Bilateral Exchange of Contingent Claims

- an adaptive, behavioral, stochastic approach

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Abstract

- exchange of contingent claims
- only via voluntary, incentive compatible bilateral barters
- fully driven by differences in gradients (or substitutions rates)
- no optimization, no coordination
- myopic & adaptive agents
- **Still**!: convergence to equilibrium

Framing the problem

Consider an **exchange economy** with **transferable utility** and **stochastic endowments**

stochastic endowment = random bundle = contingent claim

$$\text{state } s \in S \mapsto \text{commodity vector } e(s) \in \mathbb{E} \;=\; \text{Euclidean}$$

What is meant by **equilibrium**?

Can equilibrium be reached by **bilateral barters**?

Chief example: A reinsurance market (say, Loyds of London). Insurance policy = contingent claim $x(\cdot)$:

$$\text{event } s \in S \mapsto \text{ indemnity } x(s) \in \mathbb{R}$$

Everybody is

* subject to feasibility constraints
* operating with reduced non-smooth objectives.

A preview of the story: 2 main stages

1) The ex ante stage: Agents exchange contingent claims in face of uncertainty.
* They contend with repeated bilateral barters.
* These are facilitated by side payments (NB. transferable utility).
* Neither optimization nor coordination is needed.
* Nobody needs full knowledge of vision of the data.

2) The ex post stage: Nature draws the contingency/ state $s \in S$.
* Contracts are executed.
* No reneging, no default.

Stochastics enter in 2 ways

1) The objects exchanged are random vectors

2) Who trades when with whom - and how - is random.

Some background: 3 different institutions

1) **A Walrasian auctioneer** announces prices

- agents report their optimal net demands
- aggregate net demand $> 0 \iff$ increase price.
  This story is pure fiction!

2) **Market makers** announce

- ask & bid prices
- agents state demand & supply
  Used to be important, but recently less so.

3) **Order markets**: agents sumit limit or market orders

- there is computerized **bilateral matching**, random timing and adaptive design of orders.
  These platforms are increasingly popular.

The setting

- There are several agents,
- each holding his stochastic endowment = random resource bundle = contingent claim.
- They proceed by repeated bilateral barters.

Issues:

- Will equilibrium obtain? (Quit likely **yes**!)
- Is there room for "simple" agents? (**Yes**!)
- Is optimization really needed? (**No**!)
- Must prices be announced? (**No**!)

DATA

- *agent* $i \in I$
- owns *endowment* $e_i \in \mathbb{X} = L^0(S, \mathcal{F}, \mathbb{E})$, (for computation S or $\mathcal{F}$ finite)
- *faces constraint* $x_i \in X_i$ closed convex $\subseteq \mathbb{X}$, and
- *wants* to maximize quasi-linear concave utility $u_i : \mathbb{X} \to \mathbb{R}$. **But** he
- *lacks* computational competence, perfect foresight, information, global vision, ...

- There is **no** coordination, **no** auctioneer, **no** market maker,...
- Moreover: bargaining, matching, search is not made explicit
- Nonetheless: Maybe holdings converge to equilibrium?
- Inspiration from Pareto:
  "*The economy is a great computing machine.*"

Equilibrium

- (x_i) an *allocation* iff $\sum_i x_i = \sum_i e_i =: e_I$
- *feasible allocation* iff moreover, $x_i \in X_i$ for each i.

Definition (Equilibrium) A feasible allocation (x_i) and a linear price $p : \mathbb{X} \to \mathbb{R}$ constitute and **equilibrium** iff

$$u_i(x_i) + p(e_i - x_i) \geq u_i(\chi) + p(e_i - \chi) \quad \text{for each } i \in I \text{ and } \chi \in X_i.$$

In equilibrium the agent's utility + his net value of sale is maximal!

Intermezzo: the nature of equilibrium

- Recall that a linear $x^* : \mathbb{X} \to \mathbb{R}$ is a **supergradient** of the proper function $f : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$ at x, written $x^* \in \partial f(x)$, iff

$$f(\chi) \leq f(x) + x^*(\chi - x) \text{ for all } \chi \in \mathbb{X}.$$

- Also recall: a linear $n : \mathbb{X} \to \mathbb{R}$ is a **normal vector** to a proper subset $X \subseteq \mathbb{X}$ at $x \in X$ iff

$$n(\chi - x) \leq 0 \text{ for all } \chi \in X.$$

Proposition (On equilibrium) *A profile* (x_i) *alongside a price* p *constitutes an equilibrium iff*

$$p \in \partial u_i(x_i) - N_i(x_i) \text{ for each } i, \quad \text{and} \quad \sum_{i \in I} x_i = e. \quad \square$$

