The flow space associated to a CAT(0)-space (Lecture IV)

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We introduce **CAT(0)-spaces** and **CAT(0)-groups** and state their main properties.

We construct the **flow space** $FS(X)$ associated to a CAT(0)-space and collect its main properties.

We discuss the main **flow estimate**.
A CAT(0)-space or Hadamard space is a geodesic complete metric space \((X, d_X)\) such that any geodesic triangle \(\Delta\) in \(X\) satisfies the CAT(0)-inequality

\[
d_X(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y})
\]

for all \(x, y \in \Delta\) and all comparison points \(\bar{x}, \bar{y}\) in the comparison triangle \(\bar{\Delta} \subseteq \mathbb{R}^2\).

A metric space \(X\) is called geodesic if for any two points \(x, y \in X\) there exists a geodesic segment joining \(x\) and \(y\), i.e., an isometric embedding

\[
c : [0, d_X(x, y)] \rightarrow X \text{ with } c(0) = x \text{ and } c(d_X(x, y)) = y.
\]
A geodesic triangle $\Delta$ in $X$ consists of three points $p, q, r$ and a choice of geodesic segments $[p, q]$, $[p, r]$ and $[q, r]$.

A comparison triangle $\bar{\Delta}$ for geodesic triangle $\Delta$ is a geodesic triangle $\bar{\Delta} \subseteq \mathbb{R}^2$ given by three points $\bar{p}, \bar{q}$ and $\bar{r}$ such that $d_X(p, q) = d_{\mathbb{R}^2}(\bar{p}, \bar{q})$, $d_X(p, r) = d_{\mathbb{R}^2}(\bar{p}, \bar{r})$, and $d_X(q, r) = d_{\mathbb{R}^2}(\bar{q}, \bar{r})$.

If $x$ belongs to the segment $[p, q]$, then its comparison point $\bar{x}$ is the point on the geodesic $[\bar{p}, \bar{q}]$ uniquely determined by $d_X(p, x) = d_{\mathbb{R}^2}(\bar{p}, \bar{x})$ and $d_X(x, q) = d_{\mathbb{R}^2}(\bar{x}, \bar{q})$.

A simply connected complete Riemannian manifold with non-positive sectional curvature is a CAT(0)-space.
There is a unique geodesic segment joining each pair of points and this geodesic segment varies continuously with its endpoints.

If $X$ is a CAT(0)-space, then $X$ and every open ball and every closed ball in $X$ are contractible.

**Definition (Generalized geodesic)**

Let $(X, d_X)$ be a metric space. A continuous map $c : \mathbb{R} \to X$ is called a **generalized geodesic** if there are $c_-, c_+ \in \overline{\mathbb{R}} := \mathbb{R} \bigcup \{-\infty, \infty\}$ satisfying

$$c_- \leq c_+, \quad c_- \neq \infty, \quad c_+ \neq -\infty,$$

such that $c$ is locally constant on the complement of the interval $l_c := (c_-, c_+)$ and restricts to an isometry on $l_c$. 
Definition (Boundary of a metric space)

Let $X$ be a metric space. Two geodesic rays $c, c' : [0, \infty) \to X$ are called asymptotic if there exists a constant $K$ with $d_X(c(t), c'(t)) \leq K$ for all $t \in [0, \infty)$. The boundary $\partial X$ of $X$ is the set of asymptotic equivalence classes of rays. Denote by $\overline{X} = X \sqcup \partial X$ the disjoint union of $X$ and $\partial X$.

Lemma

Let $X$ be a CAT(0)-space and $c : [0, \infty) \to X$ be a geodesic ray. Then for every $x' \in X$ there is a unique geodesic ray $c' : [0, \infty) \to X$ with $c'(0) = x'$ such that $c$ and $c'$ are asymptotic.

In contrast to hyperbolic spaces it is in general not true for a CAT(0)-space that for two distinct elements $y, z \in \partial X$ there exists a geodesic $c : \mathbb{R} \to X$ joining $y$ and $z$. 
A generalized geodesic ray is a generalized geodesic $c$ that is either a constant generalized geodesic or a non-constant generalized geodesic with $c_\infty = 0$.

