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Surgery Transfer

by W. Lück and A. Ranicki

Introduction

Given a Hurewicz fibration $F \rightarrow E \rightarrow B$ with fibre an $n$-dimensional geometric Poincaré complex $F$ we construct algebraic transfer maps in the Wall surgery obstruction groups

$$p^! : L_m(\mathbb{Z}[\pi_1(B)]) \longrightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m \geq 0)$$

and prove that they agree with the geometrically defined transfer maps. In subsequent work we shall obtain specific computations of the composites $p^! p_!$, $p_! p^!$ with $p_! : L_m(\mathbb{Z}[\pi_1(E)]) \rightarrow L_m(\mathbb{Z}[\pi_1(B)])$ the change of rings maps, and some vanishing results.

The construction of $p^!$ is most straightforward in the case when $F$ is finite, with $L_\ast$ the free $L$-groups $L_\ast$. In §9 we shall extend the definition of $p^!$ to finitely dominated $F$ and the projective $L$-groups $L_\ast^p$, as well as to simple $F$ and the simple $L$-groups $L_\ast^s$, and also to the intermediate cases.

There are two main sources of applications of the surgery transfer. The equivariant surgery obstruction groups of Browder and Quinn [1] were defined in terms of the geometric surgery transfer maps of the normal sphere bundles of the fixed point sets. An algebraic version will necessarily involve the algebraic surgery transfer maps. (In this connection see Lück and Madsen [8].) The recent work of Hambleton, Milgram, Taylor and Williams [3] on the evaluation of the surgery obstructions of normal maps of closed manifolds with finite fundamental group depends on the factorization of the assembly map by twisted product formulae which are closely related to the algebraic surgery transfer.

Our construction of the quadratic $L$-theory transfer maps is by a combination of the algebraic...
surgery theory of Ranicki [14],[19] and the method used by Lück [7] to define the algebraic K-theory transfer maps \( p^! : K^*_m(\mathbb{Z}[\pi_1(B)]) \rightarrow K^*_m(\mathbb{Z}[\pi_1(E)]) \) for a fibration with finitely dominated fibre \( F \).

The algebraic surgery transfer maps \( p^! \) for a fibration are a special case of transfer maps \((C,a,U)^! : \text{Hom}_R(C,E) \rightarrow \text{Hom}_R(C,F)\) defined in abstract algebra. Here, \( A \) and \( B \) are rings with involution, \( C \) is an \( n \)-dimensional f.g. free \( B \)-module chain complex with a symmetric Poincaré duality chain equivalence \( a : C \rightarrow C^{n-*} \), and \( U : A \rightarrow R = \text{Hom}_B(C,C)^{op} \) is a morphism of rings with involution from \( A \) to \( R \). The opposition of the ring of chain homotopy classes of \( B \)-module chain maps \( f : C \rightarrow C \) with the involution on \( R \) defined by \( T(f) = a^{-1}f a \). An element of \( L_{2i}(A) \) is represented by a nonsingular \((-)^i\)-quadratic form \((M,\psi : M \rightarrow M^*)\) on a f.g. free \( A \)-module \( M \otimes A \). We define \( (C,a,U)^!(M,\psi) = (D,\theta) \in L_{n+2i}(B) \) to be the cobordism class of the \((n+2i)\)-dimensional quadratic Poincaré complex \((D,\theta)\) given by

\[
\theta \equiv \begin{cases} 
U(\psi)(\xi^{-1}) & \text{if } s=0 \\
0 & \text{if } s \neq 0
\end{cases}
\]

\[p^{n+2i-r,s} = \xi_{n+1-r,s} \quad D_r = \xi_{r-i} \]

There is a similar formula in the case \( m=2i+1 \), for which we refer to §4.

The algebraic transfer maps of fibration \( F \rightarrow E \rightarrow B \) with fibre an \( n \)-dimensional geometric Poincaré complex \( F \) are given by

\[p^! = (C(F),a,U)^! : L^*_m(\mathbb{Z}[\pi_1(B)]) \rightarrow L^*_m(\mathbb{Z}[\pi_1(E)])\]

with \( C(F) \) the cellular \( \mathbb{Z}[\pi_1(E)] \)-module chain complex of the cover \( F \) of \( F \) induced from the universal cover \( E \) of \( E \), \( \xi = ([F]^{-1}) : C(F) \rightarrow C(F)^{n-*} \) the Poincaré duality chain equivalence, and \( U \) determined by the fibre transport.

Here is the main idea in the identification of the algebraic and geometric surgery transfer. We know from the identification of the corresponding K-theory transfers in Lück [7] how to handle in algebra the lif of \( C \)-structures from the base to the total space of the fibration. We use the ultraquadratic L-theory of Ranicki [16, §7.8] both to encode the algebraic surgery data in the base spaces as \( C \)-structures, and to decode the algebraic surgery data from the lifted \( C \)-structures in the total spaces.

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The titles of the sections are:

Introduction
§1. The algebraic K-theory transfer
§2. Maps of L-groups
§3. The generalized Morita maps in L-theory
§4. The quadratic L-theory transfer
§5. The algebraic surgery transfer
§6. The geometric surgery transfer
§7. Ultraquadratic L-theory
§8. The connection
§9. Change of K-theory
Appendix 1. Fibred intersections
Appendix 2. A counterexample in symmetric L-theory

References
§1. The algebraic $K$-theory transfer

We recall from Lück [7] the construction of the algebraic $K$-theory transfer maps, and the connection with topology.

Given a ring $R$ let $R^{\operatorname{op}}$ denote the opposite ring, with the same elements and additive structure but with the opposite multiplication.

Definition 1.1 A representation $(A, U)$ of a ring $R$ in an additive category $\mathcal{A}$ is an object $A$ in $\mathcal{A}$ together with a morphism of rings $U : R \to \operatorname{Hom}_\mathcal{A}(A, A)^{\operatorname{op}}$.

Given an associative ring $R$ with 1 let $\mathcal{B}(R)$ be the additive category of based f.g. free $R$-modules $R^n$ ($n \geq 0$). A morphism $f : R^n \to R^m$ is an $R$-module morphism, corresponding to the $m \times n$ matrix $(a_{ij})$ with entries $a_{ij} \in R$, such that

$$f = \left( a_{ij} \right) : R^n \to R^m ; (x_i) \mapsto \left( \sum_{j=1}^n x_j a_{ij} \right).$$

Example 1.2 The universal representation $(R, U)$ of $R$ in $\mathcal{B}(R)$ is defined by the ring isomorphism

$$U : R \to \operatorname{Hom}_R(R, R)^{\operatorname{op}}; r \mapsto (s \mapsto sr),$$

which we shall use to identify $R = \operatorname{Hom}_R(R, R)^{\operatorname{op}}$.

A functor of additive categories $F : \mathcal{A} \to \mathcal{B}$ is required to preserve the additive structures.

Proposition 1.3 Given a ring $R$ and an additive category $\mathcal{A}$ there is a natural one-one correspondence between functors $F : \mathcal{B}(R) \to \mathcal{A}$ and representations $(A, U)$ of $R$ in $\mathcal{A}$.

Proof: Given a functor $F$ define a representation $(A, U)$ by

$$A = F(R),$$

$$U : R = \operatorname{Hom}_R(R, R)^{\operatorname{op}} \to \operatorname{Hom}_\mathcal{A}(A, A)^{\operatorname{op}};$$

$$(\rho : R \to R) \mapsto (\rho F(1) : A \to A).$$

Conversely, given a representation $(A, U)$ define a functor $F = -\Theta(A, U) : \mathcal{B}(R) \to \mathcal{A}$ by

$$F(R^n) = A^n,$$

$$F((a_{ij}) : R^n \to R^m) = (U(a_{ij})) : A^n \to A^m.$$
Given chain complexes \( C, D \) in \( A \) let \( \text{Hom}_A(C,D) \) be the abelian group chain complex defined by

\[
\text{dHom}_A(C,D) : \text{Hom}_A(C,D)_r = \sum_{q,p} \text{Hom}_A(C_q,D_p) \\
\text{Hom}_A(C,D)_{r-1} \quad \text{if} \quad \text{d}_D f + (-1)^q f d_C.
\]

There is a natural one-one correspondence between chain maps \( f : C \rightarrow D \) and 0-cycles \( f' \in \text{Hom}_A(C,D)_0 \), with

\[
f' = (-1)^n f : C_0 \rightarrow D_n \quad (n \in \mathbb{Z}).
\]

Similarly for chain homotopies and 1-chains. Thus \( H_0(\text{Hom}_A(C,D)) \) is isomorphic to the additive group of chain homotopy classes of chain maps \( C \rightarrow D \).

A chain complex \( C \) is finite if \( C_r = 0 \) for \( r < 0 \) and there exists \( n \geq 0 \) such that \( C_r = 0 \) for \( r > n \).

**Definition 1.5** Given an additive category \( A \) let \( D(A) \) be the homotopy category of \( A \), the additive category of finite chain complexes in \( A \) and chain homotopy classes of chain maps with

\[
\text{Hom}_{D(A)}(C,D) = H_0(\text{Hom}_A(C,D)).
\]

For a ring \( R \) we write \( D(B(R)) \) as \( D(R) \).

We refer to Ranicki \([17],[18]\) for an account of the algebraic K-theory groups \( K_m(A) \) of an additive category \( A \) with the split exact structure, and the application to chain complexes. In particular, the class of a finite chain complex \( C \) in \( A \) is defined by

\[
[C] = \sum_{r=0}^{\infty} (-1)^r [C_r] \in K_0(A),
\]

and the torsion of a self chain equivalence \( f : C \rightarrow C \) is defined by

\[
\tau(f) = \tau(d + \Gamma : C(f)_{\text{odd}} \rightarrow C(f)_{\text{even}}) \in K_1(A)
\]

for any chain contraction \( \Gamma : 0 \rightarrow C(f) \rightarrow C(f) \) of the algebraic mapping cone \( C(f) \).

**Definition 1.6** The generalized Morita maps \( \mu : \text{K}_m(D(A)) \rightarrow \text{K}_m(A) \) \((m = 0, 1)\) are defined for any additive category \( A \) by:

- for \( m = 0 \) \( \mu \) sends the class \([C] \in \text{K}_0(D(A))\) of object \( C \) in \( D(A) \) to the class \([C] \in \text{K}_0(A)\),

- for \( m = 1 \) \( \mu \) sends the torsion \( \tau(f) \in \text{K}_1(D(A)) \) of automorphism \( f : C \rightarrow C \) in \( D(A) \) to the torsion \( \tau(f) \in \text{K}_1(A) \) of any representative self chain equivalence.

A morphism in \( D(A) \) is a chain homotopy class and the definition of \( \mu \) involves a choice of representative chain map. The generalized Morita maps \( \mu \) are therefore not induced by a functor \( D(A) \rightarrow A \).

**Example 1.7** (Lück [7]) A Hurewicz fibration \( F \rightarrow E \) with the fibre \( F \) a CW complex determines a representation

\[
\mu : Z[\pi_1(B)] \rightarrow H_0(\text{Hom}_{Z[\pi_1(E)]}(C(\tilde{F}), C(\tilde{E})))^{op}
\]

with \( C(\tilde{F}) \) the cellular based free \( Z[\pi_1(E)] \)-module chain complex of the pullback \( \tilde{F} \) to \( F \) of the universal cover \( \tilde{E} \) of \( E \), and \( U \) the chain homotopy action \( H_0(\text{OB}) = Z[\pi_1(B)] \) on \( C(\tilde{F}) \) determined by the homotopy action of the loop space \( \text{OB} \) on \( F \). For finite \( F \) this defines a representation \( (\tilde{F}, U) \) of \( Z[\pi_1(B)] \) \( D(Z[\pi_1(E)]) \). For the identity map \( p = 1 : E \rightarrow E \) \( \mu \) \( F = \ast \) this is the universal representation \( (R, U) \) of \( Z[\pi_1(B)] = Z[\pi_1(E)] \).
The transfer map in the torsion groups associated to a representation \((C, U)\) of a ring \(R\) in \(\mathbb{D}(\bar{A})\) is the composite

\[
(C, U)^! : K_1(R) = K_1(\mathbb{D}(R)) \xrightarrow{U} K_1(\mathbb{D}(\bar{A})) \xrightarrow{\mu} K_1(\bar{A})
\]

of the map \(U\), induced by the functor \((C, U) \otimes \cdot : B(R) \to \mathbb{D}(\bar{A})\) and the generalized Morita map \(\mu\). The torsion \(\tau(f) \in K_1(R)\) of an automorphism \(f : R^k \to R^k\) is sent by \((C, U)^!\) to the torsion \(\tau(U(f)) \in K_1(\bar{A})\) of the self chain equivalence \(U(f) : \mathbb{D}(C) \to \mathbb{D}(C)\).

The idempotent completion of an additive category \(\bar{A}\) is the additive category \(\hat{\bar{A}}\) with objects pairs

\[
(A = \text{object of } \bar{A}, \ p = p^2 : A \to A)
\]

and morphisms \(f : (A, p) \to (A', p')\) defined by morphisms \(f : A \to A'\) in \(\bar{A}\) such that \(p'f = f : A \to A'\). The evident functor \(\mathbb{D}(\bar{A}) \to \mathbb{D}(\hat{\bar{A}})\) is an equivalence of additive categories, since every chain homotopy projection in \(\hat{\bar{A}}\) splits (Lück and Ranicki [9]).

For any ring \(R\) the additive category \(\mathbb{P}(R)\) of f.g. projective \(R\)-modules is equivalent to the idempotent completion \(\mathbb{D}(R)\) of the additive category \(B(R)\) of based f.g. free \(R\)-modules, with an equivalence

\[
\mathbb{B}(R) \to \mathbb{P}(R); (R^k, p) \to \text{im}(p).
\]

For any representation \((C, U)\) of a ring \(R\) in \(\mathbb{D}(\bar{A})\) the functor \((C, U) \otimes \cdot : B(R) \to \mathbb{D}(\hat{\bar{A}})\) extends to a functor \(\mathbb{P}(R) \to \mathbb{D}(\hat{\bar{A}})\) (cf. Lemma 9.3), and so determines a transfer map in the class groups

\[
(C, U)^! : K_0(R) = K_0(\mathbb{P}(R)) \xrightarrow{U^!} K_0(\mathbb{D}(\hat{\bar{A}})).
\]

The class \([\text{im}(p)] \in K_0(R)\) of a projection \(p : R^k \to R^k\) sent by \((C, U)^!\) to the projective class \([\mathbb{C} \mathbb{D}(U(p)) \mathbb{D}(\bar{A})\]

of the chain homotopy projection \(U(p) \mathbb{P}(R)) \to \mathbb{C} \mathbb{D}(\bar{A})\) is called the algebraic K-theory transfer map

\[(C, U)^! : K_m(R) = K_m(\mathbb{P}(R)) \xrightarrow{U^!} K_m(\mathbb{D}(\hat{\bar{A}})).
\]

for \(m = 0, 1\).

