Introduction to $L^2$-invariants

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Theorem

Let $G$ be a group with finite classifying space $BG$. Suppose that $G$ contains a normal infinite solvable subgroup. Then

$$
\chi(BG) = 0.
$$

Theorem

Let $M$ be a closed hyperbolic manifold of even dimension $n = 2k$. Then

$$
(-1)^k \cdot \chi(M) > 0,
$$

and every $S^1$-action on $M$ is trivial.
Theorem

Let $1 \to H \overset{i}{\to} G \overset{q}{\to} K \to 1$ be an exact sequence of infinite groups. Suppose that $G$ is finitely presented and $H$ is finitely generated. Then:

1. $\text{defi}(G) \leq 1$;
2. Let $M$ be a closed oriented $4$-manifold with $G$ as fundamental group. Then

$$|\text{sign}(M)| \leq \chi(M).$$
Conjecture (Zero-divisor Conjecture)

Let $F$ be a field of characteristic zero and $G$ be a torsionfree group. Then the group ring $FG$ has no non-trivial zero-divisors.

Theorem

Let $M$ be a closed Kähler manifold. Suppose that it admits some Riemannian metric with negative sectional curvature. Then $M$ is a projective algebraic variety.

- The point is that the statements of these theorems have nothing to do with $L^2$-invariants, but their proofs have. This list can be extended considerably.
Basic motivation

- Given an invariant for finite $CW$-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group $\pi$ into account.

- Examples:

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We want to apply this principle to (classical) Betti numbers

\[ b_n(X) := \dim_\mathbb{C}(H_n(X; \mathbb{C})). \]

Here are two naive attempts which fail:

- \( \dim_\mathbb{C}(H_n(\tilde{X}; \mathbb{C})) \)
- \( \dim_{\mathbb{C}_\pi}(H_n(\tilde{X}; \mathbb{C})) \),
  where \( \dim_{\mathbb{C}_\pi}(M) \) for a \( \mathbb{C}_\pi \)-module could be chosen for instance as
  \( \dim_\mathbb{C}(\mathbb{C} \otimes_{\mathbb{C}_\pi} M) \).

The problem is that \( \mathbb{C}_\pi \) is in general not Noetherian and \( \dim_{\mathbb{C}_\pi}(M) \) is in general not additive under exact sequences.

We will use the following successful approach which is essentially due to Atiyah and motivated by \( L^2 \)-index theory.
Given a ring $R$ and a group $G$, denote by $RG$ the group ring. Elements are formal sums $\sum_{g \in G} r_g \cdot g$, where $r_g \in R$ and only finitely many of the coefficients $r_g$ are non-zero. Addition is given by adding the coefficients. Multiplication is given by the expression $g \cdot h := g \cdot h$ for $g, h \in G$ (with two different meanings of $\cdot$). In general $RG$ is a very complicated ring.
Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

**Definition**

Define the **group von Neumann algebra**

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}$$

to be the algebra of bounded $G$-equivariant operators $L^2(G) \to L^2(G)$. The **von Neumann trace** is defined by

$$\text{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

**Example (Finite $G$)**

If $G$ is finite, then $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$. The trace $\text{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient $\lambda_e$. 
Example \((G = \mathbb{Z}^n)\)

Let \(G = \mathbb{Z}^n\). Let \(L^2(T^n)\) be the Hilbert space of \(L^2\)-integrable functions \(T^n \to \mathbb{C}\). Fourier transform yields an isometric \(\mathbb{Z}^n\)-equivariant isomorphism

\[
L^2(\mathbb{Z}^n) \cong L^2(T^n).
\]

Let \(L^\infty(T^n)\) be the Banach space of essentially bounded measurable functions \(f : T^n \to \mathbb{C}\). We obtain an isomorphism

\[
L^\infty(T^n) \cong \mathcal{N}(\mathbb{Z}^n), \quad f \mapsto M_f
\]

where \(M_f : L^2(T^n) \to L^2(T^n)\) is the bounded \(\mathbb{Z}^n\)-operator \(g \mapsto g \cdot f\).