Fix a base point $x_0 \in X$ in the CAT($0$)-space $X$. For every $x \in X$, there is a unique generalized geodesic ray $c_x$ such that $c(0) = x_0$ and $c(\infty) = x$. Define for $r > 0$ the canonical projection

$$\rho_r = \rho_{r,x_0} : \overline{X} \to \overline{B}_r(x_0)$$

by $\rho_r(x) := c_x(r)$.

**Definition (Cone topology on $\overline{X}$.)**

Let $X$ be a CAT($0$)-space. The sets $(\rho_r)^{-1}(V)$ with $r > 0$, $V$ an open subset of $\overline{B}_r(x_0)$ are a basis for the cone topology on $\overline{X}$. 

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The cone topology is independent of the choice of base point.

A map $f$ whose target is $\overline{X}$ is continuous if and only if $\rho_r \circ f$ is continuous for all $r$.

$\overline{X}$ is a compact metrizable space.

$\partial X \subseteq \overline{X}$ is closed and $X \subseteq \overline{X}$ is dense.

The inclusion $X \to \overline{X}$ is a homeomorphism onto its image which is an open subset.

If $M$ is a simply connected complete $n$-dimensional Riemannian manifold with non-positive sectional curvature, then $\partial M$ is $S^{n-1}$.

There are closed topological manifolds $M$ constructed by Davis-Januszkiewicz (1991) such that the universal covering $\tilde{M}$ admits a $\pi_1(M)$-invariant CAT(0)-metric and $\partial \tilde{M}$ is not homeomorphic to a sphere and $\tilde{M}$ is not homeomorphic to $\mathbb{R}^n$. 
CAT(0)-groups

**Definition (CAT(0)-group)**

A (discrete) group $G$ is called a **CAT(0)-group** if it acts properly cocompactly and isometrically on a CAT(0)-space of finite topological dimension.

A CAT(0)-group $G$ satisfies:

- There exists a finite model $EG$.
- There is a model for $BG$ of finite type;
- $G$ is finitely presented;
- There are only finitely many conjugacy classes of finite subgroups;
- Every solvable subgroup is virtually $\mathbb{Z}^n$;
- The direct product of two CAT(0)-groups is again a CAT(0)-group;
Limit groups in the sense of Sela are CAT(0)-groups;
Coxeter groups are CAT(0)-groups;
The word-problem and the conjugation-problem are solvable.

Question

Is every hyperbolic group a CAT(0)-group?
The flow space of a metric space

Throughout this section let \((X, d_X)\) be a metric space.

**Definition (Flow space)**

- Let \(FS = FS(X)\) be the set of all generalized geodesics in \(X\);
- We define a metric on \(FS(X)\) by

\[
d_{FS(X)}(c, d) := \int_{\mathbb{R}} \frac{d_X(c(t), d(t))}{2e^{|t|}} \, dt.
\]

- Define a flow

\[
\Phi : FS(X) \times \mathbb{R} \to FS(X)
\]

by \(\Phi_\tau(c)(t) = c(t + \tau)\) for \(\tau \in \mathbb{R}, \ c \in FS(X)\) and \(t \in \mathbb{R}\).
Lemma

The map $\Phi$ is a continuous flow and we have for $c, d \in FS(X)$ and $\tau, \sigma \in \mathbb{R}$

$$d_{FS(X)}(\Phi_\tau(c), \Phi_\sigma(d)) \leq e^{|\tau|} \cdot d_{FS(X)}(c, d) + |\sigma - \tau|.$$ 

Proof:

- We estimate for $c \in FS(X)$ and $\tau \in \mathbb{R}$:

  $$d_{FS(X)}(c, \Phi_\tau(c)) = \int_{\mathbb{R}} \frac{d_X(c(t), c(t + \tau))}{2e^{|t|}} dt$$

  $$\leq \int_{\mathbb{R}} \frac{|\tau|}{2e^{|t|}} dt$$

  $$= |\tau| \cdot \int_{\mathbb{R}} \frac{1}{2e^{|t|}} dt$$

  $$= |\tau|. $$
We estimate for \( c, d \in \text{FS}(X) \) and \( \tau \in \mathbb{R} \)

