The Farrell-Jones Conjecture for algebraic K-theory holds for word-hyperbolic groups and arbitrary coefficients.

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We explain our main Theorem that the *Farrell-Jones Conjecture for algebraic K-theory* is true for every word-hyperbolic group $G$ and every coefficient ring $R$.

It predicts the structure of the algebraic $K$-groups $K_n(RG)$.

We discuss new applications focussing on

- Vanishing of the reduced projective class group and the Whitehead group of torsionfree groups;
- Conjectures generalizing Moody’s Induction Theorem;
- Bass Conjecture;
- Kaplaky Conjecture
- Algebraic versus homotopy $K$-theory, Nil-groups;
- $L^2$-invariants;

We make a few comments about the proof.
Conjecture

The Farrell-Jones Conjecture for algebraic $K$-theory with coefficients in $R$ for the group $G$ predicts that the assembly map

$$H_n^G(E_{VCyc}(G), K_R) \to H_n^G(pt, K_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $R$ is any (associative) ring (with unit) and $G$ is discrete;
- $K_n(RG)$ is the algebraic $K$-theory of the group ring $RG$;
- $VCyc$ is the family of virtually cyclic subgroups;
- Given a family of subgroups $\mathcal{F}$, let $E_\mathcal{F}(G)$ be the classifying space associated to it;
- $H^G_*(-; K_R)$ is the $G$-homology theory with the property that for every subgroup $H \subseteq G$,

$$H_n^G(G/H; K_R) = K_n(RH).$$
The Farrell-Jones Conjecture gives a way to compute $K_n(RG)$ in terms of $K_m(RV)$ for all virtually cyclic subgroups $V \subseteq G$ and all $m \leq n$.

It is analogous to the Baum-Connes Conjecture.

**Conjecture**

_The Baum-Connes Conjecture predicts that the assembly map_

$$K_n^G(EG) = H_n^G(E_{\text{Fin}}(G), K^{\text{top}}) \rightarrow H_n^G(pt, K^{\text{top}}) = K_n(C^*_r(G))$$

_is bijective for all $n \in \mathbb{Z}$. _

Here $H_*^G(-; K^{\text{top}})$ is the $G$-homology theory with the property that for every subgroup $H \subseteq G$

$$H_n^G(G/H; K^{\text{top}}) = K_n(C^*_r(H)).$$
The formulation of the Farrell Jones Conjecture

The main result

Applications

Comments on the proof

Theorem (Bartels-L.-Reich (2006))

The (Fibered) Farrell-Jones Conjecture for algebraic K-theory with (G-twisted) coefficients in any ring $R$ is true for word-hyperbolic groups $G$.

We emphasize that this result holds for all rings $R$ and not only for $R = \mathbb{Z}$.

Corollary

If $G$ is a torsionfree word-hyperbolic group and $R$ any ring, then we get an isomorphism

$$H_n(BG; K(R)) \oplus \left( \bigoplus_{(C), C \subseteq G, C \neq 1 \atop C \text{ maximal cyclic}} NK_n(R) \right) \xrightarrow{\text{inj}} K_n(RG).$$
We are not (yet?) able to prove the $L$-theory version. The $L$-theory version implies the Novikov Conjecture.
If one knows the $K$- and $L$-theory version for a group $G$ in the case $R = \mathbb{Z}$, one gets the Borel Conjecture in dimension $\geq 5$.

Conjecture

The Borel Conjecture for $G$ predicts for two closed aspherical manifolds $M$ and $N$ with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \rightarrow N$ is homotopic to a homeomorphism and in particular that $M$ and $N$ are homeomorphic.
Let $\mathcal{FJ}(R)$ be the class of groups which satisfy the Fibered Farrell-Jones Conjecture for algebraic $K$-theory with coefficients in $R$.