Under this identification the trace becomes

\[
\text{tr}_{\mathcal{N}(\mathbb{Z}^n)} : L^\infty(T^n) \to \mathbb{C}, \quad f \mapsto \int_{T^n} f d\mu.
\]
von Neumann dimension

Definition (Finitely generated Hilbert module)

A finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is a Hilbert space $V$ together with a linear isometric $G$-action such that there exists an isometric linear $G$-embedding of $V$ into $L^2(G)^n$ for some $n \geq 0$. A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f: V \to W$ is a bounded $G$-equivariant operator.

Definition (von Neumann dimension)

Let $V$ be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a $G$-equivariant projection $p: L^2(G)^n \to L^2(G)^n$ with $\text{im}(p) \cong_{\mathcal{N}(G)} V$. Define the von Neumann dimension of $V$ by

$$\dim_{\mathcal{N}(G)}(V) := \text{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \text{tr}_{\mathcal{N}(G)}(p_{i,i}) \in \mathbb{R}_{\geq 0}.$$
Example (Finite $G$)

For finite $G$ a finitely generated Hilbert $\mathcal{N}(G)$-module $V$ is the same as a unitary finite dimensional $G$-representation and

$$\dim_{\mathcal{N}(G)}(V) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

Example ($G = \mathbb{Z}^n$)

Let $G$ be $\mathbb{Z}^n$. Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} : L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with $\chi_X$. Its image $V$ is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \text{vol}(X).$$

In particular each $r \in \mathbb{R}^{\geq 0}$ occurs as $r = \dim_{\mathcal{N}(\mathbb{Z}^n)}(V)$. 
Theorem (Main properties of the von Neumann dimension)

1. Faithfulness

We have for a finitely generated Hilbert $\mathcal{N}(G)$-module $V$

$$V = 0 \iff \dim_{\mathcal{N}(G)}(V) = 0;$$

2. Additivity

If $0 \to U \to V \to W \to 0$ is a weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules, then

$$\dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) = \dim_{\mathcal{N}(G)}(V);$$

3. Cofinality

Let $\{V_i \mid i \in I\}$ be a directed system of Hilbert $\mathcal{N}(G)$-submodules of $V$, directed by inclusion. Then

$$\dim_{\mathcal{N}(G)} \left( \bigcup_{i \in I} V_i \right) = \sup \{ \dim_{\mathcal{N}(G)}(V_i) \mid i \in I \}.$$
Definition \((L^2\text{-homology and } L^2\text{-Betti numbers})\)

Let \(X\) be a connected CW-complex of finite type. Let \(\tilde{X}\) be its universal covering and \(\pi = \pi_1(M)\). Denote by \(C_\ast(\tilde{X})\) its cellular \(\mathbb{Z}\pi\)-chain complex.

Define its cellular \(L^2\)-chain complex to be the Hilbert \(\mathcal{N}(\pi)\)-chain complex

\[
C_\ast^{(2)}(\tilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_\ast(\tilde{X}) = C_\ast(\tilde{X}).
\]

Define its \(n\)-th \(L^2\)-homology to be the finitely generated Hilbert \(\mathcal{N}(G)\)-module

\[
H_n^{(2)}(\tilde{X}) := \ker(c_n^{(2)})/\text{im}(c_{n+1}^{(2)}).
\]

Define its \(n\)-th \(L^2\)-Betti number

\[
b_n^{(2)}(\tilde{X}) := \dim_{\mathcal{N}(\pi)}(H_n^{(2)}(\tilde{X})) \in \mathbb{R}_{\geq 0}.
\]
Theorem (Main properties of $L^2$-Betti numbers)

Let $X$ and $Y$ be connected CW-complexes of finite type.

- **Homotopy invariance**
  
  If $X$ and $Y$ are homotopy equivalent, then
  
  $$b_n^{(2)}(\tilde{X}) = b_n^{(2)}(\tilde{Y});$$

- **Euler-Poincaré formula**
  
  We have
  
  $$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\tilde{X});$$

- **Poincaré duality**
  
  Let $M$ be a closed manifold of dimension $d$. Then
  
  $$b_n^{(2)}(\tilde{M}) = b_{d-n}^{(2)}(\tilde{M});$$
Theorem (Continued)