\[
\begin{align*}
\text{d}_{\text{FS}(X)}(\Phi_{\tau}(c), \Phi_{\tau}(d)) &= \int_{\mathbb{R}} \frac{d_{X}(c(t + \tau), d(t + \tau))}{2e^{t}} \, dt \\
&= \int_{\mathbb{R}} \frac{d_{X}(c(t), d(t))}{2e^{|t-\tau|}} \, dt \\
&\leq \int_{\mathbb{R}} \frac{d_{X}(c(t), d(t))}{2e^{|t-|\tau||}} \, dt \\
&= e^{\tau} \cdot \int_{\mathbb{R}} \frac{d_{X}(c(t), d(t))}{2e^{t}} \, dt \\
&= e^{\tau} \cdot \text{d}_{\text{FS}(X)}(c, d).
\end{align*}
\]
The two inequalities above together with the triangle inequality imply for \( c, d \in \text{FS}(X) \) and \( \tau, \sigma \in \mathbb{R} \)

\[
\begin{align*}
\text{d}_{\text{FS}(X)}\left( \Phi_{\tau}(c), \Phi_{\sigma}(d) \right) & = \text{d}_{\text{FS}(X)}\left( \Phi_{\tau}(c), \Phi_{\sigma - \tau} \circ \Phi_{\tau}(d) \right) \\
& \leq \text{d}_{\text{FS}(X)}\left( \Phi_{\tau}(c), \Phi_{\tau}(d) \right) + \text{d}_{\text{FS}(X)}\left( \Phi_{\tau}(d), \Phi_{\sigma - \tau} \circ \Phi_{\tau}(d) \right) \\
& \leq e^{|\tau|} \cdot \text{d}_{\text{FS}(X)}(c, d) + |\sigma - \tau|.
\end{align*}
\]
Lemma

Let \( c, d : \mathbb{R} \to X \) be generalized geodesics. Consider \( t_0 \in \mathbb{R} \).

- \( d_X(c(t_0), d(t_0)) \leq e^{|t_0|} \cdot d_{FS}(c, d) + 2 \);
- If \( d_{FS}(c, d) \leq 2e^{-|t_0|-1} \), then
  \[
  d_X(c(t_0), d(t_0)) \leq \sqrt{4e^{|t_0|+1}} \cdot \sqrt{d_{FS}(c, d)}.
  \]

In particular, \( c \mapsto c(t_0) \) defines a uniform continuous map \( FS(X) \to X \).

Proof of the first assertion

- We abbreviate \( D := d_X(c(t_0), d(t_0)) \).
- We get
  \[
  d_X(c(t), d(t)) \geq D - d_X(c(t_0), c(t)) - d_X(d(t_0), d(t)) \geq D - 2 \cdot |t - t_0|.
  \]
- This implies
\[ d_{FS(X)}(c, d') = \int_{-\infty}^{+\infty} \frac{d_X(c(t), d(t))}{2e^{|t|}} \, dt \]

\[ \geq \int_{-D/2+ t_0}^{D/2+ t_0} \frac{D - 2 \cdot |t - t_0|}{2e^{|t|}} \, dt \]

\[ = \int_{-D/2}^{D/2} \frac{D - 2 \cdot |t|}{2e^{|t|+ t_0|}} \, dt \]

\[ \geq \int_{-D/2}^{D/2} \frac{D - 2 \cdot |t|}{2e^{|t|+|t_0|}} \, dt \]

\[ = e^{-|t_0|} \cdot \int_{-D/2}^{D/2} \frac{D - 2 \cdot |t|}{2e^{|t|}} \, dt \]

\[ = e^{-|t_0|} \cdot \left(2 \cdot e^{-D/2} + D - 2\right) \]

\[ \geq e^{-|t_0|} \cdot (D - 2). \]
Lemma

The maps

\[ \text{FS}(X) - \text{FS}(X)^{\mathbb{R}} \to \mathbb{R}, \quad c \mapsto c_-; \]
\[ \text{FS}(X) - \text{FS}(X)^{\mathbb{R}} \to \mathbb{R}, \quad c \mapsto c_+, \]

are continuous.