· that is: there is a *common* price
· price "=" marginal utility (modulo normal components)

More on the nature of equilibrium: Cooperative aspects

- Suppose coalition $C \subseteq I$ uses $e_C := \sum_{i \in C} e_i$ to get

$$u_C(e_C) := \sup \left\{ \sum_{i \in C} u_i(x_i) : \sum_{i \in C} x_i = e_C \;\; \& \; x_i \in X_i \right\}.$$

Recall that a payment scheme (π_i) is in the **core** of this transferable-utility game iff

$$\begin{cases} \text{Pareto efficient:} & \sum_{i \in I} \pi_i = u_I(e_I) \\ \text{stable against blocking:} & \sum_{i \in C} \pi_i \geq u_C(e_C) \quad \text{for each } C \subset I. \end{cases}$$

Proposition (Equilibrium as core solution) *For any equilibrium price p the payment scheme*

$$i \mapsto \pi_i := \sup \left\{ u_i(\chi) + p(e_i - \chi) : \chi \in X_i \right\}$$

is in the core. □

A first simple guideline for exchange:

- **A brave approach:** When agent i meets agent j, they compare gradients: Let
$$g_i = u_i'(x_i) \quad \text{and} \quad g_j = u_j'(x_j)$$
$$\text{Is} \quad g_i \quad \neq \quad g_j'?$$
- If yes, for suitable **step-size** $\sigma > 0$, **transfer**
$$\Delta x_i := \sigma \left[g_i - g_j \right] \quad \text{to } i \quad \text{from } j.$$
- Similarly, j gets
$$\Delta x_j := \sigma \left[g_j - g_i \right] = -\Delta x_i$$
- Note, no central coordination. Fully decentralized.

- The process *stops* when all gradients are equal.
- In equilibrium
$$\text{all gradients are equal} = \text{price } p.$$

Alas, it's not that simple: what about constraints?

Constraints $x_i \in X_i$? Two alternative strategies:
1) Use *exact penalties* say *distance functions*

$$d_i(\chi) := \min \{ \|\chi - x_i\| : \; x_i \in X_i \}$$

and modified objectives

$$\hat{u}_i(x_i) = u_i(x_i) - c_i d_i(x_i), \quad \text{coefficient } c_i > 0 \text{ sufficiently large.}$$

See Flåm, Godal, Soubeyran, Optimization (2012)

2) Here: Use **projection** to enforce feasibility throughout.

Feasible exchange bewteen two agents

Actual holdings: $x_i \in X_i$ and $x_j \in X_j$.

Updated holdings:

$$x_i^{+1} := x_i + \Delta x \in X_i \quad \text{and} \quad x_j^{+1} := x_j - \Delta x \in X_j$$

When $\Delta x \neq 0$ with no loss of generality

$$\Delta x = \sigma d$$

with **positive stepsize** $\sigma > 0$ and **unit direction** $d \in \mathbb{X}$, $\|d\| = 1$.

- Which stepsize?
- Which direction?

On maintaining feasibility throughout

Cone of feasible directions of X_i at $x_i \in X_i \subseteq \mathbb{X}$

$$D_i(x_i) := \mathbb{R}_+(X_i - x_i).$$

Cone of common directions

$$D_{ij}(x_i, x_j) := D_i(x_i) \cap -D_j(x_j).$$

Tangent cone

$$T_{ij}(x_i, x_j) := clD_{ij}(x_i, x_j)$$

Maximal slope of joint improvement

$$\mathfrak{S}_{ij}(x_i, x_j) := \max_d \left\{ u_i'(x_i; d) + u_j'(x_j; -d) : d \in T_{ij}(x_i, x_j) \;\&\; \|d\| \leq 1 \right\}.$$

More on the maximal slope

$$\mathfrak{S}_{ij}(x_i, x_j) := \max_d \left\{ u_i'(x_i; d) + u_j'(x_j; -d) : d \in T_{ij}(x_i, x_j) \;\&\; \|d\| \leq 1 \right\}.$$

P_{ij} = orthogonal projection onto $T_{ij}(x_i, x_j)$:

$$\mathfrak{S}_{ij}(x_i, x_j) = \min \left\{ \| P_{ij} [g_i - g_j] \| : g_i \in \partial u_i(x_i), \; g_j \in \partial u_j(x_j) \right\}.$$

Also, with $dist(C_i, C_j) := \inf \| C_i - C_j \|$,

$$\mathfrak{S}_{ij}(x_i, x_j) = dist\left[\partial u_i(x_i) - N_i(x_i), \partial u_j(x_j) - N_j(x_j)\right].$$

Real transfers

- Fix $\varphi_{ij} \in (0,1)$ for each agent pair i, j.