- **Künneth formula**

\[ b_n^{(2)}(\tilde{X} \times \tilde{Y}) = \sum_{p+q=n} b_p^{(2)}(\tilde{X}) \cdot b_q^{(2)}(\tilde{Y}); \]

- **Zero-th \( L^2 \)-Betti number**

We have

\[ b_0^{(2)}(\tilde{X}) = \frac{1}{|\pi|}; \]

- **Finite coverings**

If \( X \to Y \) is a finite covering with \( d \) sheets, then

\[ b_n^{(2)}(\tilde{X}) = d \cdot b_n^{(2)}(\tilde{Y}). \]
Example (Finite $\pi$)

If $\pi$ is finite then

\[ b_n(\tilde{X}) = \frac{b_n(X)}{|\pi|}. \]

Example ($\pi = \mathbb{Z}^d$)

Let $X$ be a connected $CW$-complex of finite type with fundamental group $\mathbb{Z}^d$. Let $\mathbb{C}[\mathbb{Z}^d]^{(0)}$ be the quotient field of the commutative integral domain $\mathbb{C}[\mathbb{Z}^d]$. Then

\[ b_n^{(2)}(\tilde{X}) = \dim_{\mathbb{C}[\mathbb{Z}^d]^{(0)}} \left( \mathbb{C}[\mathbb{Z}^d]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}^d]} H_n(\tilde{X}) \right) \]

Obviously this implies

\[ b_n^{(2)}(\tilde{X}) \in \mathbb{Z}. \]
Some computations and results

Theorem ($S^1$-actions on aspherical manifolds, Lück)

Let $M$ be an aspherical closed manifold with non-trivial $S^1$-action. Then we get for $n \geq 0$

\[
\begin{align*}
    b_n^{(2)}(\widetilde{M}) &= 0; \\
    \chi(M) &= 0.
\end{align*}
\]

Theorem (mapping tori, Lück)

Let $f : X \rightarrow X$ be a cellular selfhomotopy equivalence of a connected CW-complex $X$ of finite type. Let $T_f$ be the mapping torus. Then

\[
    b_n^{(2)}(\widetilde{T_f}) = 0 \quad \text{for } n \geq 0.
\]
**Theorem (\(L^2\)-Hodge - de Rham Theorem, Dodziuk)**

Let \(M\) be a closed Riemannian manifold. Put

\[
\mathcal{H}_n^2(\tilde{M}) = \{ \tilde{\omega} \in \Omega^n(\tilde{M}) | \tilde{\Delta}_n(\tilde{\omega}) = 0, \|\tilde{\omega}\|_{L^2} < \infty \}
\]

Then integration defines an isomorphism of finitely generated Hilbert \(\mathcal{N}(\pi)\)-modules

\[
\mathcal{H}_n^2(\tilde{M}) \xrightarrow{\cong} H_n^2(\tilde{M}).
\]

**Corollary (\(L^2\)-Betti numbers and heat kernels)**

\[
b_n^{(2)}(\tilde{M}) = \lim_{t \to \infty} \int_F \text{tr}_\mathbb{R}(e^{-t\tilde{\Delta}_n}(\tilde{x}, \tilde{y})) \, d\text{vol}.
\]

where \(e^{-t\tilde{\Delta}_n}(\tilde{x}, \tilde{y})\) is the heat kernel on \(\tilde{M}\) and \(F\) is a fundamental domain for the \(\pi\)-action.
Theorem (hyperbolic manifolds, Dodziuk)

Let $M$ be a hyperbolic closed Riemannian manifold of dimension $d$. Then:

$$b_n^{(2)}(\tilde{M}) = \begin{cases} 
= 0 & \text{if } 2n \neq d; \\
> 0 & \text{if } 2n = d.
\end{cases}$$

Corollary

Let $M$ be a hyperbolic closed manifold of dimension $d$. Then

1. If $d = 2m$ is even, then

$$(−1)^m \cdot \chi(M) > 0;$$

2. Every $S^1$-action on $M$ is trivial $S^1$. 

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Theorem (3-manifolds, Lott-Lück)