**Theorem (Bartels-L.-Reich (2006))**

1. Every word-hyperbolic group and every virtually nilpotent group belongs to $\mathcal{FJ}(R)$;

2. If $G_1$ and $G_2$ belong to $\mathcal{FJ}(R)$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}(R)$;

3. Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}(R)$ for $i \in I$. Then $\text{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}(R)$;

4. If $H$ is a subgroup of $G$ and $G \in \mathcal{FJ}(R)$, then $H \in \mathcal{FJ}(R)$.
In order to illustrate the depth of the Farrell-Jones Conjecture, we present some conclusions which are interesting in their own right.

**Corollary**

Let $R$ be a regular ring. Suppose that $G$ is torsionfree and $G \in \mathcal{FJ}(R)$. Then

1. $K_n(RG) = 0$ for $n \leq -1$;
2. The change of rings map $K_0(R) \to K_0(RG)$ is bijective. In particular $\tilde{K}_0(RG)$ is trivial if and only if $\tilde{K}_0(R)$ is trivial;
3. The Whitehead group $\text{Wh}^R(G)$ is trivial.

The idea of the proof is to study

$$H_n(BG; \mathbf{K}(R)) = H_n^G(E_{\mathcal{T}R}(G); \mathbf{K}_R) \to H_n^G(E_{\mathcal{V}yc}(G); \mathbf{K}_R) \to K_n(RG).$$
In particular we get for a torsionfree group $G \in \mathcal{FJ}(\mathbb{Z})$

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\text{Wh}(G) = 0$;
- Every finitely dominated $CW$-complex $X$ with $G = \pi_1(X)$ is homotopy equivalent to a finite $CW$-complex;
- Every compact $h$-cobordism $W = (W; M_0, M_1)$ of dimension $\geq 6$ with $\pi_1(W) \cong G$ is trivial, i.e., diffeomorphic to $M_0 \times [0, 1]$ relative $M_0$. (For $G = \{1\}$ this implies the Poincaré Conjecture in dimensions $\geq 5$.)
The formulation of the Farrell Jones Conjecture

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Theorem

1. Let $R$ be a regular ring with $\mathbb{Q} \subseteq R$. Suppose $G \in \mathcal{FJ}(R)$. Then the map given by induction from finite subgroups of $G$

$$\colim_{\Or_{\mathcal{F} \text{in}}(G)} K_0(RH) \to K_0(RG)$$

is bijective;

2. Let $F$ be a field of characteristic $p$ for a prime number $p$. Suppose that $G \in \mathcal{FJ}(F)$. Then the map

$$\colim_{\Or_{\mathcal{F} \text{in}}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

is bijective.
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Conjecture

Let $R$ be a commutative integral domain and let $G$ be a group. Let $g \in G$ be an element in $G$. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in $R$. Then the Bass Conjecture predicts that for every finitely generated projective $RG$-module $P$ the value of its Hattori-Stallings rank $\text{HS}_{RG}(P)$ at $(g)$ is trivial.
Theorem

Let $G$ be a group. Suppose that

$$\text{colim}_{\text{Or}_{\text{Fin}}(G)} K_0(FH) \otimes_{\mathbb{Z}} \mathbb{Q} \rightarrow K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for all fields $F$ of prime characteristic. (This is true if $G \in \mathcal{FJ}(F)$ for every field $F$ of prime characteristic). Then the Bass Conjecture is satisfied for every integral domain $R$. 
The Kaplansky Conjecture says for a torsionfree group $G$ and an integral domain $R$ that $0$ and $1$ are the only idempotents in $RG$.

The Kaplansky Conjecture is related to the vanishing of $\tilde{K}_0(RG)$.

Lemma

Let $F$ be a field and let $G$ be a group with $G \in \mathcal{FJ}(F)$. Suppose that $F$ has characteristic zero and $G$ is torsionfree or that $F$ has characteristic $p$, all finite subgroups of $G$ are $p$-groups and $G$ is residually amenable. Then $0$ and $1$ are the only idempotents in $FG$. 
Conjecture

Let $R$ be a regular ring with $\mathbb{Q} \subseteq R$. Then we get for all groups $G$ and all $n \in \mathbb{Z}$ that

$$NK_n(RG) = 0$$

and that the canonical map from algebraic to homotopy $K$-theory

$$K_n(RG) \rightarrow KH_n(RG)$$

is bijective.

