{scal}_{g_2} \leq s + \varepsilon$ on $M \setminus A$, has C^0-distance $\leq \varepsilon$ from g, and whose C^1-distance from g can be controlled universally in a suitable sense. This C^1-control improves a result of Lohkamp [15]. This second step of the project is finished as well, the article is in preparation [9].

In the third step, we want to apply the other steps in order to solve the problem mentioned above, in particular to prove the conjecture. This is still (March 2013) work in progress.

3. CONCLUDING REMARKS

We greatly enjoyed the quiet atmosphere at the Hausdorff Institute, which helped us to focus on the progress of our project. We liked the distraction caused by the coffee breaks as well, though. The cake was excellent.

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