Let the 3-manifold $M$ be the connected sum $M_1 \# \ldots \# M_r$ of (compact connected orientable) prime 3-manifolds $M_j$. Assume that $\pi_1(M)$ is infinite. Then

$$b_1^{(2)}(\tilde{M}) = (r - 1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} - \chi(M)$$

$$+ \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right| ;$$

$$b_2^{(2)}(\tilde{M}) = (r - 1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|}$$

$$+ \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right| ;$$

$$b_n^{(2)}(\tilde{M}) = 0 \quad \text{for } n \neq 1, 2.$$
The fundamental square and the Atiyah Conjecture

Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let $G$ be a torsionfree finitely presented group. We say that $G$ satisfies the Atiyah Conjecture if for any closed Riemannian manifold $M$ with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\tilde{M}) \in \mathbb{Z}.$$
The **fundamental square** is given by the following inclusions of rings

\[
\begin{array}{ccc}
\mathbb{Z}G & \to & \mathcal{N}(G) \\
\downarrow & & \downarrow \\
\mathcal{D}(G) & \to & \mathcal{U}(G)
\end{array}
\]

- \(\mathcal{U}(G)\) is the algebra of affiliated operators. Algebraically it is just the **Ore localization** of \(\mathcal{N}(G)\) with respect to the multiplicatively closed subset of non-zero divisors.
- \(\mathcal{D}(G)\) is the **division closure** of \(\mathbb{Z}G\) in \(\mathcal{U}(G)\), i.e., the smallest subring of \(\mathcal{U}(G)\) containing \(\mathbb{Z}G\) such that every element in \(\mathcal{D}(G)\), which is a unit in \(\mathcal{U}(G)\), is already a unit in \(\mathcal{D}(G)\) itself.
• If $G$ is finite, it is given by

$$\mathbb{Z} G \longrightarrow \mathbb{C} G \quad \quad \text{id}$$

$$\mathbb{Q} G \longrightarrow \mathbb{C} G$$

• If $G = \mathbb{Z}$, it is given by

$$\mathbb{Z}[\mathbb{Z}] \longrightarrow L^\infty(S^1)$$

$$\mathbb{Q}[\mathbb{Z}]^{(0)} \longrightarrow L(S^1)$$
If $G$ is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.

In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.

This aspect has recently played an important role in the construction of new invariants for 3-manifolds such as the universal $L^2$-torsion or the $L^2$-polytope by Friedl-Lück.
Conjecture (Atiyah Conjecture for torsionfree groups)

Let $G$ be a torsionfree group. It satisfies the Atiyah Conjecture if $D(G)$ is a skew-field.

- A torsionfree group $G$ satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Q}G)$ the von Neumann dimension
  
  $$\dim_{\mathcal{N}(G)} \left( \ker(r_A : L^2(G)^m \to L(G)^n) \right)$$

  is an integer. In this case this dimension agrees with
  
  $$\dim_{\mathcal{D}(G)} (r_A : \mathcal{D}(G)^m \to \mathcal{D}(G)^n).$$

- The general version above is equivalent to the one stated before if $G$ is finitely presented.

- An even stronger version allows $A \in M_{m,n}(\mathbb{C}G)$. 
The Atiyah Conjecture implies the Zero-divisor Conjecture due to Kaplansky saying that for any torsionfree group and field of characteristic zero $F$ the group ring $FG$ has no non-trivial zero-divisors.

There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.

However, there exist closed Riemannian manifolds whose universal coverings have an $L^2$-Betti number which is irrational, see Austin, Grabowski.
Theorem (Linnell)

Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions.

Then every torsionfree group $G$ which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture (over $\mathbb{C}$).
Strategy to prove the Atiyah Conjecture:

1. Show that $K_0(\mathbb{C}) \to K_0(\mathbb{C}G)$ is surjective.
2. Show that $K_0(\mathbb{C}G) \to K_0(D(G))$ is surjective.
3. Show that $D(G)$ is semisimple.

Notice that the Atiyah Conjecture originally was statement about an invariant extracted from the heat kernel of the universal covering, namely about the **analytic $L^2$-Betti number**.

