

Report of
**Daniel Greb, Anne-Sophie Kaloghiros,
Vladimir Lazić, Chenyang Xu**

Participants of the Junior Hausdorff Trimester Program

**Algebraic Geometry
Group: “Birational and Hyperkähler Geometry”**

Periods of stay:

Greb: 11.02.2014 – 15.04.2014

Kaloghiros: 17.02.2014 – 25.04.2014

Lazić: 06.01.2014 – 25.04.2014

Xu: 06.01.2014 – 06.03.2014

The “Birational and hyperkähler geometry” group had a fruitful and productive time during the Bonn Junior Trimester 2014. We collaborated on several of the projects outlined in our proposal for the Trimester. In addition, we organised several events that were very successful; in our opinion, these raised the profile of the Trimester and of the Hausdorff Institute in the algebraic geometry community.

PROJECTS

Our original goal was to make progress on the following five projects.

A. The Minimal Model Program. The main outstanding problem in the Minimal Model Program is the Abundance conjecture, which predicts that the canonical bundle of a minimal variety is semiample. A more general version of Abundance is to show the existence of good minimal models (minimal models on which the Abundance holds) for varieties with the pseudoeffective canonical bundle. Our goal in the Trimester was to make progress on the easier problem of existence of good models for varieties with non-negative Kodaira dimension.

B. Building blocks of varieties with trivial canonical class. Assuming the MMP for varieties of Kodaira dimension zero, the next natural step is to

analyse the structure of varieties X with canonical singularities and trivial canonical class. A natural question to ask is to what extent the Beauville-Bogomolov-Fujiki decomposition theorem generalises in this setting. Our aim was to identify potential building blocks of varieties with canonical singularities and trivial canonical class.

C. Cones of divisors on Calabi-Yau manifolds. The Cone conjecture of Morrison and Kawamata predicts that the geometry of a Calabi-Yau manifold is controlled by the action of automorphisms and birational automorphisms on the nef and movable cones. It implies that the number of minimal models of any variety with mild singularities is finite up to isomorphism, and it predicts existence of rational curves on Calabi-Yau threefolds, at least when the automorphism group is finite. One of our aims in this part of the project was to establish more cases of the Cone conjecture in dimension 3.

D. Hyperkähler manifolds. Two of the main open problems concerning the geometry of hyperkähler manifolds are the hyperkähler Strominger-Yau-Zaslow conjecture and the classification of base spaces for holomorphic Lagrangian fibrations. We planned to attack some special cases of the generalised hyperkähler SYZ conjecture, and the classification of base manifolds of Lagrangian fibrations from the MMP point of view.

E. Uniruled log Calabi-Yau varieties. If (Y, D) is a klt pair with $K_Y + D \sim_{\mathbb{Q}} 0$, then (Y, D) is a log Calabi-Yau. We planned to study log Calabi-Yau varieties (Y, D) that are compactifications of algebraic tori $T = Y \setminus D$. Given two such pairs (Y, D) and (Y', D') and a birational map $\varphi: Y \dashrightarrow Y'$ with $\varphi_*D = D'$, the goal was to investigate conditions under which φ can be decomposed into Sarkisov links which preserve the volume form.

COLLABORATIONS

We made progress on Projects A, C, D and E from our proposal. Due to time constraints, we have not been able to make advances on our Project B, but we are committed to working on it in the future.

On Project A, we have had a concentrated effort to, above all, understand the obstacles to different approaches to the Abundance conjecture in higher dimensions. To this end, in the period 10-21 February 2014 we organised a study group on the proof of the Abundance conjecture in dimension 3 by Miyaoka and Kawamata. The aim of the study group was to understand

how much of the proof can be generalised to higher dimensions, taking into account the advances in birational geometry in the 20 years since that proof was completed. The proof was presented in a series of lectures by Chenyang Xu and Vladimir Lazić. The participants also included several members of the mathematics department and of the Max Planck Institute.

On Project C, towards the end of the trimester Artie Prendergast-Smith and Vladimir Lazić started thinking about possible generalisations of the paper [LP12] into different contexts, namely when we consider a log Calabi-Yau pair and varieties of higher rank. Some initial steps were made, and may result in a collaboration in the future.