The algebraic K-theory transfer maps of the fibration \(\mathbb{F} \to \mathbb{P}B\) with finite (or finitely dominated) fibre \(\mathbb{F}\) defined for \(m = 0, 1\) by

\[p^! = (C(\mathbb{F}), U)^! : K_m(\mathbb{Z}[\pi_1(B)]) \to K_m(\mathbb{Z}[\pi_1(\mathbb{E})])
\]

were shown in [7] to coincide with the geometric transfer maps using the following property of the functor

\[p^! = \mathbb{C} \mathbb{D}(\mathbb{Z}[\pi_1(\mathbb{E})]).
\]

Proposition 1.9 Let \((X', X)\) be a relative CW pair such that \(X'\) is obtained from \(X\) by adjoining cells of dimensions \(r, r+1\)

\[X' = X \cup \bigcup_{i \le r} \bigcup_{j \ge r+1}.\]

Given a map \(X' \to B\) to a connected space \(B\) let \((X', \tilde{X})\) the pullback to \((X', X)\) of the universal cover \(\tilde{B}\) of \(B\) and let

\[d : C(\tilde{X}', \tilde{X})'_{r+1} = \mathbb{Z}[\pi_1(B)] \to C(\tilde{X}', \tilde{X})_r = \mathbb{Z}[\pi_1(B)]\]

be the boundary map in the cellular based free
\( \mathbb{Z}[\pi_1(B)] \)-module chain complex. Let \( F \rightarrow E \rightarrow B \) be a Hurewicz fibration such that the fibre \( F \) is a CW complex. Let \( F \rightarrow (Y',Y) \rightarrow (X',X) \) be the fibration obtained from \( p \) by pullback along the map \( X' \rightarrow B \), with \( (Y',Y) \) the pullback to \( (Y',Y) \) of the universal cover \( E \) of \( E \). Then \( (Y',Y) \) is homotopy equivalent to a relative CW pair (also denoted by \( (Y',Y) \)) with cellular based \( \mathbb{Z}[\pi_1(E)] \)-module chain complex

\[
C(Y',Y) = S^r C(p^\theta(d): \mathfrak{C}(F) \rightarrow \mathfrak{C}(\tilde{F}))
\]
the \( r \)-fold suspension of the algebraic mapping cone of a chain map in the chain homotopy class

\[
p^\theta(d) : p^\theta(\mathfrak{C}[\pi_1(B)]) = \mathfrak{C}(\tilde{F})
\]

\[
\rightarrow p^\theta(\mathfrak{C}[\pi_1(B)]) = \mathfrak{C}(\tilde{F}) .
\]

Proof: See Lück [7].

\[ \square \]

\section{Maps of L-groups}

We refer to Ranicki [14],[19] for the definition of the quadratic L-groups \( L_n(A) \) \((n \geq 0)\) of an additive category \( A \) with involution \( \ast : A \rightarrow A \), as the cobordism groups of \( n \)-dimensional quadratic Poincaré complexes \((C,\mathfrak{C}_n(C))\) in \( A \), and for the proof that these groups are 4-periodic, with \( L_2i(A) \) (resp. \( L_{2i+1}(A) \)) the Witt group of nonsingular \((-1)^i\)-quadratic forms (resp. formations) in \( A \).

We now put an involution on the notions of \S 1.

\textbf{Definition 2.1} An involution on an additive category \( A \) is a contravariant functor

\[
\ast : A \rightarrow A ; M \rightarrow M^* .
\]

\textbf{Example 2.2} Given a ring \( R \) with involution

\[
- : R \rightarrow R ; r \rightarrow \tilde{r}
\]

let the additive category

\[
\mathcal{B}(R) = \text{(based f.g. free } R\text{-modules)}
\]

have the duality involution

\[
(R^n)^* = R^n , \quad (a_{ij})^* = (\tilde{a}_{ij}) ,
\]

such that

\[
L_n(\mathcal{B}(R)) = L_n(R) \quad (n \geq 0) .
\]

By definition, a quadratic Poincaré complex over \( R \) is the same as a quadratic Poincaré complex in \( \mathcal{B}(R) \).

\[ \square \]

\textbf{Notation 2.3} Let \( A \) be an additive category with involution.

\[
(f:M \rightarrow N) \rightarrow (f^*:N \rightarrow M^*)
\]
together with a natural equivalence

\[
e : \text{id}_A \rightarrow ** : A \rightarrow A ;
\]

\[
M \rightarrow (e(M):M \rightarrow M**)
\]
such that

\[
e(M^*) = (e(M)^{-1})^* : M^* \rightarrow M^{**} .
\]

\[ \square \]
1) A chain complex $C$ in $\mathcal{A}$ is $n$-dimensional if $C_r=0$ for $r<0$ and $r>n$.

ii) The $n$-dual of an $n$-dimensional chain complex $C$ is the $n$-dimensional chain complex $C^\ast$ in $\mathcal{A}$ with

$$d_C n^{-\ast} = (-)^{r} (d_C)^{\ast} ;$$

$$(C^\ast)^{\ast}_n = C^{n-r} = (C^{-r})^{\ast} (C^{n-\ast})_{r-1} .$$

iii) For $n \geq 0$ let $\mathcal{D}_n(\mathcal{A})$ be the additive category of $n$-dimensional chain complexes in $\mathcal{A}$ and chain homotopy classes of chain maps, with the $n$-duality involution $T=n^{-\ast} : \mathcal{D}_n(\mathcal{A}) \longrightarrow \mathcal{D}_n(\mathcal{A}) ; C \longrightarrow C^\ast$.

A functor of additive categories with involution $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a functor of the underlying additive categories together with a natural equivalence $G : F^{\ast} \longrightarrow F^{\ast} ; \mathcal{A} \longrightarrow \mathcal{B}$, such that for any object $M$ in $\mathcal{A}$ there is defined a commutative diagram in $\mathcal{B}$

$$
\begin{array}{ccc}
F(M) & \xrightarrow{e} & F(M)^{\ast} \\
\downarrow & & \downarrow \\
F(e_A(M)) & \xrightarrow{G(M)} & F(M)^{\ast}.
\end{array}
$$

Notation 2.4 A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ of additive categories with involution induces morphisms of the quadratic L-groups which we write as

$$F : L_n(\mathcal{A}) \longrightarrow L_n(\mathcal{B}) ;$$

$$(C, \psi) \longrightarrow (F(C), F(\psi)) (n \geq 0) .$$

Example 2.5 A morphism of rings with involution $f : R \longrightarrow S$ determines functors of additive categories with involution $F : \mathcal{B}(R) \longrightarrow \mathcal{B}(S)$ which induces change of rings morphisms in the quadratic L-groups $f : L_n(R) \longrightarrow L_n(S) (n \geq 0)$.

Definition 2.6 Given a nonsingular symmetric form $(A, d = d : A \longrightarrow A^\ast)$ in an additive category with involution $\mathcal{A}$ let the ring $\text{Hom}_A(A, A)^{\text{op}}$ have the involution

$$
\begin{array}{cccc}
\text{Hom}_A(A, A)^{\text{op}} & \longrightarrow & \text{Hom}_A(A, A)^{\text{op}} \\
(f : A \longrightarrow A) & \longrightarrow & (d^{-1} f^\ast : A \longrightarrow A^\ast \longrightarrow A) .
\end{array}
$$

By analogy with Definition 1.1:

Definition 2.7 A symmetric representation $(A, d, U)$ of a ring with involution $\mathcal{R}$ in an additive category with involution $\mathcal{A}$ is a nonsingular symmetric form $(A, d)$ in $\mathcal{A}$ together with a morphism of rings with involution $U : \mathcal{R} \longrightarrow \text{Hom}_A(A, A)^{\text{op}}$.

In particular, $(A, U)$ is a representation of $\mathcal{R}$ in the additive category $\mathcal{A}$ in the sense of 1.1.

By analogy with Example 1.2:

Example 2.8 The universal symmetric representation $(R, d, U)$ of a ring with involution $\mathcal{R}$ in $\mathcal{B}(R)$ is defined by

$$d : R \longrightarrow R^\ast ; r \longrightarrow (s \rightarrow s^r)$$

with $U$ the isomorphism of rings with involution...
Proposition 2.9 Given a ring with involution $R$ and an additive category with involution $\mathcal{A}$ there is a natural one-one correspondence between functors of pairs of additive categories with involution $F: \mathcal{B}(R) \rightarrow \mathcal{A}$ and symmetric representations $(A, \alpha, U)$ of $R$ in $\mathcal{A}$. 

Proof: Given a functor $F$ define a symmetric representation $(A, \alpha, U)$ by 

$A = F(R)$, 

$\alpha = G(R) : F(R)^* \rightarrow F(R)^* \rightarrow A^*$, 

$U : R \rightarrow \text{Hom}_R(R, R)^{op} \rightarrow \text{Hom}_{\mathcal{A}}(A, A)^{op}$; 

$(\rho: R \rightarrow R) \rightarrow (F(\rho): A \rightarrow A)$. 

Conversely, given a symmetric representation $(A, \alpha, U)$ define a functor $F = \Theta(A, \alpha, U) : \mathcal{B}(R) \rightarrow \mathcal{A}$ by 

$F(R) = A$, 

$G(R) = \alpha : F(R)^* \rightarrow F(R)^* \rightarrow A^*$, 

$F((a_{ij}) : R^n \rightarrow R^m) = (U(a_{ij})) : A^n \rightarrow A^m$. 

By definition, a nonsingular symmetric form $(C, \alpha)$ in $\mathcal{D}_n(\mathcal{A})$ is an $n$-dimensional symmetric complex $C$ in $\mathcal{A}$ together with a self dual chain homotopy class of chain equivalences $\alpha: T_0 C \rightarrow C^{n-*}$. 

---

Proposition 2.10 A symmetric representation $(C, \alpha, U)$ of a ring with involution $R$ in $\mathcal{D}_n(\mathcal{A})$ determines a function $F = \Theta(C, \alpha, U) : \mathcal{B}(R) \rightarrow \mathcal{D}_n(\mathcal{A})$ inducing morphisms in the quadratic $L$-groups 

$F_1 = \Theta(C, \alpha, U) : L_m(R) \rightarrow L_m(\mathcal{D}_n(\mathcal{A})) (m \geq 0)$. 

Proof: Immediate from 2.4 and 2.9. 

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§3. The generalized Morita maps in $L$-theory

By analogy with the algebraic $K$-theory generalization of Morita maps $\mu: K_0(\mathcal{D}(\mathcal{A})) \rightarrow K_0(\mathcal{A}) (m=0,1)$ of §1 we define generalized Morita maps in the quadratic $L$-groups: 

$\mu: L_m(\mathcal{D}_n(\mathcal{A})) \rightarrow L_{m+n}(\mathcal{A}) (m, n \geq 0)$ by passing to nonsingular quadratic forms and formations in $\mathcal{D}_n(\mathcal{A})$ quadratic Poincaré complexes in $\mathcal{A}$. The $L$-theory $L_n$ is the identity for $n=0$, since $\mathcal{D}_0(\mathcal{A}) = \mathcal{A}$. For $n \geq 1$ the maps are not isomorphisms and are not induced by functors of additive categories with involution: a morphism $\mathcal{D}_n(\mathcal{A})$ is a chain homotopy class and as in $K$-theory definition of $\mu$ involves a choice of representative chain map. 

Proposition 3.1 1) A nonsingular $(-)^i$-quadratic form
\( (M, \Theta \in \text{coker}(1-(-)^iT: \text{Hom}_{D_n}(\mathcal{A})(M, M^*)) \rightarrow \text{Hom}_{D_n}(\mathcal{A})(M, M^*)) \)

is represented by an \( n \)-dimensional chain complex \( M \) in \( \mathcal{A} \) together with a chain map \( \Theta: M \rightarrow \mathcal{M}^{n-*} \) such that \( (1+(-)^iT)\Theta = \Theta + (-)^{n+1} \Theta^* : M \rightarrow \mathcal{M}^{n-*} \) is a chain equivalence.

ii) The cobordism class \( (C_0, \psi) \in L_{n+2i}(\mathcal{A}) \) of the \( (n+2i) \)-dimensional quadratic Poincare complex in \( \mathcal{A} \) defined by \( C = \mathcal{M}^{n+i-*} \) and

\[
\psi_s = \begin{cases} 
\Theta & \text{if } s = 0 \\
0 & \text{if } s \geq 1
\end{cases}
\]

sends the class of \( \Theta \) to the quadratic structure \( \psi \in L_{n+2i}(\mathcal{C}) \).

iii) Define an \( (n+2i+1) \)-dimensional quadratic Poincare pair in \( \mathcal{A} \) \( (j: C = \mathcal{M}^{n+i-*} \rightarrow \mathcal{D} = \mathcal{L}^{n+i-*} \),

\[
5\psi_0 = \psi_0 = \psi_0 \text{ for } s \geq 1
\]

We refer to §2 of Ranicki [19] for the definition of a nonsingular \((-)^i\)-quadratic form \( (F, G) = (F, \begin{bmatrix} Y & \mu \end{bmatrix} \mu \in \mathcal{G}) \) in an additive category with involution \( \mathcal{A} \), and for the result that \( (F, G) = 0 \in L_{2i+1}(\mathcal{A}) \) if and only if there exists a \((-)^i+1\)-quadratic form in \( \mathcal{A} \) \( (H, \mu) \), and a morphism \( j: F \rightarrow H \) such that the morphism defined in \( \mathcal{A} \) by

\[
\begin{bmatrix} \mu & Y \end{bmatrix} \begin{bmatrix} \ast & \ast \\
j & \ast \end{bmatrix} : F \otimes H \rightarrow G \otimes H
\]

is an isomorphism.

Proposition 3.2 i) A nonsingular \((-)^i\)-quadratic formation \( (F, \begin{bmatrix} Y \mu \end{bmatrix}) \) in \( D_n(\mathcal{A}) \) is represented by

\[
\begin{align*}
\Theta: M & \rightarrow \mathcal{M}^{n-i-*} \\
\psi_s & \in \text{coker}(\begin{bmatrix} \ast & \ast \\
0 & \ast \end{bmatrix} : \mathcal{M} \rightarrow \mathcal{M}) \\
\psi_s & \in \text{coker}(\begin{bmatrix} \ast & \ast \\
0 & \ast \end{bmatrix} : \mathcal{M} \rightarrow \mathcal{M}) \\
\psi_s & \in \text{coker}(\begin{bmatrix} \ast & \ast \\
0 & \ast \end{bmatrix} : \mathcal{M} \rightarrow \mathcal{M}) \\
\end{align*}
\]
n-dimensional chain complexes \( F, G \) in \( \mathbb{A} \) together with chain maps \( \gamma : G \to F, \mu : G \to F^{n-1} \), \( \theta : G \to G^{n-1} \) and a chain homotopy
\[
x : \gamma^* \mu = \theta + (-)^{n+1+1} \theta^* : G \to G^{n-1}
\]
such that the chain map
\[
\begin{pmatrix}
x + (-)^{n+1} x^* \\
\gamma^* \\
0
\end{pmatrix}
: C(\mu^*)_{n+1-*} \to C(\mu^* : F \to G^{n-1})
\]
is a chain equivalence.

ii) The cobordism class \( (C, \psi) \in L_{n+2,i+1}(\mathbb{A}) \) of the \( (n+2,i+1) \)-dimensional quadratic Poincare complex \( (C = \delta^1 C(\mu^*), \psi) \) in \( \mathbb{A} \) with
\[
d_C = \begin{pmatrix}
d_F^* (-)^{r-1} \delta F \\
0
\end{pmatrix}
\]
\[
C_r = G^{n-r+i} \oplus F_{r-i+1} \to C_{r-1} = G^{n-r+i} \oplus F_{r-i+2},
\]
\[
\psi_0 = \begin{pmatrix}
(-)^{n+1}(r-1) x \\
(-)^n(r-1) \gamma
\end{pmatrix}
: C^{n+2,i+1-r} \to C^{n+1,i+1}
\]
\[
C^{n+2,i+1-r} = G_{r-i} \oplus F^{n-r+1}
\]
\[
C^{n+1,i+1} = G_{r-i} \oplus F^{n-r+1},
\]
\[
\psi_1 = \begin{pmatrix}
(-)^{(n+1)r+i} \theta \\
0
\end{pmatrix}
: C^{n+2,i+1-r} \to C^{n+1,i+1}
\]
\[
C^{n+1,i+1} = G_{r-i} \oplus F^{n-r+i+1}
\]
is a chain equivalence. Then \( (C, \psi) \in L_{n+2,i+1}(\mathbb{A}) \).