Lemma

Let \((c_n)_{n \in \mathbb{N}}\) be a sequence in \(\text{FS}(X)\). Then it converges uniformly on compact subsets to \(c \in \text{FS}(X)\) if and only if it converges to \(c\) with respect to \(d_{\text{FS}(X)}\).
Lemma

The flow space $FS(X)$ is sequentially closed in the space of all maps $\mathbb{R} \to X$ with respect to the topology of uniform convergence on compact subsets.

Definition (Proper metric space)

A metric space is called proper if every closed ball is compact.

Lemma

If $(X, d_X)$ is a proper metric space, then $(FS(X), d_{FS(X)})$ is a proper metric space.
Proof:

- Let $R > 0$ and $c \in \text{FS}(X)$.
- It suffices to show that the closed ball $\overline{B}_R(c)$ in $\text{FS}(X)$ is sequentially compact.
- Let $(c_n)_n \in \mathbb{N}$ be a sequence in $\overline{B}_R(c)$. There is $R' > 0$ such that $c_n(0) \in \overline{B}_{R'}(c(0))$. By assumption $\overline{B}_{R'}(c(0))$ is compact.
- Now we can apply the Arzelà-Ascoli Theorem.
- Thus after passing to a subsequence there is $d : \mathbb{R} \to X$ such that $c_n \to d$ uniformly on compact subsets.
Lemma

Let \((X, d_X)\) be a proper metric space and \(t_0 \in \mathbb{R}\). Then the evaluation map \(FS(X) \rightarrow X\) defined by \(c \mapsto c(t_0)\) is uniformly continuous and proper.

Proof:

- We have already shown that the map is uniformly continuous.
- To show that is is also proper, it suffices to show that preimages of closed balls have finite diameter.
- If \(d_X(c(t_0), d(t_0)) \leq r\), then \(d_X(c(t), d(t)) \leq r + 2|t - t_0|\). Thus

\[
    d_{FS}(c, d) \leq \int_{\mathbb{R}} \frac{r + 2|t - t_0|}{2e^{|t|}} \, dt,
\]

provided \(d_X(c(t_0), d(t_0)) \leq r\). 

\[\square\]
Lemma

Let $G$ act isometrically, properly and cocompactly on the proper metric space $(X, d_X)$. Then action of $G$ on $(FS(X), d_{FS})$ is also isometric, proper and cocompact.

Proof:

- The action of $G$ on $FS(X)$ is isometric.
- The map $FS(X) \to X$ defined by $c \mapsto c(0)$ is $G$-equivariant, continuous and proper.
- The existence of such a map implies that the $G$-action on $FS(X)$ is also proper and cocompact.

Lemma

The subspace $FS(X)\mathbb{R}$ is closed in $FS(X)$.
Let $X$ be a metric space. For $c \in \text{FS}(X)$ and $T \in [0, \infty]$, define $c|_{[-T,T]} \in \text{FS}(X)$ by

$$c|_{[-T,T]}(t) := \begin{cases} 
  c(-T) & \text{if } t \leq -T; \\
  c(t) & \text{if } -T \leq t \leq T; \\
  c(T) & \text{if } t \geq T.
\end{cases}$$

We denote by

$$\text{FS}(X)_f := \{ c \in \text{FS}(X) - \text{FS}(X)^\mathbb{R} \mid c_- > -\infty, c_+ < \infty \} \cup \text{FS}(X)^\mathbb{R}$$

the subspace of finite geodesics.

**Lemma**

The map

$$H: \text{FS}(X) \times [0, 1] \rightarrow \text{FS}(X)$$

defined by $H_\tau(c) := c|_{[\ln(\tau), -\ln(\tau)]}$ is continuous and satisfies $H_0 = \text{id}_{\text{FS}(X)}$ and $H_\tau(c) \in \text{FS}(X)_f$ for $\tau > 0$. 
The flow space of a CAT(0)-space

Example (Flow space of a manifold of non-positive sectional curvature)

- Let $M$ be a simply connected complete Riemannian manifold of non-positive sectional curvature.
- Recall that $M$ is a CAT(0)-space.
- Put

$$P := \{(a_-, a_+) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid a_- < \infty, a_+ > -\infty, a_- \leq a_+\};$$

$$\Delta = \{(a, a) \in \overline{\mathbb{R}} \times \overline{\mathbb{R}} \mid -\infty < a < \infty\}.$$