On Project D, we concentrated on proving that the base of a Lagrangian fibration of a hyperkähler 4-fold is the projective plane. We have made substantial effort towards this goal, and (partly in discussions with Sönke Rollenske and other participants of the program) we have developed several strategies which deepened our understanding of this difficult problem. All these strategies employ recent advances in the Minimal Model Program, and it is an exciting prospect that the MMP techniques can be used further in hyperkähler geometry, as implied by recent works [GLR13, Mat14]. This is a collaboration between Daniel Greb, Anne-Sophie Kaloghiros and Vladimir Lazić.

As an offshoot, this gave connections with our Project E. Indeed, the base of a Lagrangian fibration is, in a natural way, a Fano variety which is, at the same time, a log Calabi-Yau pair. One of our approaches to Project D involves using Sarkisov Program on the base, and the connection to Project E is immediate. We look forward to exploring this connection in the future work.

RESULTING PAPERS

Our activities during the Trimester resulted in the following papers.

Daniel Greb, *Complex-analytic quotients of algebraic G -varieties*, Math. Ann. 363 (2015), no. 1-2, 77-100.

Tobias Dorsch and Vladimir Lazić, *A note on the abundance conjecture*, Algebraic Geometry 2 (2015), no. 4, 476-488.

Alessio Corti and Anne-Sophie Kaloghiros, *The Sarkisov program for Mori fibred lc Calabi-Yau pairs*, arxiv:1504.00557.

Johannes Nicaise and Chenyang Xu, *Poles of maximal order of motivic zeta functions*, to appear in Duke Math. J., arXiv:1403.6792.

ACTIVITIES ORGANISED BY THE GROUP

We organised several highly successful activities during the Bonn Junior Trimester 2014.

In the period 6-22 January 2014 we organised, together with Daniel Huybrechts, the visit of Robert Lazarsfeld (Stony Brook University) as a Felix Klein lecturer. Even though this is a traditional yearly event at the Hausdorff Center for Mathematics, Daniel Huybrechts and we thought that the speaker fits perfectly within the scope of the Trimester. The topic was on the recent advances in the syzygy theory on projective varieties, and was an event which brought all the groups of the Trimester together.

The central event of our group at the Trimester was the workshop *Birational geometry and foliations*, in the period 24-28 February 2014. This was again a highly successful event, at which we had members of all groups at the Trimester as participants. The speakers included world's leading experts in birational and hyperkähler geometry, and the workshop was on par with any other similar event in the world. The speakers were: Arend Bayer, Frédéric Campana, Paolo Cascini, Alessio Corti, Jun-Muk Hwang, Stefan Kebekus, Sándor Kovács, Brian Lehmann, Christian Lehn, Christian Liedtke, James M'Kernan, Mircea Mustața, Gianluca Pacienza, Thomas Peternell, Taro Sano, Karl Schwede and Mingmin Shen.

Finally, we had three short-term visitors: Thomas Eckl (25-28 March 2014), Junyan Cao (30 March-4 April 2014) and John-Christian Ottem (6-12 April 2014). With Eckl we discussed recent analytic techniques of Siu related to the Abundance conjecture; with Cao we discussed analytic techniques related to extension of sections from subvarieties and the Ohsawa-Takegoshi theorem; with Ottem we discussed recent developments on the behaviour of subvarieties of projective varieties with ample normal bundles. The visitors gave seminars in the realm of the Trimester Seminar session, which we also shared with the usual Seminar Algebraic Geometry of the University of Bonn: Eckl spoke on 27 March, Cao on 3 April, and Ottem on 10 April 2014.

References

- [GLR13] D. Greb, C. Lehn, and S. Rollenske, *Lagrangian fibrations on hyperkähler manifolds—on a question of Beauville*, Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 3, 375–403 (2013).
- [LP12] V. Lazić and Th. Peternell, *On the Cone conjecture for Calabi-Yau manifolds with Picard number two*, to appear in Math. Res. Lett., arXiv:1207.3653.
- [Mat14] D. Matsushita, *On almost holomorphic Lagrangian fibrations*, Math. Ann. **358** (2014), no. 3-4, 565–572.