Proof: i) The inclusion of the lagrangian \( (G,0) \to H(\cdot)^i(F) \) extends to an isomorphism of \( (-)^i \)-quadratic forms in \( \mathbb{A} \)
\[
\begin{pmatrix}
\gamma \\
\mu
\end{pmatrix}
: H(\cdot)^i(G) \to H(\cdot)^i(F)
\]
which is represented by a chain equivalence in \( \mathbb{A} \)
\[
\begin{pmatrix}
\gamma \\
\mu
\end{pmatrix}
: G \oplus G^{n-*} \to F \oplus F^{n-*}
\]
with chain homotopy inverse
\[
\begin{pmatrix}
\gamma \\
\mu
\end{pmatrix}
^{-1} = \begin{pmatrix}
\mu^* \\
(-)^i \gamma^*
\end{pmatrix}
: F \oplus F^{n-*} \to G \oplus G^{n-*}
\]
For any representative chain map \( \tilde{\mu} : G^{n-*} \longrightarrow F^{n-*} \) there exist a chain map \( \mu : F \longrightarrow F^{n-*} \) and a chain homotopy
\[
\eta : \tilde{\mu} \mu = \nu + (-)^{n+1} \mu \quad : F \longrightarrow \quad F^{n-*} .
\]
The chain maps in \( \mathcal{A} \) defined by
\[
f = \left[ \begin{array}{ccc} x^{*} (-)^{n+1} & \gamma \end{array} \right] : C(\mu^{*})^{n+1-*} \longrightarrow C(\mu^{*})
\]
\[
g = \left[ \begin{array}{ccc} \eta^{*} & \eta \end{array} \right] : C(\mu^{*})^{n+1-*} \longrightarrow C(\mu^{*})^{n+1-*}
\]
are such that there are defined chain homotopies
\[
\left[ \begin{array}{ccc} \epsilon & \gamma \end{array} \right] : gf \approx \left[ \begin{array}{ccc} 1 & 0 \end{array} \right] = \text{automorphism}
\]
\[
\left[ \begin{array}{ccc} \gamma & \delta \end{array} \right] : fg \approx \left[ \begin{array}{ccc} 1 & \delta \end{array} \right] = \text{automorphism}
\]
are such that there are defined chain homotopies
\[
\delta = \tilde{\mu} \gamma^{*} (-)^{n+1} + \gamma (-)^{n+1} \eta^{*} \gamma, \quad \text{and} \quad \epsilon, \delta \in \text{chain homotopies}
\]
\[
\delta : \tilde{\mu} \gamma^{*} (-)^{n+1} \mu \gamma = 1 : F^{n-*} \longrightarrow F^{n-*},
\]
\[
\epsilon : \tilde{\mu} \gamma^{*} (-)^{n+1} \gamma \mu = 1 : G \longrightarrow G .
\]
Thus both \( fg \) and \( gf \) are chain equivalences, and \( f \) is a chain equivalence with chain homotopy inverse \( (gf)^{-1}g \approx g(fg)^{-1} \).

ii) With \( \left( \begin{array}{ccc} \gamma & \tilde{\mu} \end{array} \right) \) as in i) there exist chain maps in \( \mathcal{A} \)
\[
\gamma : G^{n-*} \longrightarrow F, \quad \tilde{\mu} : G^{n-*} \longrightarrow F^{n-*},
\]
\[
\theta : G^{n-*} \longrightarrow G
\]
and a chain homotopy
\[
\tilde{\chi} = \gamma \tilde{\mu} \gamma = \theta + (-)^{n+1} \theta^{*} \gamma : G^{n-*} \longrightarrow G
\]
such that the chain map
\[
\left[ \begin{array}{ccc} \gamma^{*} (-)^{n+1} & \tilde{\mu}^{*} \end{array} \right] : C(\mu^{*})^{n+1-*} \longrightarrow C(\mu^{*})
\]
is a chain equivalence in \( \mathcal{A} \). Let \( (C, \psi) \) be the \((n+2i+1)\)-dimensional quadratic Poincaré complex derived from \( F, G^{n-*}, \gamma, \mu, \tilde{\mu}, \theta, \tilde{\chi} \) in the way \( (C, \psi) \) is derived from \( (F, G, \gamma, \mu, \tilde{\mu}, \theta, \tilde{\chi}) \). Define an \((n+2i+2)\)-dimensional quadratic Poincaré cobordism \((f, \tilde{f}) : C \phi C \longrightarrow D, (\delta \psi, \tilde{\psi}) \) by
\[
\tilde{D} = s^{i+1} F, \quad \tilde{\psi} = 0 .
\]
\[
f = (0, 1) : C \phi = G^{n-*} \phi F_{r-1} \longrightarrow D_{r} = F_{r-1-1}
\]
\[
\tilde{f} = (0, 1) : \tilde{C} \phi \tilde{G} = G_{r-1} \phi F_{r-1} \longrightarrow D_{r} = F_{r-1-1} .
\]
Thus \( (C, \psi, \tilde{C}, \tilde{\psi}) \in L_{n+2i+1}(\mathcal{A}) \). Since \( \theta \) and \( \tilde{\chi} \) can be chosen independently of \( \mu \) and \( \tilde{\mu} \) it follows that the cobordism is independent of these choices also. Given \( (F, G, \gamma, \mu, \tilde{\mu}, \theta, \tilde{\chi}) \) and chain equivalences \( h : F \longrightarrow F', k : G \longrightarrow G' \) it is possible to define \( (F', G', \gamma', \mu', \tilde{\mu}', \theta', \tilde{\chi}') \) such that the corresponding quadratic Poincaré complex \( (C', \psi') \) is homotopy equivalent to \( (C, \psi) \), and so \( (C', \psi') \in L_{n+2i+1}(\mathcal{A}) \).

iii) Define an \((n+2i+2)\)-dimensional quadratic Poincaré pair \( f : C \longrightarrow D, (\delta \psi, \tilde{\psi}) \) by
\[ D = H^{n+i} - \Phi \]
\[ f = (0, j) : \]
\[ C_r = C^{n-r} + 1 \Phi R_{r-1} \quad \longrightarrow \quad D_r = H^{n+i} - 1 - r \]
\[ \Delta \psi_0 = \tau \quad \:
\[ \phi^{n+2i+2-r} = H_{r-1} \quad \longrightarrow \quad \Phi_r = H^{n+i} - 1 - r \]
\[ \Delta \psi_r = 0 \text{ for } s \geq 1 \]

This is a quadratic Poincaré null-cobordism of \((C, \psi)\), so that \((C, \psi) = 0 \in H_{n+2i+1}(A)\).

**Definition 3.3** For any additive category with involution \(A\), define the generalized Morita maps

\[ \mu : L_m(D_n(A)) \longrightarrow L_{m+n}(A) \quad (m, n \geq 0) \]

for \(m = 2i\) (resp. \(2i+1\)) by sending a nonsingular \((-)^i\)-quadratic form \((M, \Theta)\) (resp. formation \((F, G)\) in \(D_n(A)\) to the cobordism class of the \((m+n)\)-dimensional quadratic Poincaré complex \((C, \psi)\) in \(A\) defined in Proposition 3.1 ii) (resp. 3.2 ii)). The verification that the maps \(\mu\) are well-defined is contained in Propositions 3.1 iii) (resp. 3.2 iii)).

For a ring with involution \(R\) apply 3.3 to \(A = B(R)\) to obtain generalized Morita maps \(\mu : L_m(D_n(R)) \longrightarrow L_{m+n}(R) \quad (m, n \geq 0)\).

§4. The quadratic L-theory transfer

As before, let \(A\) be an additive category with involution, and let \(D_n(A)\) be the chain homotopy category of \(n\)-dimensional chain complexes in \(A\) with the \(n\)-duality involution.

**Definition 4.1** The quadratic L-theory transfer maps of a symmetric representation \((C, \sigma, U)\) of a ring with involution \(R\) in \(D_n(A)\)

\[ (C, \sigma, U)^! : L_m(R) \longrightarrow L_{m+n}(A) \quad (m \geq 0) \]

are the composites

\[ (C, \sigma, U)^! : L_m(R) = L_m(B(R)) \longrightarrow L_m(D_n(A)) \quad \mu \longrightarrow L_{m+n}(A) \]

of the maps \(-\Theta(C, \sigma, U)\) of 2.10 and the generalized Morita maps \(\mu\) of 3.3.

**Example 4.2** Let \(A\) be the additive category \(B(S)\) of based \(\pi\)-free \(S\)-modules with the duality involution for a ring with involution \(S\). The transfer map \(\Omega\) is determined by an \(n\)-dimensional symmetric representation \((C, \sigma, U)\) of a ring with involution \(R\) in \(D_n(A) = D_n(S)\) and the morphisms of quadratic L-groups

\[ (C, \sigma, U)^! : L_m(R) \longrightarrow L_{m+n}(A) = L_{m+n}(S) \quad (m, n \geq 0) \quad \square \]

**Example 4.3** Given a Hurewicz fibration \(F \longrightarrow E \rightarrow B\) with the fibre \(F\) a finite \(n\)-dimensional geometric Poincaré complex we shall define in §5 below a symmetric representation \((C', \sigma', U')\) of \(\mathbb{Z}[\pi_1(B)]\) in \(D_n(\mathbb{Z}[\pi_1(E)])\) with \(F\) the pullback to \(F\) of the universal cover \(E\) of \(F\), and \(\sigma = (F|\pi_1)^* : C(F) \longrightarrow C(F)^{n-*}\) the Poincaré dual chain equivalence. The algebraic surgery transfer map will be defined in §5 to be
\[ p_{\text{alg}} = (C(\tilde{F}), \sigma, U)^! = L_m(\mathbb{Z}\pi_1(B)) \rightarrow L_{m+k}(\mathbb{Z}\pi_1(E)) \quad (m \geq 0). \]

In §6 we shall recall the definition via the lifting of normal maps of the geometric surgery transfer map \( p_{\text{geo}} \), which will be identified with \( p_{\text{alg}} \) in §8.

\[ \text{Example 4.4} \quad \text{Given a morphism of rings with involution } f : R \rightarrow S \text{ define a symmetric representation } (C, \sigma, U) \text{ of } \mathbb{D}_0(S) \ast \mathbb{B}(S) \text{ by} \]

\[ \sigma_0 : C_0 = S \rightarrow C^0 = S^*; \]

\[ s \rightarrow (t \rightarrow ts^{-1}) \]

\[ C_r = 0 \text{ for } r \neq 0, \]

\[ U = f : R \rightarrow H_0(\text{Hom}_S(C, C))^{op} = S. \]

In this case the transfer maps are just the change of rings morphisms \((C, \sigma, U)^! = f_! : L_m(R) \rightarrow L_m(S)\). For \( f = 1 : R \rightarrow S \), \((C, \sigma, U)\) is the universal symmetric representation (2.8) of \( R \) in \( \mathbb{B}(R) \).

\[ \text{Example 4.5} \quad \text{Given a ring with involution } S \text{ and an integer } k \geq 1 \text{ let } R = M_k(S) \text{ be the ring of } k \times k \text{ matrices} \]

\[ (s_{ij}) \mid i, j \in k \text{ with entries } s_{ij} \in S, \text{ with the involution} \]

\[ \sigma : R \rightarrow R; (s_{ij}) \rightarrow (s_{ji}). \]

Define a symmetric representation \((C, \sigma, U)\) of \( R \) in \( \mathbb{D}_0(S) \ast \mathbb{B}(S) \) by

\[ C_0 = \Sigma_S, \quad C_r = 0 \text{ for } r \neq 0, \]

\[ d : C_0 = \Sigma_S \rightarrow C^0 = (\Sigma_S)^*; \]

\[ (s_1, s_2, \ldots, s_k) \rightarrow (t_1 s_1 + t_2 s_2 + \ldots + t_k s_k), \]

\[ U = 1 : R = M_k(S) \rightarrow H_0(\text{Hom}_S(C, C))^{op} = M_k(S). \]

The generalized Morita maps \( \mu : L_*(R) \rightarrow L_*(S) \) in this case are just the usual Morita maps, which are isomorphisms for the projective and round L-groups. See Hambleton, Taylor and Williams [5] and Hambleton, Ranicki and Taylor [4] for Morita maps in quadratic L-theory.

\[ \text{Example 4.6} \quad \text{Let } F = \bigvee \rightarrow B \text{ be a } k \text{-sheeted finite covering, so that } \pi_1^k(B) \text{ is a subgroup of } \pi_1^k(B) \text{ of index } k. \text{ There are evident identifications of spaces} \]

\[ \tilde{F} = \pi_1(B) = \bigvee_{k} \pi_1^k(E) \subset \tilde{E} = E, \]

and also of \( Z \)-module chain complexes

\[ C(F) = \mathbb{Z}\pi_1(F) = \mathbb{Z}\pi_1(E)_k. \]

The symmetric representation \((C(F), \sigma, U)\) of \( \mathbb{Z}\pi_1(B) \) in \( \mathbb{D}_0(\mathbb{Z}\pi_1(B)) \ast \mathbb{B}(\mathbb{Z}\pi_1(B)) \) associated to \( p : F \rightarrow B \) (as in 4.3) is given by

\[ U : \mathbb{Z}\pi_1(B) = H_0(\text{Hom}_S(\mathbb{Z}\pi_1(B), C(F), C(F)))^{op} \]
$$H_0\left(\text{Hom}_{Z[\pi_1(E)]}(C(F), C(F))\right) = M_k(Z[\pi_1(E)])$$

$$\alpha = \Phi^1_{k} : C(F) = \Phi Z[\pi_1(E)]$$

$$\text{Hom}_{Z[\pi_1(E)]}(C(F), Z[\pi_1(E)]) = \Phi Z[\pi_1(E)]^*$$

The algebraic transfer maps in this case are the composites

$$P^l_{\text{alg}} : L_m(Z[\pi_1(B)]) \xrightarrow{U^l} L_m(M_k(Z[\pi_1(E)])) \xrightarrow{\mu} L_m(Z[\pi_1(E)])$$

with $U^l$ induced by $U$ as in 2.5 and $\mu$ the Morita maps of 4.5. In this case $P^l_{\text{alg}}$ can be described more directly by the restrictions of $Z[\pi_1(B)]$-module actions to $Z[\pi_1(E)]$-module actions, and it is clear that $P^l_{\text{alg}} = P^l_{\text{geo}}$.

Example 4.7 The algebraic $S^1$-bundle transfer maps of Munkholm and Pedersen [10] and Ranicki [16, §7.8] $P^l_{\text{alg}} : L_m(R) \rightarrow L_{m+1}(S)$ are defined for any ring with involution $S$, with $R = S/(1-t)$ for a central element $t \in S$ such that $t^2 = 1$. (We are only dealing with the orientable case here). From our point of view these are the quadratic $L$-theory transfer maps $P^l_{\text{alg}} = (C, \alpha, U)^l$ of 4.1 with $(C, \alpha, U)$ the symmetric representation of $R$ in $Z(S)$ given by

$$d = 1-t : C_1 = S \rightarrow C_0 = S$$

for an $S^1$-bundle $s^1 \rightarrow E \rightarrow B$ one takes $R = Z[\pi_1(B)]$, $s = Z[\pi_1(E)]$, $t = \text{fibre} \in \pi_1(E)$.

$$\square$$

§5. The algebraic surgery transfer

A map $p : E \rightarrow B$ of connected spaces with homotopy fibre of the homotopy type of a finite (or finitely dominated) CW complex $F$ determines a representation of $Z[\pi_1(B)]$ in $D(Z[\pi_1(E)])$ (as in 1.7). We shall now show that if $F$ is a finite $n$-dimensional geometric Poincaré complex then for any choice of orientation map $w(B) : \pi_1(B) \rightarrow \mathbb{Z}_2$ in the base there is defined a symmetric representation $(C(F), \alpha, U)$ of $Z[\pi_1(B)]$ in $D(Z[\pi_1(E)])$, and hence obtain from §4 quadratic $L$-theory transfer maps

$$P^l_{\text{alg}} = (C(F), \alpha, U)^l :$$

$$L_m(Z[\pi_1(B)]) \rightarrow L_{m+n}(Z[\pi_1(E)])$$

In §8 below we shall identify these algebraic surgery transfer maps with the geometric surgery transfer maps.

There is no loss of generality in assuming that $F \rightarrow E \rightarrow B$ is a Hurewicz fibration with the fibre $F = p^{-1}(*)$ a finite CW complex $F$. If $F$ is disconnected then $p : E \rightarrow B$ is the composite of a Hurewicz fibration
Definition 5.2 An orientation map for a group π is a morphism \( w: \pi \to \mathbb{Z}_2 \otimes (\mathbb{Z}_2^\pi) \). Let \( Z[\pi]^w \) denote the ring \( Z[\pi] \) with the \( w \)-twisted involution.

Given a chain complex \( C \) in \( \mathbb{B}(Z[\pi]) \) let \( w^\text{cn} \) denote the \( n \)-dual chain complex \( C^\text{cn} \) in \( \mathbb{B}(Z[\pi]) \) defined using the \( w \)-twisted involution on \( Z[\pi] \). If \( w \) is trivial \( w^\text{cn} \) is written as \( C^\text{cn} \). Let \( Z^w \) denote the right \( Z[\pi] \)-module with additive group \( Z \) and

\[
Z^w \times Z[\pi] \longrightarrow Z^w ; (m, \Sigma n g) \longmapsto m \Sigma w(g)n g .
\]

Let \( Z^w \) denote the left \( Z[\pi] \)-module defined in the same way.