Theorem

Let $R$ be a regular ring with $\mathbb{Q} \subseteq R$. If $G \in \mathcal{FJ}(R)$, then the conjecture above is true.
Conjecture

If $X$ and $Y$ are det-$L^2$-acyclic finite $G$-CW-complexes, which are $G$-homotopy equivalent, then their $L^2$-torsion agree:

$$\rho^{(2)}(X; \mathcal{N}(G)) = \rho^{(2)}(Y; \mathcal{N}(G)).$$

- The $L^2$-torsion of closed Riemannian manifold $M$ is defined in terms of the heat kernel on the universal covering. If $M$ is hyperbolic and has odd dimension, its $L^2$-torsion is up to dimension constant its volume.
- The conjecture above allows to extend the notion of a volume to word-hyperbolic groups whose $L^2$-Betti numbers all vanish.
Theorem

Suppose that $G \in \mathcal{FJ}(\mathbb{Z})$. Then $G$ satisfies the Conjecture above.

- Deninger can define a $p$-adic Fuglede-Kadison determinant for a group $G$ and relate it to $p$-adic entropy provided that $\text{Wh}^{\mathbb{F}_p}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial.
- The surjectivity of the map

$$\text{colim}_{\text{Or}_{\mathcal{F}\text{in}}(G)} K_0(\mathbb{C}H) \to K_0(\mathbb{C}G)$$

plays a role in a program to prove the Atiyah Conjecture which predicts for a closed Riemannian manifold with torsionfree fundamental group that the $L^2$-Betti numbers of its universal covering are all integers.
There is no group known for which the Farrell-Jones Conjecture, the Fibered Farrell-Jones Conjecture or the Baum-Connes Conjecture is false.

However, Higson, Lafforgue and Skandalis have constructed counterexamples to the Baum-Connes-Conjecture with coefficients. They describe precisely what properties a group $\Gamma$ must have so that it does not satisfy the Baum-Connes Conjecture with coefficients. Gromov outlines the construction of such a group $\Gamma$ as a colimit over a directed system of groups $\{G_i \mid i \in I\}$ such that each $G_i$ is word-hyperbolic.

Our main result implies that the Fibered Farrell-Jones Conjecture for algebraic $K$-theory with twisted coefficients in any ring does hold for $\Gamma$. 
Here are the basic steps of the proof of the main Theorem.

**Step 1**: Interprete the assembly map as a forget control map.

**Step 2**: Show for a finitely generated group $G$ that $G \in \mathcal{FJ}(R)$ holds for all rings $R$ if one can construct the following geometric data:

- A $G$-space $X$, such that the underlying space $X$ is the realization of an abstract simplicial complex;
- A $G$-space $\overline{X}$, which contains $X$ as an open $G$-subspace. The underlying space of $\overline{X}$ should be compact, metrizable and contractible,

such that the following assumptions are satisfied:
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Comments on the proof

- **Z-set-condition**
  There exists a homotopy $H: \overline{X} \times [0, 1] \to \overline{X}$, such that $H_0 = \text{id}_{\overline{X}}$ and $H_t(\overline{X}) \subset X$ for every $t > 0$;

- **Long thin covers**
  There exists an $N \in \mathbb{N}$ that only depends on the $G$-space $\overline{X}$, such that for every $\beta \geq 1$ there exists an $\mathcal{VCyc}$-covering $\mathcal{U}(\beta)$ of $G \times \overline{X}$ with the following two properties:
    - For every $g \in G$ and $x \in \overline{X}$ there exists a $U \in \mathcal{U}(\beta)$ such that $\{g\}^\beta \times \{x\} \subset U$. Here $g^\beta$ denotes the $\beta$-ball around $g$ in $G$ with respect to the word metric;
    - The dimension of the covering $\mathcal{U}(\beta)$ is smaller than or equal to $N$.

**Step 3:** Prove the existence of the geometric data above.