Definition: While holding $x_i \in X_i$ and $x_j \in X_j$, agents i, j, make a **real transfer** $\Delta x = \sigma d$, with $\sigma > 0$ and $\|d\| = 1$, if

$$x_i + \sigma d \in X_i \quad \text{and} \quad x_j - \sigma d \in X_j$$

and

$$\Delta u_{ij} := u_i(x_i + \sigma d) + u_j(x_j - \sigma d) - u_i(x_i) - u_j(x_j)$$

satisfies

$$\Delta u_{ij} \geq \sigma \varphi_{ij} \mathfrak{S}_{ij}(x_i, x_j).$$

Bilateral exchange "as algorithm":

- **Start** at any feasible allocation: $\sum_{i \in I} x_i = e_I$, $x_i \in X_i$.

- **Pick two agents** i, j. Actually these hold $x_i \in X_i$ and $x_j \in X_j$. **If** $\mathfrak{S}_{ij}(x_i, x_j) = 0$, **select two new agents**.

- **Otherwise, they make a real transfer. That is, they ppdate their holdings**

$$x_i \leftarrow x_i + \sigma d \in X_i \quad \text{and} \quad x_j \leftarrow x_j - \sigma d \in X_j$$

- **Continue** to **Pick two agents** until **Convergence.**

On side payments

$\Delta u_{ij} > 0 \Longrightarrow \exists$ sidepayments Δr_i and Δr_j such that $\Delta r_i + \Delta r_j = 0$ and

$$u_i(x_i^{+1}) + \Delta r_i > u_i(x_i) \quad \& \quad u_j(x_j^{+1}) + \Delta r_j > u_j(x_j)$$

is solvable.

• Money "oils" the transaction machinery.

• Deals and incentives are compatible.

On common price and joint improvement

Recall that in equilibrium

$$p \in \cap_{i \in I} \left[\partial u_i(x_i) - N_i(x_i) \right].$$

We say agents i, j **see a common price** iff

$$[\partial u_i(x_i) - N_i(x_i)] \cap [\partial u_j(x_j) - N_j(x_j)] \neq \emptyset.$$

Proposition *i, j see a common price iff*

$$\mathfrak{S}_{ij}(x_i, x_j) = 0. \quad \square$$

Thus joint improvement is possible as long as $\mathfrak{S}_{ij}(x_i, x_j) > 0$.

Complete trade (no more joint improvement)

Trade is complete in the set

$$\mathbb{C} := \{\text{feasible allocations } (x_i) : \text{ each } \mathfrak{S}_{ij}(x_i, x_j) = 0\}$$

Two questions:
1) Will (x_i^k) "converge" to $\mathbb{C}$?

2) Will each profile $(x_i) \in \mathbb{C}$ be an equilibrium?

Standing hypotheses now: *The set of feasible allocations is bounded*, and

$$T_{ij}(x_i, x_j) = cl\,[D_i(x_i) \cap -D_j(x_j)]\,.$$

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Convergence and equilibrium

Proposition (On convergence) *Suppose real transfers, and that*
* *agents meet in quasi cyclical manner*
* *or that periodically* $\mathfrak{S}_{ij}(x_i, x_j)$ *is maximal. Then* (x_i^k) *clusters to the set*

$$\mathbb{C} := \{\text{feasible allocations } (x_i) : \text{ each } \mathfrak{S}_{ij}(x_i, x_j) = 0\}.$$

Proposition (On equilibrium) Each profile $(x_i) \in \mathbb{C}$ is an equilibrium if
1) $\mathbb{X}$ is one-dimensional or
2) Some agent i has $x_i \in intX_i$ and u_i differentiable at x_i.