When \( w \) is clear we abbreviate \( Z[\pi]^w \) to \( Z[\pi] \).

An \( n \)-dimensional geometric Poincaré complex \( X \) is a (connected) finite CW complex together with an orientation map \( w(X): \pi_1(X) \to \mathbb{Z}_2 \) and a fundamental class

\[
[X] \in H_n(X; \mathbb{Z}^w(X)) = H_n(Z^w \otimes Z[\pi_1(X)]^c(\tilde{X}))
\]

such that the \( Z[\pi_1(X)] \)-module chain map \( [X]: w(X) \otimes (\tilde{X})^n \to \tilde{C}(\tilde{X}) \) is a chain equivalence, with \( \tilde{X} \) the universal cover. See Wall [21] for the general theory.

The orientation map \( w=w(X): \pi=\pi_1(X) \to \mathbb{Z}_2 \) of
Let $f : X \rightarrow X$ be a pointed self homotopy equivalence, inducing an automorphism $f_* : \pi_1(X) \rightarrow \pi_1(X)$ of the fundamental group $\pi_1(X)$. A lift $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$ of $f$ to the universal cover $\tilde{X}$ induces a $\mathbb{Z}$-module chain equivalence $f : C(\tilde{X}) \rightarrow C(\tilde{X})$ which is $f_*$-equivariant.
Definition 5.5 An n-dimensional Poincaré fibration $F \to E \to B$ is a Hurewicz fibration with the fibre $F$ an n-dimensional geometric Poincaré complex, together with an orientation map $\omega(B) : \pi_1(B) \to \mathbb{Z}_2$. The lift of $\omega(B)$ is the orientation map

$$p^*\omega(B) = \omega(E) : \pi_1(E) \to \mathbb{Z}_2$$

with $U^+$ as in 5.1 and $\hat{\omega}$ as in 5.3.

Proposition 5.6 An n-dimensional Poincaré fibration $F \to E \to B$ determines a symmetric representation $(C(F),\alpha,U)$ of $\mathbb{Z}[\pi_1(B)]^\omega(B)$ in $D_n(\mathbb{Z}[\pi_1(E)]^\omega(E))$ with $G^* = ([F] \wedge)^{-1} : C(F) \to C(F)^G$ the Poincaré duality chain equivalence and $(C(F),\alpha)$ the representation of $\mathbb{Z}[\pi_1(E)]$ in $D_n(\mathbb{Z}[\pi_1(E)])$ associated to $p$.

Proof: We have to show that

$$U : \mathbb{Z}[\pi_1(B)]^{\omega(B)} \to H_0(Hom_{\mathbb{Z}[\pi_1(E)]}(C(F),\hat{C}(F))^G)$$

is a morphism of rings with involution, or equivalently that for every $g \in \pi_1(B)$ there is defined a chain homotopy commutative diagram of $\mathbb{Z}[\pi_1(E)]$-module chain complexes

This follows from 5.4 and the $\pi_1(E)$-equivariant $\omega(B)$ used to transport along the fibre $U : \pi_1(B) \to [F,F] \pi_1(E)$ used to define the ring morphism $U$ in Lück [7].

Definition 5.7 The algebraic surgery transfer maps of an n-dimensional Poincaré fibration $F \to E \to B$ are the quadratic L-theory transfer maps of 4.1 associated to the symmetric representation $(C(F),\alpha,U)$ of $\mathbb{Z}[\pi_1(B)]^{\omega(B)}$ in $D_n(\mathbb{Z}[\pi_1(E)]^{\omega(E)})$ given by 5.6

$$p^\theta : L_m(\mathbb{Z}[\pi_1(B)]) \to L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m \geq 0)$$

By definition, the algebraic surgery transfer maps are the composites

$$L_m(\mathbb{Z}[\pi_1(B)]) \to L_m(D_n(\mathbb{Z}[\pi_1(E)])) \to L_{m+n}(\mathbb{Z}[\pi_1(E)]) \quad (m \geq 0)$$

of the maps induced as in §2 by the functor of additive categories with involution.
§6. The geometric surgery transfer

Wall [22] defined the relative surgery obstruction
\[ \sigma_*(f,b) \in L_m(\mathbb{Z}[\pi_1(X)]) \]
for a normal pair \((f,b): (M,\partial M) \hookrightarrow (X,\partial X)\) from a compact m-manifold with boundary \((M,\partial M)\) to a finite m-dimensional geometric Poincaré pair \((X,\partial X)\) with \(\partial f = \partial g : \Omega \rightarrow \partial X\) a homotopy equivalence, and \(b : \Omega \rightarrow \partial X\) a map from the stable normal bundle of \(M\) to a topological reduction of the Spivak normal fibration \(\nu\) of \(X\), with the \(\omega(X)-\)twisted involution on \(\mathbb{Z}[\pi_1(X)]\). The surgery obstruction has the property that \(\sigma_*(f,b) = 0\) if (and only if) \((f,b)\) is normal bordant relative to a homotopy equivalence of pairs. Given a connected space \(B\) with finite orientation map \(w(B) : \pi_1(B) \rightarrow \mathbb{Z}_2\), it is possible to realize every element \(x \in L_m(\mathbb{Z}[\pi_1(B)])\) as the surgery obstruction of an m-dimensional normal map \((f,b): (M,\partial M) \hookrightarrow (X,\partial X)\) with a \(\pi_1\)-isomorphism reference map \(\nu : X \hookrightarrow B\) and orientation map \(w(X) : \pi_1(X) \rightarrow \pi_1(B)\).

The total space \(E\) of an n-dimensional Poincaré fibration \(F \xrightarrow{p} E \xrightarrow{q} B\) over an m-dimensional geometric Poincaré complex \(B\) is homotopy equivalent to an \((m+n)\)-dimensional geometric Poincaré complex, with the orientation map the lift \(\nu = p^*w(B) : \pi_1(E) \rightarrow \mathbb{Z}_2\) in the sense of §5.5 of the orientation map \(w(B): \pi_1(B) \rightarrow \mathbb{Z}_2\) (Quinn [12], Gottlieb [2]).

Quinn [11] used the realization theorem for surgery obstructions to define geometric transfer maps \(\Phi_{\text{geo}}: L_m(\mathbb{Z}[\pi_1(B)]) \rightarrow L_{m+n}(\mathbb{Z}[\pi_1(E)])\); for (or even in the quadratic L-groups for a fibre bundle (or even in the quadratic L-groups) \(F \rightarrow E \rightarrow B\) with the fibre \(F\) a compact \(n\)-manifold.

Given a (m+n)-dimensional normal map equipped with a reference map \(Y \rightarrow E\) obtained from the n-dimensional normal map \((f,b): M \rightarrow X\) by the pullback of \(p\) along a reference map \(X \rightarrow B\).

The surgery obstruction of Wall [22] was defined using geometric intersection numbers on the homology groups remaining after surgery below the middle dimension. The theory of Ranicki [14], [15] associates an invariant \(L_m(\mathbb{Z}[\pi_1(X)])\) to a normal map \((f,b): (M,\partial M) \hookrightarrow (X,\partial X)\) of m-dimensional geometric Poincaré pairs, with \(b : \Omega \rightarrow \partial X\) a map of the Spivak normal fibrations and \(\partial f : \Omega \rightarrow \partial X\) a homotopy equivalence. The quadratic kernel of \((f,b)\) is an m-dimensional quadratic Poincaré complex \((C(f^!),\partial)\) over \(\mathbb{Z}[\pi_1(X)]\). Here, \(C(f^!\partial)\) is the algebraic mapping cone of the Umkehr \(\mathbb{Z}[\pi_1(X)]\)-module chain map

\[ f^! : C(\tilde{X},\partial\tilde{X}) \xrightarrow{((X)\Lambda)^{-1}} C(\tilde{X}) \xrightarrow{\partial^*} C(\tilde{M}) \xrightarrow{\partial^*} C(\tilde{M}) \xrightarrow{\partial^*} \]

with \(\tilde{X}\) the universal cover of \(X\), \(\tilde{M}\) the \(\pi_1(X)\)-equivariant lift of \(f\) to the pullback cover \(\tilde{M} = f^!\tilde{X}\) of \(M\). The Poincaré duality chain equivalence given up to chain homotopy by the composite
(1+T)\psi_0 : C(f^1) \to C(\tilde{M}, \partial \tilde{M}) \to e^* \\
[M]\eta \\
C(\tilde{M}) \to C(\tilde{M}, \partial \tilde{M}) \to C(f^1)

with \( e: \text{C}(\tilde{M}, \partial \tilde{M}) \to \text{C}(f^1) \) the inclusion. The quadratic signature of \((f, b)\) is the cobordism class

\[
\sigma_*(f, b) = (C(f^1), \psi) \in L_m(\text{Z}[\pi_1(X)]) .
\]

A normal map from a manifold to a geometric Poincaré complex determines a normal map of geometric Poincaré complexes with quadratic signature the surgery obstruction.

**Definition 6.1** The geometric surgery transfer maps \( \sigma_* \) of an \( n \)-dimensional Poincaré fibration \( F \to E \to B \) with finitely presented \( \pi_1(B) \)

\[
P_{\text{geo}} : L_m(\text{Z}[\pi_1(B)]) \to L_{m+n}(\text{Z}[\pi_1(E)]) ;
\]

\[
\sigma_*((f, b): M \to X) \to \sigma_*((g, c): N \to Y) \quad (m \geq 3)
\]

are defined using the quadratic signature of normal maps of geometric Poincaré complexes. Here, \((g, c): N \to Y\) is the \((m+n)\)-dimensional normal map obtained from an \( m \)-dimensional normal map \((f, b): M \to X\) by the pullback of \( p \) along a reference map \( X \to B \).

\[
\sigma_*((f, b) X \to X \to F) = \sigma_*(f, b) \otimes \sigma_*(F)
\]

which expressed the quadratic signature of a product \((f, b) X \to X \to F\) as the tensor product of the quadratic signature of \((f, b)\) and the symmetric signature \( \sigma_*(F) = (C(F), \phi) \in L_n^N(\text{Z}[\pi_1(F)]) \). However, this would require the development of a fair amount new technology, translating the homotopy action of \( \Omega B \) on the geometric Poincaré complex \( F \) into a chain homotopy action of \( C(\Omega B) \) on the symmetric Poincaré complex \( C(F), \phi \) over \( \text{Z}[\pi_1(E)] \). For the purpose at hand we can assume by the realization theorem that the \( m \)-dimensional normal map \((f, b): M \to X\) is \([(m-2)/2]\)-connected. In the highly-connected case we can give a chain level geometric interpretation of both the element \( \psi_* \sigma_*(f, b) \in L_m(\text{D}_n(\text{Z}[\pi_1(E)]) \) and its image...
under the generalized Morita map $\mu: \pi_0(D_n(A)) \to \pi_0(\pi_1(E))$. For a fibre bundle $E \to B$ it is possible to dispense with some of the algebra, using instead the fibred intersection theorem below.

§7. Ultrasound L-theory

Ultrasound L-theory was developed in §7.8. Ranicki [16] in connection with the algebraic theory. We use it here to recognize quadratic Poincaré complexes in the image of the algebraic and geometric surgery transfer maps.

Let $A$ be an additive category with involution. In Ranicki [15], [19] define for any finite chain complex $C$ in $A$ and $s \geq 1$ the $Z$-module chain complex

$W_s^C = W \otimes Z[Z_2] \cdot \text{Hom}_A(C^*, G)$,

with the generator $T \in Z_2^2$ acting on $\text{Hom}_A(C^*, G)$ by the $e$-transposition involution $T e = sT$ and $W$ the standard free $Z[Z_2]$-module resolution of $Z$.

An $m$-chain $\psi \in W_{s-1}^C$ is a collection of morphisms $\psi = (\psi_s \in \text{Hom}_A(C^*, C)_{m-s})_{s \geq 0}$ such that for a cycle there is defined a chain map $\psi : C^* \to C$. An $m$-dimensional $e$-quadratic (Poincaré) complex $(C, \psi)$ in $A$ is an $m$-dimensional complex $C$ in $A$ together with an element $\psi \in \pi_0(\text{Hom}_A(C^*, C))$ such that $(1 + e_\pi)^2\psi : C^* \to C$ is a chain equivalence. The skew-suspension isomorphisms are defined by $(\tilde{\psi})_s = \psi_s (s \geq 0)$, for any finite chain complex $C$ in $A$. The skew-suspension map $s : L_m(A, e) \to L_{m+2}(A, e)$ ($m \geq 0$) in the $\pm e$-quadratic L-groups are also isomorphisms, so that

$L_m(A, e) = L_{m+2}(A, e) = L_{m+4}(A, e) (m \geq 0)$.

For $e = 1$ we write $Q_m(C, e) = Q_m(C)$, $L_m(A, 1) = L_m(A)$, and $L$-quadratic = quadratic.

Poincaré complexes are $e$-quadratic complexes $(C, \psi)$ with $\psi_s = 0$ for $s \geq 1$.

For any finite chain complex $C$ in $A$ define the abelian group

$\tilde{\psi} : e : C^* \to C$.

Definition 7.1. An $m$-dimensional $e$-ultrasound (Poincaré) complex $(C, \hat{\psi})$ in $A$ is an $m$-dimensional chain complex $C$ in $A$ together with an element $\hat{\psi} \in \pi_0(\text{Hom}_A(C^*, C))$ such that $(1 + e_\pi)^2\hat{\psi} : C^* \to C$ is a chain equivalence.

There is a corresponding notion of cobordism.
\[ \hat{L}_m(\mathbb{A}, \varepsilon) = \hat{L}_{m+2}(\mathbb{A}, -\varepsilon) = \hat{L}_{m+4}(\mathbb{A}, \varepsilon) \quad (m \geq 0) \]

by skew-suspension isomorphisms, just like for \( L_\varepsilon \). If \( \mathbb{A} = \mathbb{R}(R) \) for a ring \( R \) with involution \( \ast \) we write \( \hat{L}_m(\mathbb{A}) \) as \( \hat{L}_m(R) \).

Define a map \( \mapsto: \hat{Q}_m(C) \rightarrow \hat{Q}_m(C, \varepsilon) \) by \( \psi \mapsto \hat{\psi} \). An \( m \)-dimensional \( \varepsilon \)-ultraquadratic \( \hat{P} \) complex \( (C, \hat{\psi}) \) determines an \( m \)-dimensional \( \varepsilon \)-ultraquadratic \( \hat{P} \) complex \( (C, \psi) \). The forgetful maps in the cobordism groups

\[ \hat{L}_m(\mathbb{A}, \varepsilon) \rightarrow \hat{L}_m(\mathbb{A}, \varepsilon) \quad ; \quad (C, \hat{\psi}) \rightarrow (C, \psi) \quad (m \geq 0) \]

are surjective for even \( m \) and injective for odd \( m \).

The ultraquadratic L-group \( \hat{L}_m(\mathbb{Z}) \) was identified in §7.8 of [16] with the cobordism group \( C_{m-1} \) of knots \( k: S^{m-1} \xrightarrow{\partial} S^m \) \( (m \geq 4) \). A Seifert surface for a knot \( k: S^{m-1} \xrightarrow{\partial} S^m \) is a codimension 1 framed submanifold \( M \xrightarrow{\partial} S^m \) with boundary \( \partial M = k(S^{m-1}) \). Inclusion defines an \( m \)-dimensional normal map \((f, b): (M, \partial M) \rightarrow (S^{m+2}, S^{m-1})\) with quadratic kernel \( \sigma_{k}(f, b) = (C, \psi) \) such that \( H_*(C) = H_*(M) \). The framing determines a map \( M \rightarrow S^{m+1} \) which induces a chain map \( \hat{\psi}: C^{m-1} \rightarrow C \)

defining an \( m \)-dimensional ultraquadratic \( \hat{P} \) complex \((C, \hat{\psi})\) over \( \mathbb{Z} \). The knot complement \( U = S^{m+1} \) - \( \text{nbhd. of } k(S^{m-1}) \) has boundary \( \partial U = S^{m-1} \times S^1 \), and there is defined an \( (m+1) \)-dimensional normal map \((U, \partial U) \rightarrow (D^{m+2}, S^{m-1}) \times S^1 \) which is a \( Z \)-homology equivalence. Let \((L_{m+1}; M^m, \partial M^m)\) be the fundamental

triads. The inclusions \( j: M \rightarrow L, \ k: 2L \rightarrow L \) induce \( \text{chain maps } j, k: C \rightarrow C\) \( (f, b): (D_m(A), \varepsilon) \rightarrow \hat{L}_m(A, \varepsilon) \)

be the corresponding \( (m+1) \)-dimensional normal map of \( \text{module chain maps } j, k: C \rightarrow C\) \( (f, b): (D_m(A), \varepsilon) \rightarrow \hat{L}_m(A, \varepsilon) \)

such that \( j, k: C \rightarrow C \) is a chain equivalence. The ultraquadratic structure \( \psi E\hat{Q}_m(C) \) is determined by the symmetric structure \( (1+T)\hat{\psi}: C^{m-1} \rightarrow C \) and \( j, k \), since up to chain homotopy

\[ (j-k)^{-1}j = \hat{\psi}((1+T)\hat{\psi})^{-1}: C \rightarrow C \]

\[ (j-k)^{-1}k = -T((1+T)\hat{\psi})^{-1}: C \rightarrow C \].