- Define maps

$$f: STM \times P \rightarrow \text{FS}(M), \quad (v, a_-, a_+) \mapsto c(v)[a_-, a_+];$$

$$p: STM \times \Delta \rightarrow M, \quad (v, a) \mapsto c_v(a),$$

where $c(v): \mathbb{R} \rightarrow M$ is the geodesic determined by $v$. 
Example (continued)

- The map \( f \) is compatible with the obvious flows.
- Then we obtain pushout

\[
\begin{align*}
STM \times \Delta & \xrightarrow{i} STM \times P \\
p & \downarrow \quad \downarrow f \\
M & \xrightarrow{j} FS(M)
\end{align*}
\]

where \( i \) is the inclusion and \( j : M \to FS(X) \) sends \( x \) to \( \text{const}_x \).

- In particular \( f \) induces a homeomorphism

\[
STM \times (P - \Delta) \xrightarrow{\mathbb{R}} FS(M) - FS(M) \oplus \mathbb{R}
\]
Definition (End points of a geodesic)

For \( c \in \text{FS}(X) \) we define \( c(\infty) \in \overline{X} \) by

\[
\lim_{t \to \infty} c(t) = \begin{cases} 
  c(c_+) & \text{if } c_+ < \infty; \\
  [c|_{[0,\infty)}] & \text{if } c_+ = \infty.
\end{cases}
\]

Define \( c(-\infty) \) analogously.

Lemma

The maps

\[
\begin{align*}
\text{FS}(X) - \text{FS}(X)^{\mathbb{R}} & \to \overline{X}, \quad c \mapsto c(-\infty); \\
\text{FS}(X) - \text{FS}(X)^{\mathbb{R}} & \to \overline{X}, \quad c \mapsto c(\infty),
\end{align*}
\]

are continuous.
The two maps appearing above cannot be continuously extended to $\text{FS}(X)$ by the following observation.

Let $c$ be a generalized geodesic with $c_+ < \infty$ and $c_- = \infty$. Then

$$c(\infty) \neq c(-\infty);$$

$$d_{\text{FS}}(c, \text{const}_c(\infty)) \leq e^{c_+}/2;$$

$$\lim_{\tau \to \infty} \Phi_\tau(c) = \text{const}_c(\infty);$$

$$\left( \lim_{\tau \to \infty} \Phi_\tau(c) \right)(-\infty) = c(\infty);$$

$$\Phi_\tau(c)(-\infty) = c(-\infty) \quad \text{for all } \tau > 0;$$

$$\lim_{\tau \to \infty} \left( \Phi_\tau(c)(-\infty) \right) = c(-\infty).$$
**Theorem (Embedding the flow space)**

If $X$ is proper as a metric space, then the map

$$E : \text{FS}(X) - \text{FS}(X)^{\mathbb{R}} \to \mathbb{R} \times \overline{X} \times X \times \overline{X} \times \overline{X}$$

defined by $E(c) := (c_-, c(-\infty), c(0), c(\infty), c_+)$ is injective and continuous. It is a homeomorphism onto its image.

**Lemma**

If $X$ is proper as a metric space and its covering dimension $\dim X$ is $\leq N$, then $\dim \overline{X} \leq N$.

**Proof:**

- Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of $\overline{X}$.
- For every $x \in \overline{X}$ there are $r_x$, $W_x \subseteq \overline{B}_{r_x}(x_0)$ and $U_x \in \mathcal{U}$ such that $x \in \rho_{r_x}^{-1}(W_x) \subset U_x$.
- Since $\overline{X}$ is compact, a finite number of the sets $\rho_{r_x}^{-1}(W_x)$ cover $\overline{X}$.
- Note that $\rho_r = \rho_r|_{\overline{B}_{r'}(x_0)} \circ \rho_{r'}$ and hence $\rho_{r_x}^{-1}(W) = \rho_{r_x}^{-1}(\rho_r|_{\overline{B}_{r'}(x_0)}(W))$ if $r' > r$. 