A simple example: 3 agents and 3 contingencies

Agents $i = 1, 2, 3$, **sample space** $S = \{1, 2, 3\}$; $\mathbb{X} = \mathbb{R}^S$, $X_i = \mathbb{R}^S_+$,

$$\boxed{\begin{array}{rlrl} u_1(x_1) = & 2x_{11} + 1x_{12} + 0x_{13}, & e_1 = & (1,1,0), \\ u_2(x_2) = & 0x_{21} + 2x_{22} + 1x_{23}, & e_2 = & (0,1,1), \\ u_3(x_3) = & 1x_{31} + 0x_{32} + 2x_{33}, & e_3 = & (1,0,1). \end{array}}$$

Trade sequence: first $\{1, 2\}$, second $\{1, 3\}$, third $\{2, 3\}$. Start from $\left[x_i^0\right] = [e_i]$, use direction $d = P_{ij}\left[u_i' - u_j'\right]$ and step-size $\sigma = 1$, to get

$$\begin{array}{lll} d = P_{12}\left[u_1' - u_2'\right] = (0, -1, 0) & \Rightarrow & \left[x_i^1\right] = \left[(1,0,0),(0,2,1),(1,0,1)\right], \\ d = P_{13}\left[u_1' - u_3'\right] = (1, 0, 0) & \Rightarrow & \left[x_i^2\right] = \left[(2,0,0),(0,2,1),(0,0,1)\right], \\ d = P_{23}\left[u_2' - u_3'\right] = (0, 0, -1) & \Rightarrow & \left[x_i^3\right] = \left[(2,0,0),(0,2,0),(0,0,2)\right]. \end{array}$$

$\left[x_i^3\right]$ = the efficient allocation. The equilibrium price $p = (2, 2, 2)$ becomes common by choosing normals
$[n_i] = -\left[(0,1,0),(0,0,1),(1,0,2)\right] \in \Pi_i N_i(x_i)$.

Example extended: Stochastic LP & exchange

Linear production games. Let $\mathbb{X} = \mathbb{R}^S$ for some finite state space S, and

$$u_i(x_i) := \sup\{y_i^* \cdot y \mid x_i \geq A_i y_i \ \& \ y_i \geq 0\}.$$

Then by LP-duality

$$x_i^* \in \partial u_i(x_i) \Longleftrightarrow x_i^* \in \arg\min\left\{\chi_i^* \cdot x_i \;\middle|\; A_i^T \chi_i^* \geq y_i^* \ \& \ \chi_i^* \geq 0\right\}.$$

The cone $D_i(x_i)$ is closed convex and easily computable. Let binding constraints

$$S_i(x_i) := \{s \in S \mid [x_i - A_i y_i]_s = 0 \text{ and } y_i \text{ is primal optimal}\}$$

Then where

$$D_i(x_i) = \Pi_{s \in S_i(x_i)} \mathbb{R}_+ \times \Pi_{s \notin S_i(x_i)} \mathbb{R}.$$

Asymmetric information

$\mathbb{X} = L^0(S, \mathcal{F}, \pi, \mathbb{E})$ with $\mathcal{F}$ finite.

$X_i = L^0(S, \mathcal{F}_i, \pi, \mathbb{E})$ with $\mathcal{F}_i \subseteq \mathcal{F}$.

$$x_i \in X_i \Longrightarrow D_i(x_i) = X_i.$$

and $d \in T_{ij}(x_i, x_j) = X_i \cap X_j \Longrightarrow$

d constant on $A_i \cup A_j$ when atoms $A_i \in \mathcal{F}_i$ and $A_j \in \mathcal{F}_j$ intersect.

Projection = conditional expectation:

$$\Pr(x)_s = \frac{\sum_{s \in A} x_s \pi_s}{\sum_{s \in A} \pi_s} \quad \text{for each} \quad s \in \text{atom } A.$$

Concluding remarks on equilibrium and dynamics?

- How can players arrive at equilibrium - if any?
- While underway, how much competence, coordination, and foresight is required?
- What are the roles of cognition and perception?

Lacunes in economic theory

only concerned with equilibrium,

modestly interested in computation, uniqueness, stability, or attainability

most often out-of-equilibrium behavior gets no mention,

cognition and perception are hardly in focus.

Here! emergence of market equilibrium requires little

- prices need not come from somewhere; they rather emerge
- price-taking or maximization is neither necessary nor quite realistic
- agents can do without public prices; no posting
- agents merely seek own improvements, avoiding set-backs
- everybody can contend with idiosyncratics, local information
- no coordination, central agency, or global konowledge is ever required

Bilateral barters as viewed here

requires no coordination, experience, foresigth, or optimization,..

It's totally decentralized.

Fully driven by low-complexity adaptive agents

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