More generally:

**Proposition 7.2** Let \( (C, \psi) \) be an \( m \)-dimensional \( \varepsilon \)-quadratic \( \hat{P} \) complex in \( \mathbb{A} \). A cobordism \((j, k): C \rightarrow D, (\delta \psi, \psi \gamma - \psi)\) with \( j, k: C \rightarrow D \) a chain equivalence determines an \( \varepsilon \)-ultraquadratic structure \( \psi E\hat{Q}_m(C) \) with image \( \psi E\hat{Q}_m(C, \varepsilon) \), such that

\[ (C, \psi) = \mu(C^{m-1}, \hat{\psi}) \in \text{im}(\mu: L_0(D_m(A), \varepsilon) \rightarrow L_m(A, \varepsilon)) \]

with \((C^{m-1}, \hat{\psi})\) a nonsingular \( \varepsilon \)-quadratic form in \( D_m(A) \).

**Proof:** Define a morphism in \( D_m(A) \)

\[ h = (j-k)^{-1}j: C \rightarrow D, (j-k)^{-1} \rightarrow C. \]

By the chain homotopy invariance of the Q-groups we can replace \((j, k), (\delta \psi, \psi \gamma - \psi)\) by a homotopy equivalent
corresponds to the chain level

$$h_2(\psi) - (h-1)\zeta(\psi) = d(5\psi) \in (W_2C)_m$$

so that there is defined a chain homotopy

$$(1 + t)\delta \psi_0 : (1 + t)\psi_0 \simeq (h-1)\zeta(\psi) = d(5\psi) \in (W_2C)_m$$

The $\varepsilon$-ultraquadratic Poincaré complex $(C, \hat{\psi})$ in $A$ defined by the chain map

$$\hat{\psi} = h(1 + t)\psi_0 : C^{m-1} \xrightarrow{(1 + t)\delta \psi_0} C$$

is such that $\hat{\psi} + \varepsilon \hat{\psi} = (1 + t)\psi_0 : C^{m-1} \xrightarrow{\delta \psi_0} C$. Define a chain $x(5\psi_0) \in (W_2C)_m$ such that $\varepsilon \psi = \mu(5\delta \psi_0) \in (W_2C)_m$ by

$$x_s = \begin{cases} 0 & \text{if } s = 0 \\ hT^s \varepsilon \psi - 1 & \text{if } s \geq 1 \end{cases} : C^{m+1-r-s} \xrightarrow{\varepsilon - 1} C.$$ 

Thus $\hat{\psi} = \varepsilon \psi \in Q_m(C, \varepsilon)$ and

$$(C, \hat{\psi}) = (C, \psi) = \mu(C^{m-1}, \psi) \in L_m(A, \varepsilon).$$

**Corollary 7.3** Let $(f, b) : M \to X$ be an $(i-1)$-connected normal map of $(n+2i)$-dimensional geometric Poincaré complexes, and let

$$(e, f, zf) : (L; M, zM) \to XX([0, 1]; (0), (1))$$

be an $(i-1)$-connected normal bordism between $(f, b)$ and a disjoint copy $(zf, zb)$. If the $(i-1)$-connected normal
$Z[\pi_1(X)]$ satisfying the hypothesis of 7.2. It follows that $\psi \in \mathbf{Q}^n_{+1}(C)$ is the image of the element $\hat{\psi} \in \mathbf{E}_{0}(\mathbf{H}_d(c^{n+21-k}, C))$ defined by the composite chain map

$$\hat{\psi} : c^{n+21-k} \to C \to D \to (j-k)^{-1} \to C$$

with $\vartheta_0 = [M]_0 : C^{n+21-k} \to C$ the Poincaré duality chain equivalence. The nonsingular $(-1)^i$-quadratic form $(S^{-1}c^{n+21-k}, \hat{\psi})$ in $D_n(Z[\pi_1(X)])$ is such that

$$\sigma_x(f, b) = (C, \psi) = (S^{-1}c^{n+21-k}, \hat{\psi})$$

$$\in \text{im}(\mu : L_0(D_n(Z[\pi_1(X)]), (-1)_n) \to L_n(Z[\pi_1(X)], (-1)_n))$$

$$= \text{im}(\mu : L_0(D_n(Z[\pi_1(X)])) \to L_n(Z[\pi_1(X)])) .$$

Proposition 7.4 Let $((j, j') : C \otimes g : D, (\delta \psi, \psi_0 : \psi'))$ be a cobordism of $m$-dimensional $\varepsilon$-quadratic Poincaré complexes in $A$, such that $D, C(j)$ and $C(j')$ are the suspensions of $(m-1)$-dimensional chain complexes (up to chain equivalence), with $m \geq 1$. The chain homotopy classes of the chain maps

$$Y = \text{inclusion} : G = S^{-1}D \to S^{-1}C(j') = F$$

$$\mu = \text{inclusion} :$$

$$G = S^{-1}D \to S^{-1}C(j) = C(j')^{m-1} = \pi^{m-1}$$

are the components of a morphism of $\varepsilon$-symmetric forms in $D_{m-1}(A)$

$$\begin{bmatrix} \gamma \end{bmatrix} : (G, 0) \to H_\varepsilon (F) = (\mathbf{F}_\varepsilon \pi^{m-1}, \begin{bmatrix} 0 & 1 \end{bmatrix})$$

such that $\gamma^* \mu = (1 + T_{\varepsilon} - \psi_0^0 : \psi_0 : \psi')$ for a certain element $\theta \in \mathbf{Q}^{m-1}(G^{m-1-k}, \varepsilon)$ determined by $(\delta \psi, \psi_0 : \psi')$.

If the morphism $\mathbf{f}_0 : G \to \mathbf{F}_\varepsilon \pi^{m-1}$ is a split injection in $D_{m-1}(A)$ and if $\text{im}(Q_{m-1}(G^{m-1-k}) : O_{m-1}(G^{m-1-k}, \varepsilon))$ then $G$ is a lagrangian of the hyperbolic $\varepsilon$-quadratic form

$$H_\varepsilon (F) = (\mathbf{F}_\varepsilon \pi^{m-1}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$$

and $(F, G)$ is a nonsingular $\varepsilon$-quadratic formation in $D_{m-1}(A)$ such that

$$(C, \psi) = \mu(F, G) \in \text{im}(\mu : L_1(D_{m-1}(A), \varepsilon) \to L_m(A, \varepsilon)).$$

Proof: Let $(D^{m+1-k}, \theta)$ be the $(m+1)$-dimensional $\varepsilon$-quadratic complex in $A$ (not in general Poincaré) defined by the algebraic Thom construction, the image of $(\delta \psi, \psi_0 : \psi') \in \mathbf{Q}_{m+1}(C(j, j')$, $\varepsilon$) under the isomorphism

$$((1 + T_{\varepsilon})(\delta \psi_0, \psi_0 : \psi'))^{-1} :$$

$$Q_{m+1}(C(j, j'), \varepsilon) \to Q_{m+1}(D^{m+1-k}, \varepsilon) = Q_{m-1}(G^{m-1-k}, \varepsilon).$$

Up to chain homotopy

$$Y^* \mu : G = S^{-1}D \to S^{-1}C(j, j') = D^{m-1} = G^{m-1-k},$$

$S^{-1}C(j, j') = D^{m-1} = G^{m-1-k}$,
so that there exists a chain homotopy

\[ \gamma^* \mu \simeq (1 + T_\varepsilon) \theta_0 : G \longrightarrow G^{m-1,*}, \]

and

\[ (\gamma^* \mu^*) \otimes 1, (\gamma \otimes 0) \]

as required for \( G \) to be a lagrangian in \( H^c(F) \).

\( \varepsilon \in D_{n+1}(G^{m-1,*},\varepsilon) \) is the image of \( \varepsilon \tilde{G}_{m-1}(G^{m-1,*}) \) through the hessian \( -\varepsilon \)-quadratic form in \( D_{m-1}(G^{m-1,*},\varepsilon) \), required for \( G \) to be a lagrangian in \( H^c(F) \). The algebraic Thom construction defines a one-one correspondence between the homotopy equivalence classes of \((m+1)\)-dimensional \( \varepsilon \)-quadratic Poincaré pairs and \((m+1)\)-dimensional \( \varepsilon \)-quadratic complexes in \( D_{m+1}(G^{m-1,*},\varepsilon) \), denoted \( ((j', j) : \mathbb{C} \to D_{m+1}(G^{m-1,*},\varepsilon)) \).

Thus, \( ((j', j) : \mathbb{C} \to D, (\delta \psi, \psi @ \psi')) \) is homotopy equivalent to the \((m+1)\)-dimensional \( \varepsilon \)-quadratic Poincaré pair \( ((0 \pm 1) : \mathbb{D} \to D, (0, \varepsilon \tilde{\theta})) \) defined by

\[ \begin{cases} \partial \mathbb{D} = \tilde{d}_\mathbb{D} \varepsilon, & \mathbb{D} = (-)F(1 + T_\varepsilon) \hat{\theta} \\ \partial r \mathbb{D} = d_r, & \mathbb{D} = (-r) \hat{\theta} \end{cases} \]

so that there is defined a chain equivalence \( f : C(\mu^*) \to C(\mu^*) \).

Choosing a representative chain map \( \hat{\theta} : D \to D^{m+1,*} \) and a chain homotopy \( \hat{\theta} : D \to D^{m+1,*} \) defines a chain map \( \hat{\theta} : D \to C(\mu^*) \) by

\[ \hat{\theta} = \begin{pmatrix} 1 & X \\ 0 & \gamma \end{pmatrix} \]

such that

\[ \partial_{\mathbb{D}} = d_{\mathbb{D}} \varepsilon, \partial_{\mathbb{D}} = d_{\mathbb{D}} \varepsilon + 1 \]

Now \( (C(\mu^*), \varepsilon) \) is the \( m \)-dimensional \( \varepsilon \)-quadratic Poincaré complex in \( \mathbb{A} \) constructed in 3.2 from the nonsingular \( \varepsilon \)-quadratic form \( (F, G) \) in \( D_{m-1}(\mathbb{A}) \), so that

\[ (C(\psi), (C(\mu^*), f_\varepsilon(\psi)) = (C(\mu^*), \varepsilon) \]
(1-1)-connected 2i-dimensional normal map \((f,b)\) is an \((i-1)\)-connected 2i-dimensional quadratic Poincaré complex over \(\mathbb{Z}[\pi_i(B)]\) which is homotopy equivalent to \((S^i \mathbb{K}_i(M), \hat{\psi})\). Thus we can identify and up to chain homotopy

\[(1+T)^{i} \psi_0 = \hat{\psi} + (-)^i \hat{\psi} = \lambda^{-1} : \]

\[C(f)^{2i-k} = S^i \mathbb{K}_i(M)^* \xrightarrow{\psi} \mathbb{K}_i(M)* \to \mathbb{K}_i(M) \]

The quadratic structure \(\psi \in \mathcal{Q}_2i(C(f'))\) is the equivalence class of \(\mathbb{Z}[\pi_i(B)]\)-module morphisms \(\hat{\psi} : \mathbb{K}_i(M)* \to \mathbb{K}_i(M)*\) described above. A choice of representative \(\hat{\psi}\) is a choice of ultragquadratic structure \(\hat{\psi} \in \mathcal{Q}_2i(C(f'))\) for \(\mathbb{Z}[\pi_i(B)]\)-module chain homotopy classes of \(\mathbb{K}_i(M)*\). We now fix a choice of representative \(\hat{\psi}\).

Let \((v_1, v_2, \ldots, v_k)\) be a basis for the \(f\)-g. of \(\mathbb{Z}[\pi_i(B)]\)-module \(\mathbb{K}_i(M)*\), and use the dual to define a basis for \(\mathbb{K}_i(M) = \mathbb{K}_i(M)*\). The functor of addition in categories with involution

\[p^\# = \Theta(C(\hat{\psi}), d, \eta) : \]

\[\mathcal{B}(\mathbb{Z}[\pi_i(B)]) \to \mathcal{D}_n(\mathbb{Z}[\pi_i(B)])\]

sends the morphisms in \(\mathcal{B}(\mathbb{Z}[\pi_i(B)])\)

\[\hat{\psi}, \hat{\psi}^* : \]

\[\mathbb{K}_i(M)* = \Theta \mathbb{Z}[\pi_i(B)] \to \mathbb{K}_i(M) = \Theta \mathbb{Z}[\pi_i(B)]\]

to chain homotopy classes of \(\mathbb{Z}[\pi_i(B)]\)-module chain maps

\[p^\#(\hat{\psi}), p^\#(\hat{\psi}^*) : \]

\[\mathcal{Q}(\mathbb{Z}[\pi_i(B)]) \to \mathcal{Q}(\mathbb{Z}[\pi_i(B)])\]

such that there is defined a chain homotopy commutative diagram

\[\begin{array}{ccc}
\mathcal{Q}(\mathbb{Z}[\pi_i(B)]) & \xrightarrow{p^\#(\hat{\psi})} & \mathcal{Q}(\mathbb{Z}[\pi_i(B)]) \\
\mathcal{Q}(\mathbb{Z}[\pi_i(B)]) & \xrightarrow{p^\#(\hat{\psi}^*)} & \mathcal{Q}(\mathbb{Z}[\pi_i(B)]) \\
\mathcal{Q}(\mathbb{Z}[\pi_i(B)]) & \xrightarrow{p^\#(\hat{\psi})} & \mathcal{Q}(\mathbb{Z}[\pi_i(B)]) \\
\mathcal{Q}(\mathbb{Z}[\pi_i(B)]) & \xrightarrow{p^\#(\hat{\psi}^*)} & \mathcal{Q}(\mathbb{Z}[\pi_i(B)]) \\
\end{array}\]

and such that

\[p^\#(\hat{\psi}) + (-)^i p^\#(\hat{\psi}^*) = p^\#(\lambda^{-1}) : \]

\[\mathcal{Q}(\mathbb{Z}[\pi_i(B)]) \to \mathcal{Q}(\mathbb{Z}[\pi_i(B)])\]

is a chain equivalence.