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Theorem (Dimension of the flow space)

Assume that $X$ is proper and that $\dim X \leq N$. Then

$$\dim (FS(X) - FS(X)^R) \leq 3N + 2.$$ 

Proof:

- Every compact subset $K$ of $FS(X) - FS(X)^R$ is homeomorphic to a compact subset of $\overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R}$.

- Hence its topological dimension satisfies

  $$\dim(K) \leq \dim(\overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R})$$

  $$= 2 \dim(\overline{R}) + 2 \dim(\overline{X}) + \dim(X) \leq 3N + 2.$$ 

- One shows that $FS(X) - FS(X)^R$ has a countable basis for its topology.

- Now $\dim(FS(X) - FS(X)^R) \leq 3N + 2$ follows from standard result of dimension theory.
The homotopy action on $B_r(x_0)$

- The $G$-action on $X$ induces an $G$-action on $X$.
- For technical reasons we will not take the space $\overline{X}$ as the space appearing in the axiomatic approach as we have done it for hyperbolic groups. We will take the closed ball $B_R(x_0)$ for some base point $x_0$ and some very large real number $R$.
- The prize to pay is that we do not obtain a $G$-action on $B_R(x_0)$ but at least the following homotopy $G$-action.
**Definition (The homotopy $G$-action on $\overline{B}_R(x_0)$)**

Define a homotopy $G$-action $(\varphi^R, H^R)$ on $\overline{B}_R(x)$ as follows.

- For $g \in G$, we define the map
  \[ \varphi^R_g : \overline{B}_R(x_0) \to \overline{B}_R(x_0) \]
  by $\varphi^R_g(x) := \rho_{R,x_0}(gx)$.

- For $g, h \in G$ we define the homotopy
  \[ H^R_{g,h} : \varphi^R_g \circ \varphi^R_h \simeq \varphi^R_{gh} \]
  by $H^R_{g,h}(x, t) := \rho_{R,x_0}(t \cdot (ghx) + (1 - t) \cdot (g \cdot \rho_{R,x_0}(hx)))$. 
\[ \varphi^R_g(x) \]

\[ B_R(x_0) \]

\[ x \]

\[ gx \]

\[ x_0 \]
\[
\begin{align*}
    y & := \rho_{R, x_0}(h x) \\
    H_{g, h}^R(x, t) & = gy \\
    \overline{B}_R(x_0) & \\
    x & \\
    x_0 &
\end{align*}
\]
It turns out that the more obvious homotopy given by convex combination \((x, t) \mapsto t \cdot \varphi_{gh}^R(x) + (1 - t) \cdot \varphi_g^R \circ \varphi_h^R(x)\) is not appropriate for our purposes.

Notice that \(H^R_{g,h}\) is indeed a homotopy from \(\varphi_g^R \circ \varphi_h^R\) to \(\varphi_{gh}\) since

\[
H^R_{g,h}(x, 0) = \rho_{R,x_0}(0 \cdot (ghx) + 1 \cdot (g \cdot \rho_{R,x_0}(hx))) = \rho_{R,x_0}(g \cdot \rho_{R,x_0}(hx)) = \varphi_g^R \circ \varphi_h^R(x),
\]

and

\[
H^R_{g,h}(x, 1) = \rho_{R,x_0}(1 \cdot (ghx) + 0 \cdot (g \cdot \rho_{R,x_0}(hx))) = \rho_{R,x_0}(ghx) = \varphi_{gh}^R(x).
\]
The map \( \iota \)

**Definition (The map \( \iota \))**

Define the map

\[
\iota : G \times X \rightarrow \text{FS}(X)
\]

by sending \((g, x) \in G \times X\) to the generalized geodesic \(c_{g x_0, g x}\) from \(g x_0\) to \(g x\).
The flow estimate

**Theorem (The flow estimate)**

Let $\beta, L > 0$. For all $\delta > 0$ there are $T, r > 0$ with the following property:

For $x_1, x_2 \in X$ with $d_X(x_1, x_2) \leq \beta$, $x \in \overline{B}_{r+L}(x_1)$ there is $\tau \in [-\beta, \beta]$ such that

$$d_{FS}(\Phi_T(c_{x_1, \rho_r, x_1}(x)), \Phi_{T+\tau}(c_{x_2, \rho_r, x_2}(x))) \leq \delta.$$