BRILL-NOETHER METHODS IN THE STUDY OF HYPER-KÄHLER AND CALABI-YAU MANIFOLDS

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Our group participated in the Junior Hausdorff Trimester Program with the project "Brill-Noether methods in the study of hyper-Kähler and Calabi-Yau manifolds". Examples of hyper-Kähler varieties are provided by Hilbert schemes of points on $K3$ surfaces and generalized Kummer manifolds. Their geometry is strictly connected with the study of line bundles on curves lying on symplectic surfaces (i.e., abelian or $K3$ surfaces), which goes under the name of Brill-Noether theory.

Research activities. A first research topic concerns curves on abelian surfaces and was carried out by Lelli-Chiesa and Mongardi together with Knutsen, who visited HIM in January. Severi varieties and Brill-Noether theory of curves on $K3$ surfaces are well understood. Quite little was known for curves on abelian surfaces and the paper "Severi varieties and Brill-Noether theory of curves on K3 surfaces" partially redresses this imbalance. Given a general abelian surface S with polarization L of type $(1, n)$, non-emptiness and regularity of the Severi variety parametrizing δ-nodal curves C in the linear system $|L|$ is proved. The gonality of the normalization $\widetilde{C}$ of C (i.e., the minimal integer k such that $\widetilde{C}$ has a $k : 1$ map to $\mathbb{P}^1$) is then studied: even in the smooth case, this is not constant when moving C in $|L|$. The second part of the paper is focused on linear series of type g^r_d with $r \geq 2$; roughly speaking, these correspond to rational maps $\varphi : C \dashrightarrow \mathbb{P}^r$ with image of degree d. It turns out that in some unexpected cases the locus $|L|^r_d$ of smooth curves in $|L|$ possessing a g^r_d is nonempty and has the expected dimension. As an application, one obtains the existence of a component of the Brill-Noether locus $M^r_{g,d}$ having the expected codimension in the moduli space of curves M_g.

In the paper "Wall divisors and algebraically coisotropic subvarieties of irreducible holomorphic symplectic manifolds", the above results are used in order to construct rational curves on Hilbert schemes of points on $K3$ surfaces and generalised Kummer manifolds. All wall divisors (i.e., divisors describing the birational geometry of these manifolds) can be obtained, up to isometry, as dual divisors to such rational curves. The locus covered by the rational curves is then described, thus exhibiting algebraically coisotropic subvarieties that deform to general small deformations of the manifold. This provides strong evidence for a conjecture by Voisin concerning the Chow ring of irreducible holomorphic symplectic manifolds.

During his stay, Mongardi made also advances in the topic of his Ph.D thesis: the paper "Towards a classification of symplectic automorphisms on manifolds of $K3^{[n]}$-type" extends results about automorphisms on Hilbert schemes of two points on a $K3$ surface to Hilbert schemes of an arbitrary number of points and their deformations. This topic is deeply linked with derived autoequivalences of $K3$ surfaces.

While staying at the HIM, Frank Gounelas started interactions with Yohan Brunebarbe at the Max Planck Institute, leading over the past year and a half to three projects. The first one (joint with John Christian Ottem) is related to the geometry of Calabi-Yau varieties and consists in studying the cone of nef and effective divisors on the projectivization of their cotangent bundle. This is still work in progress and a preprint will be posted online over the next months. Other projects include structure results for surfaces with nef cotangent bundles and positivity results for foliations.

The interaction with other groups was also important. For instance, Mongardi and Saccà collaborated with Rapagnetta in studying a particular birational morphism on a hyperkähler manifold that is deformation-equivalent to the Hilbert scheme of three points on a $K3$ surface. The quotient of the manifold by this morphism turns out to have a symplectic resolution which is deformation-equivalent to O'Grady's six dimensional example: one can use this fact in order to compute its cohomology. This project is still going on.

Organization activities. We actively participated at the trimester scientific life and organized the following activities:

- Three mini-courses held by E. Arbarello, A. L. Knutsen and E. Sernesi, who lectured about "Curves on $K3$ surfaces", "Linear series on singular curves on $K3$ surfaces: vector bundle methods and degenerations"and "Syzygies of special line bundles on curves", respectively.
- A workshop on our research topic with nine invited talks by G. Farkas, P. Frediani, M. Kemeny, G. Pacienza, G. Sankaran, J. Sawon, P. Stellari, A. Verra, C. Voisin

Papers.

(1) A. L. Knutsen, M. Lelli-Chiesa, G. Mongardi, *Severi Varieties and Brill-Noether theory of curves on abelian surfaces*, `arXiv:1503.04465v2`.
(2) A. L. Knutsen, M. Lelli-Chiesa, G. Mongardi, *Wall divisors and algebraically coisotropic subvarieties of irreducible holomorphic symplectic manifolds*, `arXiv:1507.06891`.
(3) G. Mongardi, *Towards a classification of symplectic automorphisms on manifolds of $K3^{[n]}$-type*, `arXiv:1405.3232`, to appear in Math. Z..