In Lemmas 8.1, 8.3 below we shall show that the quadratic kernel \(\sigma_i(g,c) = (C(g'), \eta)\) of the pullback \((n+1)-\text{connected} (n+2i)-\text{dimensional} \) normal map

\((g,c) : (\mathbb{N}, \partial \mathbb{N}) \to (Y, \partial Y)\) is homotopy equivalent to the \((n+2i)-\text{dimensional} \) quadratic Poincaré complex \((D,Y)\) defined by the \(\mathbb{Z}[\pi_i(E)]\)-module chain complex

\[D = S^i p^\#(\mathbb{K}_i(M)) = \Theta S^i \mathcal{Q}(\hat{\psi})\]

with the (ultra)quadratic structure

\[\eta_0 : D^{n+2i-} \xrightarrow{\Theta S^i \mathcal{Q}(\hat{\psi})} \mathcal{Q}(\mathbb{Z}[\mathbb{K}_i(M)]) \xrightarrow{\Theta^i \mathcal{Q}(\hat{\psi})} \mathcal{Q}(\mathbb{Z}[\mathbb{K}_i(M)]) \]

\[\begin{array}{ccc}
\mathcal{Q}(\mathbb{Z}[\mathbb{K}_i(M)]) & \xrightarrow{\Theta S^i \mathcal{Q}(\hat{\psi})} & \mathcal{Q}(\mathbb{Z}[\mathbb{K}_i(M)]) \\
\mathcal{Q}(\mathbb{Z}[\mathbb{K}_i(M)]) & \xrightarrow{\Theta S^i \mathcal{Q}(\hat{\psi})} & \mathcal{Q}(\mathbb{Z}[\mathbb{K}_i(M)]) \\
\mathcal{Q}(\mathbb{Z}[\mathbb{K}_i(M)]) & \xrightarrow{\Theta S^i \mathcal{Q}(\hat{\psi})} & \mathcal{Q}(\mathbb{Z}[\mathbb{K}_i(M)]) \\
\mathcal{Q}(\mathbb{Z}[\mathbb{K}_i(M)]) & \xrightarrow{\Theta S^i \mathcal{Q}(\hat{\psi})} & \mathcal{Q}(\mathbb{Z}[\mathbb{K}_i(M)]) \\
\end{array}\]
\[ n_s = 0 : \ b^{n+2i-r-s} \rightarrow \ b_r \ (s \geq 1). \]

It will follow that the nonsingular \((-1)^s\)-quadratic \((p^\theta(K_i(M)^s), (\Theta(F) \cap -)^{-1} p^\theta(\tilde{\psi}))\) in \(D_n(Z[\pi_1(E)])\) de-

\[ \Theta(F) \cap - \rightarrow \Theta_1^iC(\tilde{\psi}) = S^i p^\theta(K_i(M)^s) = \Theta_1^i Z[\pi_1(B)] \]

**Proof:** Represent the base elements \(v_j \in K_i(M) (1 \leq j \leq k)\) by framed immersions \(v_j : S^1 \rightarrow \text{int}(M^2_j)\) with nullhomotopies in \(X\), and with \(\pi_1(B)\)-equivariant lifts \(\tilde{v}_j : \tilde{\pi}_1(B) \times S^1 \rightarrow \tilde{N}\). Replace \(i : X \rightarrow X\) by the inclusion of \(\tilde{N}^i \rightarrow X^i\) in the CW complex \(N \cup V^i + 1\) homotopy equivalent to \(X\) (which is also denoted by \(X\)), so that \(C(f^i) \simeq S^i C(X, \tilde{N}) = S^i K_i(M) = \Theta_1^i Z[\pi_1(B)]\).

In the total spaces of the pullbacks \(g : N \rightarrow Y\) is replaced by the inclusion of \(N\) in the CW complex \(N \cup V^i + 1\), so that \(C(g^i) \simeq S^i C(Y, \tilde{N}) = S^i p^\theta(K_i(M)) = \Theta_1^i C(\tilde{\psi})\).

The Poincaré duality chain equivalence is given up to chain homotopy by the composite
\[
(1+T)g_0 : C(g^i)^{n+2i} \rightarrow C(\tilde{N}, \partial \tilde{N})^{n+2i} \rightarrow C(\tilde{N}) 
\]
with \(e\) the inclusion.

For a sufficiently large number \(q \geq 0\) the framed immersions can be approximated by framed embedding.
\( v_j: \Sigma^i \rightarrow \text{int}(M \times D^q) \) with nullhomotopies in \( X \), i.e.,
\[ \text{P}^j = \text{closure}(M \times D^q - v_j), \]
so that
\[ M \times D^q = v_j \cup \partial v_j P^j, \quad \partial P^j = \partial v_j \cup \partial (M \times D^q). \]

\( (v_j, \partial v_j) = v_j(\Sigma^i) \times (D^{i+q}, \Sigma^i + q - 1). \)

The intersection number
\[ \lambda_{j, j'} = \lambda(\tilde{v}_j, \tilde{v}_j') \in \mathbb{Z}[\pi_1(B)] \quad (1 \leq j, j' \leq k) \]
is the image of \( 1 \in \mathbb{Z}[\pi_1(B)] \) under the composite
\[ H_1(\Sigma^i) = \mathbb{Z}[\pi_1(B)] \xrightarrow{\tilde{v}_j} H_1(\tilde{M}) \cong H_i(\tilde{M}) \cong H_i(\tilde{M} \times S^q - 1) \]
\[ \cong H_{i+q}(\tilde{M} \times D^q, \tilde{M} \times S^{q+1}) \]
\[ = H_0(\Sigma^i) = \mathbb{Z}[\pi_1(B)]. \]

which can also be expressed as
\[ H_1(\Sigma^i) = \mathbb{Z}[\pi_1(B)] \xrightarrow{\tilde{v}_j} H_1(\tilde{M}) \cong H_i(\tilde{M}) \cong H_i(\tilde{M} \times S^q - 1) \]
\[ \cong H_{i+q}(\tilde{M} \times D^q, \tilde{M} \times S^{q+1}) \]
\[ = H_0(\Sigma^i) = \mathbb{Z}[\pi_1(B)]. \]

The pullbacks from the \( n \)-dimensional Poincaré fibrations
\[ F \longrightarrow B \]
define framed Poincaré immersions
\[ w_j: F^i \times S^i \longrightarrow N^{n+2i} \]
with nullhomotopies in \( Y \), and with \( \pi_1(E) \)-equivariant lifts \( \tilde{w}_j: F^i \times S^i \longrightarrow \tilde{N} \) \((1 \leq j \leq k) \). Let
\[ \text{C}_k M \times D^q \text{ be the total spaces of the fibrations over } \text{C}_k \]
\[ \text{M} \times D^q, \]
so that
\[ \text{N} \times D^q = w_j \cup \partial w_j Q_j, \quad \partial Q_j = \partial w_j \cup \partial (N \times D^q). \]

\[ (w_j, \partial Q_j) = w_j(F^i \times S^i) \times (D^{i+q}, S^{i+q} - 1). \]

for any embedding \( D^{i+q} \subset \text{C}_k \text{M} \times D^q \) the pair
\[ \text{C}_k \text{M} \times D^q - \text{int}(D^{i+q}) \cup v_j x D^{i+1} \times D^q, \]
\[ P_j \cup D^{i+1} \times S^{q-1} \]
has a relative CW structure with one \((i+q)\)-cell and one \((i+q+1)\)-cell, such that the cellular chain complex in \( B(\pi_1(B)) \) is \( \lambda_{j, j'}: \mathbb{Z}[\pi_1(B)] \longrightarrow \mathbb{Z}[\pi_1(B)]. \) By 1.9 the cellular chain homotopy class of the \( \mathbb{Z}[\pi_1(E)] \)-module chain map
\[ p(\lambda, j, j') : C(F) \longrightarrow C(F) \]
conicides with the composite
\[ C(F) \longrightarrow s^{-1}C(F^i) \xrightarrow{\tilde{w}_j} \]
\[ s^{-1}C(\tilde{N}, \partial \tilde{N}) \cong s^{-1}q_1C(\tilde{N} \times D^q, \partial (\tilde{N} \times D^q)) \]
\[ s^{-1}q_1C(\tilde{N} \times D^q, \tilde{Q}_j) \cong s^{-1}q_1C(\tilde{w}_j, \partial \tilde{w}_j) = C(F) \]
\[ \longrightarrow C(F), \]
and hence also with the composite
\[ C(F) \longrightarrow s^{-1}C(F^i) \xrightarrow{\tilde{w}_j} \]
\[ s^{-1}C(\tilde{N}, \partial \tilde{N}) \]
\[ \cong s^{-1}q_1C(\tilde{N} \times D^q, \partial (\tilde{N} \times D^q)) \]
\[ \cong s^{-1}q_1C(\tilde{N} \times D^q, \tilde{Q}_j) \cong s^{-1}q_1C(\tilde{w}_j, \partial \tilde{w}_j) = C(F) \]
\[ \longrightarrow C(F). \]
The $(i,j')$-component of the $\mathbb{Z}[\pi_1(E)]$-module equivalence

$$(1+T)\eta_0^{-1} : C(g') = \oplus_{\lambda^2} C(\tilde{F}) \to C(g')^{n+2i-1} = \oplus_{\lambda^3} C(\tilde{F})^n$$

is thus the composite

$$\eta^{-1} \circ \tilde{C} \circ \eta^{-1} : (\tilde{F})$$

and up to chain homotopy

$$(1+T)\eta_0 : C(g')^{n+2i-1} = \oplus_{\lambda^2} C(\tilde{F})^n \to (\tilde{F})$$

We extend the description of the symmetric structure of $\sigma_*(\mathfrak{g},c)$ given by 8.1 to the quadratic structure, using the ultradimensional $I$-theory of §7. A choice of ultradimensional structure $\tilde{\psi}: K_1(M) \to K_1(M)$ for $\sigma_*(\mathfrak{g},c)$ is used to construct a normal bordism between $(f,b): M \to X$ and a copy $(zf,zb): zM \to \tilde{X}$, which encodes the quadratic self-intersection form $\mu$ in the structure. The quadratic structure of $\sigma_*(\mathfrak{g},c)$ is then decoded from the CW structure of the pullback normal bordism between $(\mathfrak{g},c): N \to Y$ and a copy $(z\mathfrak{g},zc): zN \to zY$ using 1.9 and 7.3. The construction of the bordism is motivated by the way in which the infinite cyclic cover

$$(M,\lambda,\mu)$$

of a choice of ultradimensional structure $\tilde{\psi}$ for $(M,\lambda,\mu)$ can be realized by an $(i-1)$-connected $(2i+1)$-dimensional normal bordism

$$(\eta: (\eta: f: zI, (a: b, zb)): (L: M, zM) \to XX([0,1]; (0); (1)))$$

between $(f,b): M \to X$ and a disjoint copy $(zf,zb): zM \to \tilde{X}$, such that the difference of the $\mathbb{Z}[\pi_1(B)]$-module morphisms $j,k: K_1(M) \to K_1(L)$ induced by the inclusions $j: M \to L$, $k: zM \to L$ is an isomorphism $j-k: K_j(M) \to K_j(L)$ with

$$(j-k)^{-1}j = \psi^*\lambda : K_j(M) \to K_j(L)$$

$$(j-k)^{-1}k = (-)^{i+1} \psi^*\lambda : K_j(M) \to K_j(L).$$

The $(i-1)$-connected $(2i+1)$-dimensional normal map

$$(e/(f: zI, a/(b: zb)): (L: (M: zM), \partial MX)^l \to (X: \partial X)\tilde{X}^l$$

is a $\mathbb{Z}[\pi_1(B)]$-homology equivalence, with the homotopy equivalence $\partial X: \partial MX \to \tilde{X}^l$ on the boundary.

**Proof:** Every based $f$-free lagrangian of the form $k(\mu_f(M),\lambda,\mu)\oplus K_1(M,\lambda,\mu)$ can be realized by disjoint framed embeddings of $S^1$ in $X$. The $(-)^i$-quadratic form $(K_1(M),\lambda,\mu)$ on $K_1(M,\lambda,\mu)$ is realized by nullhomotopies in $X$, such that the $MV: \partial MX([0,1]) \to zM$ with nullhomotopies in $X$, such that the $MV: \partial MX([0,1]) \to zM$ with nullhomotopies in $X$, such that the trace of the surgeries on these framed embedded $i$-spheres defines a normal bordism between $(f,b)$ and $(zf,zb)$. The realization of the lagrangian

$$(f: zI, a: b: zb): (L: M, zM) \to XX([0,1]; (0); (1)))$$

is constructed using 1.9 and 7.3.
has the required properties. (This lagrangian direct complement of the diagonal lagrangian \( \text{im}(\{\lambda\}) : K_i(M) \rightarrow K_i(M) \otimes K_i(M) \). The realization of normal map \( (f,b)X_1 : MX(0,1);(0),(1) \rightarrow XX(0,1);(0),(1) \). The required normal map \( (e,a) \) can also be obtained from \( (f,b)X_1 \) by surgeries on \( i \)-spheres in the interior \( MX(0,1) \) representing a base of \( K_i(MX(0,1)) = K_i(M) \).)

We can now extend Lemma 8.1 to the quadratic structure:

**Lemma 8.3** The quadratic kernel \( \sigma_*(g,c) = \langle C(g'), \eta \rangle \) is such that up to chain homotopy

\[
\eta_0 : C(g')^{n+2i-1} \rightarrow \mathcal{H}^1_p(K_i(M)^* \otimes K_i(M)^*) \simeq \mathcal{H}^1_p(C(F)) \simeq \mathcal{H}^1_p(K_i(M)^* \otimes K_i(M)^*)
\]

\[
\eta_s = 0 : C(g')^{n+2i-r-s} \rightarrow C(g')_r \quad (s \geq 1).
\]

**Proof:** Let \( \lambda, (e,a),(f,b), j,k \) be as in 8.2, and let

\[
(h/g=zf), (d/c,zc) : \mathcal{X}(0,1);(0),(1) \rightarrow \mathcal{X}(0,1);(0),(1)
\]

be a \( \mathbb{Z}[\pi_1(E)] \)-homology equivalence. By 7.3 the quadratic kernel \( \sigma_*(g,c) \) is determined by the chain homotopy classes of the \( \mathbb{Z}[\pi_1(E)] \)-module chain maps \( \mathcal{H}^1_p(C(F)) \rightarrow \mathcal{H}^1_p(K_i(M)^* \otimes K_i(M)^*) \) and the Poincaré duality chain equivalence \( \mathcal{H}^1_p(C(g'))^{n+2i-1} \rightarrow \mathcal{H}^1_p(C(g')) \). We shall now arrange CW structures for \((e,a)\) in such a way that only cells in dimensions \( i+1 \) occur in the relevant pairs and 1.9 applies to obtain the \( \mathbb{Z}[\pi_1(E)] \)-module chain homotopy data in the total spaces of the pullbacks from \( \mathcal{F} \rightarrow \mathcal{P} \rightarrow \mathcal{B} \) as the algebraic transfers of \( \mathbb{Z}[\pi_1(B)] \)-module data.

\( L \) is the trace of surgeries on \((i-1)-\) and \(i\)-spheres in \( M \), so that \((L,N)\) has a relative CW structure with \(i\)- and \((i+1)\)-cells, with the cellular chain complex in \( B(\mathbb{Z}[\pi_1(B)]) \) given by

\[
d = j : C(\tilde{L}, \tilde{M})_i+1 = K_i(M) \rightarrow C(\tilde{L}, \tilde{M})_i = K_i(L).
\]

Replacing \( e:L \rightarrow XX(0,1) \) by the inclusion of \( L \) in the mapping cylinder it may be assumed that \( L \) is a subcomplex of \( X \), such that \((X,L)\) and \((X,M)\) have cellular chain complexes in \( B(\mathbb{Z}[\pi_1(B)]) \).

\[
C(\tilde{X}, \tilde{L}) = S^{i+1}K_i(L),
\]

\[
d = (j, 1) : C(\tilde{X}, \tilde{M})_i+1 = K_i(M) \otimes K_i(L).
\]
The kernel chain complexes $C(t')$, $C(e')$ are equivalent to $S^{-1}C(\bar{X}, M)$, $S^{-1}C(\bar{X}, \bar{L})$ respectively, under the inclusion $C(t') \to C(e')$ by the equivalence of the pullbacks to replacing by $S^{-1}C(\bar{X}, \bar{N}) \to S^{-1}C(\bar{Y}, \tilde{Y}, \tilde{P})$, and by L.

Thus up to chain homotopy the inclusion $C(zg') \to C(h')$ may be identified with the $\mathbb{Z}[\pi_1(E)]$-module chain map

$$p^\theta(j) : C(g') = S^1p^\theta(K_1(M)) \to C(h') = S^1p^\theta(K_1(L)).$$

Similarly, up to chain homotopy $C(zg') \to C(h')$ may be identified with

$$p^\theta(k) : C(zg') = S^1p^\theta(K_1(M)) \to C(h') = S^1p^\theta(K_1(L)).$$

The $\mathbb{Z}[\pi_1(B)]$-module isomorphism $j-k : K_1(M) \to K_1(L)$ such that $(j-k)^{-1} = \bar{\omega}_\lambda : K_1(M) \to K_1(M)$ lifts to the chain homotopy class of a $\mathbb{Z}[\pi_1(E)]$-module chain equivalence

$$p^\theta(j-k) = p^\theta(j) - p^\theta(k) : C(g') \to C(h')$$

such that up to chain homotopy

$$p^\theta(j-k)^{-1}p^\theta(j) = p^\theta(\bar{\omega}_\lambda) : C(g') \to \mathbb{Z}[\pi_1(F)] \mathbb{Z}[\pi_1(F)]$$

Applying 8.1 and 7.3 we have that the quadratic kernel $\sigma_*(g,c)$ is homotopy equivalent to the multidimensional quadratic Poincaré complex $(\tilde{F}, \eta)$ over $\mathbb{Z}[\pi_1(E)]$ with

$$(1+t)\eta_0 : C(g')n+2i-1 = \mathbb{Z}[\pi_1(F)]n-1 \quad \mathbb{Z}[\pi_1(F)]n-1 \mathbb{Z}[\pi_1(F)]n-1$$

This completes the proof of Theorem 6.2 in the case $m=2i+1$, and we proceed to the case $m=2i+1$.