However, the strategy described above is based on and requires $K$-theoretic and ring theoretic input.
In general there are no relations between the Betti numbers $b_n(X)$ and the $L^2$-Betti numbers $b_n^{(2)}(\tilde{X})$ for a connected $CW$-complex $X$ of finite type except for the Euler Poincaré formula

$$
\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\tilde{X}) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).
$$
Theorem (Approximation Theorem, Lück)

Let $X$ be a connected CW-complex of finite type. Suppose that $\pi$ is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\bigcap_{i \geq 1} G_i = \{1\}$. Let $X_i$ be the finite $[\pi : G_i]$-sheeted covering of $X$ associated to $G_i$.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\tilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$
Ordinary Betti numbers are not multiplicative under finite coverings, whereas the $L^2$-Betti numbers are. With the expression
\[ \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}, \]
we try to force the Betti numbers to be multiplicative by a limit process.

The theorem above says that $L^2$-Betti numbers are asymptotic Betti numbers. It was conjectured by Gromov.
Let $p$ be a prime and $\mathbb{F}_p$ be the field with $p$ elements.

**Conjecture (Approximation Conjecture in characteristic $p$)**

Let $X$ be a connected *aspherical* CW-complex of finite type. Suppose that $\pi$ is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\bigcap_{i \geq 1} G_i = \{1\}$. Let $X_i$ be the finite $[\pi : G_i]$-sheeted covering of $X$ associated to $G_i$.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\tilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i; \mathbb{F}_p)}{[G : G_i]}.$$
Schick used approximation techniques to prove the Atiyah Conjecture for matrices over $\mathbb{Q}G$ for a large class of groups.

A lot of work has been done by Jaikin-Zapirain to extend this from $\mathbb{Q}G$ to $\mathbb{C}G$ using ring theoretic methods.
The Singer Conjecture

Conjecture (Singer Conjecture)

If $M$ is an aspherical closed manifold, then

$$b_n^{(2)}(\tilde{M}) = 0 \quad \text{if } 2n \neq \dim(M).$$

If $M$ is a closed Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\tilde{M}) \begin{cases} = 0 & \text{if } 2n \neq \dim(M); \\ > 0 & \text{if } 2n = \dim(M). \end{cases}$$

- The computations presented above do support the Singer Conjecture.
- Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.
Because of the Euler-Poincaré formula

$$\chi(M) = \sum_{n \geq 0} (-1)^n \cdot b_{2n}(\tilde{M})$$

the Singer Conjecture implies the following conjecture provided that $M$ has non-positive sectional curvature.

**Conjecture (Hopf Conjecture)**

*If $M$ is a closed Riemannian manifold of even dimension with sectional curvature $\sec(M)$, then*

$$(-1)^{\dim(M)/2} \cdot \chi(M) > 0 \quad \text{if} \quad \sec(M) < 0;$$

$$(-1)^{\dim(M)/2} \cdot \chi(M) \geq 0 \quad \text{if} \quad \sec(M) \leq 0;$$

$$\chi(M) = 0 \quad \text{if} \quad \sec(M) = 0;$$

$$\chi(M) \geq 0 \quad \text{if} \quad \sec(M) \geq 0;$$

$$\chi(M) > 0 \quad \text{if} \quad \sec(M) > 0.$$
Theorem (Gromov)

Let $M$ be a closed Kähler manifold of complex dimension $c$. Suppose that it admits some Riemannian metric with negative sectional curvature. Then

$$b_n^{(2)}(\tilde{M}) = 0 \quad \text{if } n \neq c;$$

$$b_n^{(2)}(\tilde{M}) > 0 \quad \text{if } n = c;$$

$$(−1)^m \cdot \chi(M) > 0;$$

Moreover, $M$ is a projective algebraic variety.
Further important problems or connections

- Conjecture about the equality of the first $L^2$-Betti number, cost, and rank gradients of groups, e.g., the Fixed Prize Conjecture.
- $L^2$-invariants and measured and geometric group theory
- Determinant Conjecture.
- The Conjecture of Bergeron-Venkatesh for the growth of the torsion part of the homology and $L^2$-torsion.
- Conjecture about the vanishing of all $L^2$-invariants for closed aspherical manifolds with vanishing simplicial volume.
- $L^2$-invariants and $K$-theory.
- $L^2$-invariants and entropy.
- $L^2$-invariants and graph theory.
- Twisted $L^2$-invariants and 3-manifolds.
- Applications to von Neumann algebras.
- and so on.