By Chapter 6 of Wall [22] every element $x \in L_{2i+1}(\mathbb{Z}[\pi_1(B)])$ $(i \geq 2)$ is the Witt class of the kernel nonsingular $(-)^2$-quadratic formation over $\mathbb{Z}[\pi_1(B)]$

$$(F,G) = (K_{i+1}(U, \partial U), K_{i+1}(M_0, \partial U))$$

of an $(i+1)$-connected $(2i+1)$-dimensional normal map $(f,b):(M, \partial M) \to (X, \partial X)$ with $\partial f: \partial M \to \partial X$ a homotopy equivalence, and with a $\pi_1$-isomorphism reference map $X \to B$ such that $w(X): \pi_1(X) \to \pi_1(B)$ $w(B)$. Here, $U$ is the connected sum of a sufficiently large number $k>0$ of framed embeddings $S^1C(M)$ with nullhomotopies in $X$ to generate the f.g. $\mathbb{Z}[\pi_1(B)]$-module $K_1(M)$, and $M_0$ is the closure $(M-U)$. Thus $F=K_{i+1}(U, \partial U)$ is a based f.g. free $\mathbb{Z}[\pi_1(B)]$-module, and $G=K_{i+1}(M_0, \partial U)$ is a based f.g. free...
Lagrangian of the hyperbolic \((-1)^i\)-quadratic
form
\[ H_{(-)}(F) = (\mathbb{F} \otimes \mathbb{F}^*) \]
with
\[ F \ast G = \otimes \mathbb{Z}[\pi_i(B)]. \]

The inclusion \([\mu], G \to F \otimes \mathbb{F}^*\) extends to an isomorphism of hyperbolic \((-1)^i\)-quadratic forms
\[ \left\{ \begin{array}{c} \gamma \quad \gamma \\ \mu \\ \mu \end{array} \right\} : H_{(-)}(G) \longrightarrow H_{(-)}(F). \]

Surgery on the framed embedded \(i\)-spheres in \(U\), defines an \((i+1)\)-connected \((2i+2)\)-dimensional normal map
\[ ((e,f,f'),(a;b,b')) : (\mathbb{L}^{2i+2}, \mathbb{M}^{2i+1}, \mathbb{M}, 2i+1) \]
with \((F^*, G)\) the kernel nonsingular \((-1)^i\)-quadratic form of \((f', b')\), and
\[ G = K_{i+1}(L), \quad F = K_{i+1}(L, M^*), \quad F^* = K_{i+1}(L, M), \]
\[ C(e^i) = S^{i+1} G, \quad C(e^i, f') = S^{i+1} F, \]
\[ C(e^i, f') = S^{i+1} F^*. \]

The quadratic kernel \(\sigma^*(\mathbb{E}; f', (a;b,b'))\) is an \((i+1)\)-connected \((2i+2)\)-dimensional quadratic Poincaré
homotopy equivalent to the \(i\)-connected \((2i+2)\)-dimensional quadratic complex \((S^{i+1} G^*, \Theta)\)
with algebraic Thom complex
\[ (G, \Theta) \] such that \( \gamma \ast \mu = \Theta \ast (-1)^{i+1} \Theta^* : G \to G^*. \] The base elements of the \(F\)-free \(\mathbb{Z}[\pi_i(B)]\)-module \(G = \otimes \mathbb{Z}[\pi_i(B)]\) can be represented by immersed \((i+1)\)-spheres in \(\text{int}(L^{2i+2})\) with nullhomotopies in \(X\), so that the form \((G, \Theta)\) can be
\[ \left\{ \begin{array}{c} \gamma \quad \gamma \\ \mu \\ \mu \end{array} \right\} : (\mathbb{F} \otimes \mathbb{F}^*) \longrightarrow H_{(-)}(\mathbb{F} \otimes \mathbb{F}^*), \]

in terms of geometric intersection and
of geometric intersection numbers exactly as in Chapter 5 of

(22).

The pullback of \(((e,f,f'),(a;b,b'))\) from \(F \to B\)
the reference map \(X \to B\) is an \((i+1)\)-connected
map of \((n+2i+2)\)-dimensional geometric Poincaré
triads
\[ ((h; g, g'), (d; c, c')) : (p^{n+2i+2}; n^{n+2i+1}, n, n+2i+1) \]

\[ \longrightarrow \text{XX}([0,1]; (0), (1)). \]

the \(\otimes \mathbb{Z}[\pi_i(E)]\)-module chain maps \(C(h^i) \longrightarrow C(h^i, g^i)\),
\[ (\otimes \mathbb{Z}[\pi_i(E)])^{-1} p^\theta(\mu) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]
with chain homotopy by
\[ p^\theta(\gamma) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]

\[ (\otimes \mathbb{Z}[\pi_i(E)])^{-1} p^\theta(\mu) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]

\[ \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta(\mu) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]

\[ \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta(\mu) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]

\[ \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta(\mu) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]

\[ \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta(\mu) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]

\[ \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta(\mu) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]

\[ \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta(\mu) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]

\[ \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta(\mu) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]

\[ \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta(\mu) : C(h^i) = S^{i+1} p^\theta G = \otimes \mathbb{Z}[\pi_i(E)]^{-1} p^\theta F^* \]
The inclusion of a lagrangian of the \((-)^i\)-quadratic hyperbolic form $H^{(-)^i}(p^\theta F)$ in $D_n(z_\pi(E))$ becomes an isomorphism of \((-)^i\)-quadratic forms as:

$$
\begin{align*}
(p^\theta)_!(\hat{\gamma}) & \mapsto (\Theta[F]\hat{\gamma}_!^{-1})^{-1}p^\theta(\hat{\mu})(\Theta[F]\hat{\gamma}_!^{-1})^{-1} \\
(p^\theta)_!(\hat{\gamma}) & \mapsto (\Theta[F]\hat{\gamma}_!^{-1})^{-1}p^\theta(\hat{\mu})(\Theta[F]\hat{\gamma}_!^{-1})^{-1}
\end{align*}
$$

\[ H^{(-)^i}(p^\theta G) \cong H^{(-)^i}(p^\theta F). \]

Working as in the proof of Lemma 8.1, the hessians of \((-)^{i+1}\)-quadratic forms in $D_n(z_\pi(E))$ may be expressed as $(p^\theta G, (\Theta[F]^{-1})^{-1} p^\theta(\hat{\theta}))$, with $\Theta: G \to G^{(-)^{i+1}}$.

Proposition 7.4: The nonsingular \((-)^i\)-quadratic form $(p^\theta F, p^\theta G)$ in $D_n(z_\pi(E))$ is such that

$$
p^!_{\text{geo}}(f,b) = \sigma_{\text{geo}}(g,c) = \mu(p^\theta F, p^\theta G)
$$

where $\sigma_{\text{geo}}$ satisfies the conditions of Proposition 7.4.

The dual algebraic K-theory.

We now extend the definition of the algebraic K-theory to the category of $R$-modules, and show that they are compatible with the Rothenberg exact sequences.

An involution $R \to R^*$ on a ring $R$ determines a duality involution $*: P(R) \to P(R)$, $x \mapsto x^*$ on the additive category $P(R)$ of f.g. projective $R$-modules by

$$
R \times P \to P^*; (r,f) \mapsto (x \mapsto f(x), r),
$$

and $e(P): P \to P^{**}; x \mapsto (f \mapsto f(x))$.

The duality involution on $P(R)$ determines involutions on the algebraic K-groups

$$
*: K_0(R) \to K_0(R); [P] \mapsto [P]^*,
$$

$$
*: K_1(R) \to K_1(R);
$$

and also on the reduced K-groups

$$
\tilde{K}_1(R) = \text{coker}(K_1(z_\pi) \to K_1(R)) (i=0,1).
$$

The intermediate quadratic L-groups $L^X(R)$ of a ring with involution $R$ are defined for $\ast$-invariant subgruops $X \subseteq K^*_i(R)$, such that $x \ast X$ for all $x \in X$. The intermediate L-groups for $X = \{0\}$, $\tilde{K}_1(R)$ are written as

$$
\begin{align*}
\tilde{K}_0(R) & \to \tilde{K}_1(R) & (0) \subseteq & \tilde{K}_1(R) \\
L^0(R) & = L^0(R) & L^0(R) & = L^0(R) \\
L^0(R) & = L^0(R) & L^0(R) & = L^0(R)
\end{align*}
$$

We now write the surgery transfer maps unambiguously as

$$
p^! : L_m(Z_\pi(\ast)) \to L_{m+n}(Z_\pi(E)) (m \geq 0).
$$
For \( \ast \)-invariant subgroups \( X \subseteq \tilde{X}'_1(R) \) there is defined a Rothenberg exact sequence

\[
\cdots \rightarrow \tilde{L}^X_n(R) \rightarrow \tilde{L}^X_n(R) \rightarrow \hat{H}^n(\tilde{Z}_2; X'/X) \rightarrow \cdots
\]

with

\[
\hat{H}^n(\tilde{Z}_2; X'/X) =
\]

\[(a \in X'/X | a \ast = (-)^n a) / \langle b + (-)^n b \ast | b \in X' \rangle \]

See Ranicki [13, 14] for further details.

We consider first the torsion case \( X \subseteq \tilde{X}'_1(R) \).

A representation \( (C, U) \) of \( R \) in \( D(S) \) determines a transfer map in the absolute torsion group \( K^*_1(R) \rightarrow K^*_1(S) \) (Example 1.8), and also in the reduced torsion groups \( (C, U)^! : K^*_1(R) \rightarrow K^*_1(S) \). In the definition, \( D(S) \) is the homotopy category of finite chain complexes of based f.g. free \( S \)-modules. We shall now make use of these bases.

**Proposition 9.1** Let \( (C, U, \psi) \) be a symmetric representation of \( R \) in \( D_1(S) \), for some rings \( R, S \).

1) For any \( \ast \)-invariant subgroups \( X \subseteq \tilde{X}'_1(R), \tilde{Y} \subseteq \tilde{X}'_1(S) \), such that \( (C, U)^!(X) \subseteq \tilde{Y} \), and \( \tau(a : C \rightarrow \tilde{C} \ast) \) there are defined transfer maps in the intermediate torsion L-groups

\[ (C, U)^! : \tilde{L}_m^X(R) \rightarrow \tilde{L}^Y_{m+n}(S) \quad (n \geq 0) . \]

2) For any \( \ast \)-invariant subgroups \( X \subseteq \tilde{X}'_1(R), \tilde{Y} \subseteq \tilde{X}'_1(S) \), such that \( (C, U)^!(X) \subseteq \tilde{Y} \), \( (C, U)^!(X') \subseteq \tilde{Y}' \), \( \tau(a : C \rightarrow \tilde{C} \ast) \) there is defined a morphism of Rothenberg exact sequences

\[
I^X_m(R) \rightarrow L^X_m(R) \rightarrow \hat{H}^m(\tilde{Z}_2; X'/X) \rightarrow \cdots
\]

\[
1^Y_m(R) \rightarrow L^Y_m(R) \rightarrow \hat{H}^m(\tilde{Z}_2; Y'/Y) \rightarrow \cdots
\]

**Proof:** The transfer map in the reduced torsion group \( (C, U)^! : K^*_1(R) \rightarrow K^*_1(S) \) is such that

\[
*(C, U)^! = (-)^n (C, U)^! : K^*_1(R) \rightarrow K^*_1(S)
\]

Let \( m = 2i \). For any nonsingular \( (-)^i \)-quadratic form \( (M, \psi) \) on a based f.g. free \( R \)-module \( M \otimes R \) the \( n \)-dimensional \( (-)^i \)-quadratic Poincare complex \( (\otimes C, \psi) \) representing \( (C, U)^!(M, \psi) \) has reduced torsion

\[
\tau((1+T)\theta_0 : \tilde{C} \otimes \tilde{C} \ast) \rightarrow \tilde{K}_1(S)
\]

\[
= (C, U)^! \tau(\psi(-)^i \psi : M \rightarrow M) \subseteq \tilde{K}_1(S)
\]

the image of \( \tau(\psi(-)^i \psi) \tilde{E}_1^C(R) \). Similarly for \( m = 2i + 1 \) and formations.

Next, we consider the projective case \( X \subseteq \tilde{X}'_0(R) \). It is more convenient to work with the preimage of \( X \) is more convenient to work with the preimage of \( X \) in \( E(S) \), the homotopy category of finite-dimensional f.g. projective \( S \)-modules, and only for the case of projective S-modules. A representation \( (C, U) \) of a ring \( R \) in \( E(S) \) determines transfer maps in the algebraic K-groups

\[ (C, U)^! : K^*_1(R) = K^*_1(S) \]
\[ K_i(S) = K_i(F(S)) \quad (i = 0, 1) \]

(Example 1.8). For \( n \geq 0 \) let \( \mathbb{E}_n(S) = D_n(F(S)) \), the projective \( n \)-dimensional \( f.g. \) modules determine the \( n \)-duality chain complexes. An involution \( C \to C^* \) on \( \mathbb{E}_n(S) \)

**Proposition 9.2** Let \((C, \alpha, U)\) be a symmetric representation of \( R \) in \( \mathbb{E}_n(S) \), for some rings \( S \).

i) For any \( \ast \)-invariant subgroups \( X \in K_0(R) \), \( Y \in K_0(S) \) such that \([R] \in \mathbb{E}_n \), \([S] \in \mathbb{E}_n \), \((C, U)(X) \in \mathbb{E}_n \) there are defined transfers maps in the intermediate class L-groups

\[
(C, \alpha, U) : L^X_m(R) \to L^Y_{m+n}(S) \quad (n \geq 0).
\]

ii) For any \( \ast \)-invariant subgroups \( X \in K_0(R) \), \( Y \in K_0(S) \) such that \([R] \in \mathbb{E}_n \), \([S] \in \mathbb{E}_n \), \((C, U)(X) \in \mathbb{E}_n \) there is defined a morphism of Rothenberg exact sequences

\[
\[ L^X_m(R) \to L^X_{m+n}(S) \to L^Y_{m+n}(S) \to L^X_{m+n}(S) \to \cdots \]
\]

The proof of 9.2 is somewhat more involved than that of 9.1.

A splitting \((B, r, i)\) in \( A \) of an object \((A, p)\) in the idempotent completion \( \hat{A} \) is an object \( B \) in \( A \) together with morphisms \( r : A \to B \), \( i : B \to A \) in \( A \) such that

\[
ri = 1 : B \to B, \quad ir = p : A \to A
\]

**Lemma 9.3** A functor of additive categories \( F : A \to B \)

An additive category \( A \) is idempotent complete if the functor \( \hat{A} \to A \) is an equivalence of categories. Applying 9.3 to \( 1 : A \to A \) we have that \( A \) is idempotent complete if and only if every object \((A, p)\) in \( A \) splits in \( A \). If \( B \) is idempotent complete every functor \( F : A \to B \) extends to a functor \( \hat{F} : \hat{A} \to \hat{B} \), namely the composite of \( \hat{F} : \hat{A} \to \hat{B} \) and an equivalence \( \hat{B} \to B \).

For any ring \( S \) the additive category \( F(S) \) of \( f.g. \) projective \( S \)-modules is idempotent complete, with every object \((A, p)\) in \( F(S) \) split by the triple \((B, r, i)\) defined by

\[
r : A \to B = \text{im}(p) ; \quad x \to p(x),
\]

\[
i = \text{inclusion} : B \to A.
\]

This is the special case \( n=0 \) of:

**Lemma 9.4** For any ring \( S \) and any \( n \geq 0 \) the homotopy category \( E_n(S) \) of \( n \)-dimensional \( f.g. \) projective \( S \)-module chain complexes is idempotent complete.

**Proof:** For every chain homotopy projection \( p \in \mathbb{P}_2(D) \to \) of an object \( D \) in \( E_n(S) \) there exists by Lemma 3.4 of Lück [7] an \((n+1)\)-dimensional infinitely generated
Projective $S$-module chain complex $C$ with chain homotopies $\partial:C \to D$, $\iota:D \to C$ and chain homotopies $\alpha, \beta$.

Since $C$ is dominated by an object in $E_n(S)$ (namely Proposition 3.1 of Ranicki [17]),

The idempotent completion of an additive category $\mathbb{A}$ with an involution $*:\mathbb{A} \to \mathbb{A}$ is an additive category $\mathbb{A}$ with the involution $*:\mathbb{A} \to \mathbb{A}$.

For a ring $R$ with involution $*$ the functor $\mathbb{A} \to \mathbb{A}$ is an equivalence of additive categories with involution. Both 9.3 and 9.5 have evident versions for additive categories with involution.

**Definition 9.5** Let $R, S$ be rings with involution. The projective surgery transfer maps of a symmetric representation $(C, d, U)$ of $R$ in $E_n(S)$ are defined by

$$(C, d, U)^{!} : L^{p}(R) \to L^{p}(S) \quad (m \geq 0)$$

are the composites

$$(C, d, U)^{!} : L^{p}(R) = L^{p}(F(R)) \xrightarrow{\sim} L_{m}(E_{n}(S)) \xrightarrow{\mu} L_{m}(S) \to L_{m+n}(S)$$

with $\mu$ the generalized Morita maps of 3.3 for $\mathbb{A} = P(S)$ with involution $F: P(R) \to P(S)$ associated to the functor $F: \mathbb{B}(R) \to \mathbb{E}_{n}(S)$. The proof of 9.2 is now completed by observing that the transfer map in the projective class groups $[ (C, d, U)^{!}] : K_{0}(R) \to K_{0}(S)$ is such that $[ (C, d, U)^{!}] = (-)^{n}(C, d, U)^{*} : K_{0}(R) \to K_{0}(S)$.

**Remark 9.6** Our methods also apply to construct algebraic surgery transfer maps in the round $L$-group $L_{m+n}^{+}(R)$ of Hambleton, Ranicki and Taylor [4], which are defined for $*$-invariant subgroups $X_{K_{1}(R)}$. For an $*$-invariant subgroup $X_{K_{1}(R)}$, $Y_{K_{1}(S)}$ such that $Y_{K_{1}(S)}$ there are defined round $L$-theory transfer maps

$$(C, d, U)^{!} : L_{m}^{+}(R) \to L_{m+n}^{+}(S) \quad (m \geq 0)$$

which are compatible with the round $L$-theory Rothenberger exact sequences.

**Remark 9.7** The connection established in §8 between the algebraic and geometric surgery transfer maps extends to the intermediate cases, and also to round $L$-theory.

**Remark 9.8** Our algebraic constructions apply also to the $e$-quadratic $L$-groups $L_{e}(R, e)$, which are defined for a ring with involution $R$ and a central unit $e$ in $R$ such that $e^{2} = 1$. $L_{2}^{+}(R, e)$ (resp. $L_{2}^{+}(R, e)$) is the Witt group of nonsingular $(-)^{e}$-quadratic forms (resp. formations) over $R$. A symmetric representation $(C, d, U)$ of $R$ in $E_{n}(S)$ such that $U(c) = \sigma_{C}^{-1}C \to C$ for a central unit
\[ \eta \in \mathbb{S} \text{ with } \eta \eta = 1 \text{ induces transfer maps } \]
\[ (C, \alpha, \nu) : L_m(\mathbb{R}, \mathbb{C}) \longrightarrow L_{m+n}(S, \eta) \quad (m \geq 0). \]

Hitherto we considered the case \( \epsilon = 1 \in \mathbb{R} \) for \( L_*(\mathbb{R}, 1) = L_*(\mathbb{R}) \), with \( \eta = 1 \in \mathbb{S} \).

**Appendix 1. Fibred intersections**

The proof of \( \rho \geq \rho \text{Alg} \) in §8 makes heavy use the algebraic properties of the \( L \)-groups. For a fibred bundle \( F \rightarrow E \rightarrow B \) with the fibre \( F \) a compact \( n \)-dimensional manifold it is possible to verify the algebraic and geometric surgery transfer matrices coincide more directly, using the bordism intersection theory of Hatcher and Quinn [6] to obtain fibrations of the geometric intersection forms (representations) used by Wall [22] to define the surgery obstruction of a highly-connected even (resp. odd) dimensional normal map. The quadratic kernel of the pullback normal map is the fibred intersection if (resp. formation) both algebraically and geometrically.

We now sketch the argument for the intersection pair \( \lambda \) in the even-dimensional case, leaving the self-intersection function \( \mu \) and the odd-dimensional case to the interested reader.

Given two maps \( v_1 : Q_1 \longrightarrow M \) (i=1,2) let \( E(v_1, v_2) \) the pointed space of triples \( (x_1, x_2, \omega) \) defined by \( x_1, x_2 \in Q_i \) and a path \( \omega : [0, 1] \longrightarrow M \) from \( \omega(0) = v_1(x_1) \), \( \omega(1) = v_2(x_2) \), so that there is defined a homotopy fibre square

\[
\begin{array}{ccc}
E(v_1, v_2) & \longrightarrow & Q_1 \\
\downarrow & & \downarrow \\
Q_2 & \longrightarrow & M.
\end{array}
\]

Now suppose that \( M \) is an \( m \)-manifold, and that \( v_1 : Q_1 \longrightarrow M \) is an immersion of a \( q_1 \)-manifold \( Q_1 \) (i=1,2) such that \( v_1(Q_1) \) intersects \( v_2(Q_2) \) in general position. Hence \( E(v_1, v_2) \) denotes the corresponding \( \Omega Q_1 \Omega Q_2 \).

Let \( E(v_1, v_2) \) be the \( (6, 2, 1) \)-dimensional submanifold of \( M \). The bordism \( \lambda(v_1, v_2) = [Q_1 \Omega Q_2] \).

If \( Q_1 \) and \( Q_2 \) are \( (q_1 + q_2 - m) \)-connected, the map \( \tilde{E}(v_1, v_2) \) induces an isomorphism \( (6, 3, 1) \) of bordism classes

\[
\lambda(v_1, v_2) = [Q_1 \Omega Q_2] \quad \tilde{E}(v_1, v_2) \quad \lambda(v_1, v_2) \quad \tilde{E}(v_1, v_2) \quad \lambda(v_1, v_2).
\]

which is used as an identification.

Let \( (f, b) : M \longrightarrow X \) be an \((i-1)\)-connected 2i-dimensional normal map with a \( \pi_1 \)-isomorphism \( f \) and the surgery obstruction \( \sigma_0(f, b) = (K_i(M), \lambda, \mu) \in L_{2i+2}(Z[\pi_1(B)]) \) as defined in Chapter 5 of Wall [22]. Let \( v_1, v_2, \ldots, v_k \) be a base of the kernel of free \( Z[\pi_1(B)] \)-module \( K_i(M) = \pi_{i+1}(f) \)

Represent each \( v_j \in K_i(M) \) by a pointed framed immersion \( v_j : S^i \longrightarrow M \) with a null homotopy in \( X \). The values taken by the \((-)^l\)-symmetric form \( (K_i(M), \lambda) \) on the base elements are just the bordism intersections.
Now let \((g, c): N \to Y\) be the \((i-1)\)-connected reference map \(Y \to X\) obtained from \((f, b): M \to X\) by pullback of the fibre bundle \(F \to E \to B\) along \(X \to B\). Poincaré duality isomorphisms in \(X\) lift to pointed framed immersions \(w_j: S^i \to X\) with nullhomotopies in \(Y\). On the chain level this corresponds to lifting the kernel \(Z[\pi_1(B)]\)-module chain complex \(C(\ell) = S^i K_i(M) \otimes S^i Z[\pi_1(B)]\) to the kernel \(Z[\pi_1(E)]\)-module chain complex

\[
C(g) = S^i C(\ell) = S^i C(\ell) = \bigoplus_{k} S^i \cdot \bigoplus_{k} S^i C(\ell)
\]

under the geometric bordism transfer map

\[
\lambda = \bigoplus_{k} S^i \cdot \bigoplus_{k} S^i C(\ell)
\]

are the images of the bordism intersections\(\lambda(w_j, w_j) \in \Omega^0_e(E(w_j, w_j), S^1 X \otimes S^1 X \otimes S^1 Y)\) under the geometric bordism transfer map

\[
p! = -X F : \Omega^0_f(\Omega M) \to \Omega^0_{fr}(\Omega MXF, Y_F)
\]

The Poincaré duality isomorphism of based f.g. free \(Z[\pi_1(B)]\)-modules\(\lambda(w_j, w_j) \in \Omega^0_f(\Omega M) \to \Omega^0_f(\Omega MXF, Y_F)\) is lifted to the Poincaré duality chain equivalence of the complexes of based f.g. free \(Z[\pi_1(E)]\)-modules

\[
\lambda(w_j, w_j) : C(\ell) = \bigoplus_{k} S^i C(\ell) = \bigoplus_{k} S^i C(\ell)
\]

The anticlockwise composition gives the geometric surgery transfer \(p_{geo}\) on the level of intersections, while the clockwise composition gives the algebraic surgery transfer \(p_{alg}\).

Appendix 2. A counterexample in symmetric L-theory

An \(n\)-dimensional Poincaré fibration \(F \to E \to B\) does not in general induce transfer maps in the symmetric L-groups \(\ell : L^n(Z[\pi_1(B)]) \to L^{n+m}(Z[\pi_1(E)])\), either algebraically or geometrically. It is not possible to define \(p_!\) geometrically since the symmetric L-groups are not geometrically realizable (Ranicki [16, 7.6.8]). There are two obstructions to an algebraic definition of \(p_!\), which requires the lifting of an \(n\)-dimensional symmetric Poincaré complex \((C, \gamma)\) over \(Z[\pi_1(B)]\) representing an element \((C, \gamma) \in L^n(Z[\pi_1(B)])\) to an \((m+n)\)-dimensional symmetric Poincaré complex \((C', \gamma')\) over \(Z[\pi_1(E)]\) representing the putative transfer \(p_!(C, \gamma) = (C', \gamma') \in L^{m+n}(Z[\pi_1(E)])\). The symmetric L-groups are not 4-periodic, so it cannot be assumed that \((C, \gamma) \to (C', \gamma')\).
is highly-connected as in the quadratic case. In the finite or infinite, and that the chain complex C consists of free module \( \mathbb{Z}[\pi_1(B)] \)-modules. The two obstruction liftings \((C,\phi)\) to \((C',\phi')\) are given by:

1) it may not be possible to lift \( C \) to a bi-filter filtration \( F_0 C' \subseteq C' \subseteq \cdots \subseteq F_m C' = C' \) with connecting chain maps between successive quotients are given up to chain homotopy by

\[
\sigma = p^\theta(d_C) : F_r C' / F_{r-1} C' = S^r p^\theta(C_r) \quad \sigma(S(F_r-1 C' / F_{r-2} C')) = S^r p^\theta(C_r-1) \quad (1 \leq r \leq m)
\]

where \( S^r \) denotes the \( r \)-fold dimension shift and \( p^\theta \) the functor of \( \mathbb{Z}[\pi_1(B)] \)

\[
p^\theta = \varnothing(C(F),C) : \mathbb{B}(\mathbb{Z}[\pi_1(B)]) \rightarrow \mathbb{B}(\mathbb{Z}[\pi_1(E)])
\]

2) even if \( C' \) exists, it may not be possible to lift the \( n \)-dimensional symmetric Poincare structure \( \phi' \) on \( C \) to an \( (n+m) \)-dimensional symmetric Poincare structure \( \phi' \) on \( C' \).

If \( C \) can be assembled over \( B \) in the sense of Ranich and Weiss [20] then it can be lifted to \( C' \), but in general it is not possible to assemble \( \mathbb{Z}[\pi_1(B)] \)-module chain complexes, so already 1) presents a non-trivial obstruction to the existence of transfer in symmetric L-theory. Even if the obstruction of 1) vanishes (e.g., if \( B \) is an Eilenberg-MacLane space \( K(\pi_1(B),1) \)) then it may present a non-trivial obstruction. This is illustrated by the following example, which exhibits the failure of a projection of rings with involution \( \varphi : S \rightarrow \mathbb{S} \) to \( \mathbb{S} \) into \( \mathbb{S} \) analogous to the \( S^1 \)-bundle quadratic L-theory transfer map \( p^1 : L^0(\mathbb{S}) \rightarrow L(S) \) (cf. 4.7). The transfer is not defined for the \( n \)-dimensional symmetric Poincare complex \( (C,\phi) = (R^m,\phi) \) over \( R \), for although \( C \) can be lifted to \( C' \) and \( \phi' \) there does not exist a symmetric \( \phi' \). Both the obstruction to 1) and 2) vanish for the visible symmetric L-groups \( VL^m(\mathbb{Z}[\pi_1(B)]) \) of Weiss [23] provided that \( B \) is an Eilenberg-MacLane space \( K(\pi_1(B),1) \), in which case there are defined transfer maps

\[
p^1 : VL^m(\mathbb{Z}[\pi_1(B)]) \rightarrow VL^{m+n}(\mathbb{Z}[\pi_1(E)])
\]

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SOME REMARKS ON THE KIRBY-SIEBENMANN CLASS
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In this note we study the relations that hold between the Kirby-Siebenmann class $[K S] \in H^4(B_{STOP}; Z/2)$ and the first Pontrjagin class.

The first result is that the natural map $p_0: B_{STOP} \to B_{SG}$ does not detect $[K S]$, no matter what coefficients might be used. However, the homology dual of $(K S)$ in the image of the Hurewicz map $\pi_4(B_{STOP}) \to H_4(B_{STOP}; Z/2)$.

In fact there is a unique non-zero element $\{K S\} \in \pi_4(B_{STOP})$ of order 2, and $p_0(\{K S\})$ is non-zero. To understand this we introduce an intermediate classifying space, $B_TSG$ on which we have a factorization

$$p_0 = p \cdot f: B_{STOP} \to B_TSG \to B_{SG}.$$  

$B_{SG}$ is universal for the vanishing of transversality obstructions through dimension 4, and the Kirby-Siebenmann class maps to $B_{TSG}$ 24 times the second generator. Thus, this transversality theory does detect $(K S)$. But note also the $Z/48$. Our main question is the extent to which it gives rise to a fiber homotopy invariant of topological $R^4$-bundles. The general result is

Theorem I: Let $\xi, \psi$ be two stable $R^4$-bundes over $X$, and suppose they are fiber homotopy equivalent. Then there is $a \in H^4(X; Z/2)$ and

$$24a^2 + P_0(\xi) + 24\{K S(\xi)\} = P_0(\psi) + 24\{K S(\psi)\}$$

in $H^4(X; Z/48)$ where $P_0(\xi)$ is the $Z/48$ reduction of the first Pontrjagin class.

In other words, there is an element $a \in H^4(B_{TSG}; Z/48)$ with $f^*(a) = P_0 + 24\{K S\}$ and (I) gives the effect of different liftings of a map $p_0 g: X \to B_{STOP} \to B_{SG}$ on $H^4(B_{STOP}; Z/2) = Z/2$ with generator $w_2$, so the possible factorizations through $B_{TSG}$ differ in their effect on $a$ only by $24w_2$. In particular this gives

Corollary: If $M^4$ is a compact closed topological manifold with even index, and $\nu$ is a stable normal bundle, then $w_2 = 0 \in H^2(M; Z/2)$ and

$$f^*(a) = P_0(\nu) + 24\{K S(\nu)\}$$

is independent of the choice of $f$ factoring $p_0$.

This note came about in answer to a question of Frank Quinn. He pointed out that, in $[M - M]$, the exact structure of $B_{STOP}$, and the various surgery maps in dimension 4 would never work out. But currently it appears very useful to understand them. Of course, do not attempt to work out explicit geometric methods for evaluating the new invariants. But knowing what they are and how they fit together should make that fairly direct.