We briefly explain homology theories and how they arise from spectra.

We state the Farrell-Jones-Conjecture and the Baum-Connes Conjecture for torsionfree groups.

We discuss applications of these conjectures such as the Kaplansky Conjecture, Novikov Conjecture and the Borel Conjecture.

We explain that the formulations for torsionfree groups cannot extend to arbitrary groups and state the general versions.

We give a report about the status of the Farrell-Jones Conjecture.
A homology theory $\mathcal{H}_*$ is a covariant functor from the category of $CW$-pairs to the category of $\mathbb{Z}$-graded abelian groups together with natural transformations

$$\partial_n(X, A): \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- Homotopy invariance
- Long exact sequence of a pair
- Excision

If $(X, A)$ is a $CW$-pair and $f: A \to B$ is a cellular map, then

$$\mathcal{H}_n(X, A) \cong \mathcal{H}_n(X \cup_f B, B).$$
Homology theory

Definition (Homology theory)

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Definition (continued)

- **Disjoint union axiom**

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\bigoplus_{i \in I} \mathcal{H}_n(X_i) \xrightarrow{\cong} \mathcal{H}_n\left( \bigsqcup_{i \in I} X_i \right).
\]

If the CW-complex \( X \) is the union of two subcomplexes \( X_1 \) and \( X_2 \) and we put \( X_0 = X_1 \cap X_2 \), then there is a long exact Mayer-Vietoris sequence

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Theorem (Homology theories and spectra)

Let $E$ be a spectrum. Then we obtain a homology theory $H_*(-; E)$ by

$$H_n(X, A; E) := \pi_n ((X \cup_A \text{cone}(A)) \wedge E).$$

It satisfies

$$H_n(pt; E) = \pi_n(E).$$

Any homology theory arises in this way.

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Any homology theory arises in this way.

- The following conjectures are motivated by computations which reveal a homological flavour of $K$ and $L$-theory of group rings.
The Baum-Connes Conjecture for the torsionfree group predicts that the assembly map

\[ K_n(BG) \to K_n(C^*_r(G)) \]

is bijective for all \( n \in \mathbb{Z} \).

- \( BG \) is the classifying space of the group \( G \).
- \( K_n(BG) \) is the topological \( K \)-homology of \( BG \).
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- There is also a real version of the Baum-Connes Conjecture

\[ KO_n(BG) \to K_n(C^*_r(G; \mathbb{R})) \].
The Isomorphism Conjectures for torsionfree groups

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Conjecture (\textit{K-theoretic Farrell-Jones Conjecture for torsion-free groups})

The \textit{K-theoretic Farrell-Jones Conjecture} with coefficients in the regular ring $R$ for the torsion-free group $G$ predicts that the assembly map

$$H_n(BG; K_R) \rightarrow K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $K_n(RG)$ is the algebraic $K$-theory of the group ring $RG$;
- $K_R$ is the (non-connective) algebraic $K$-theory spectrum of $R$;
- $H_n(pt; K_R) \cong \pi_n(K_R) \cong K_n(R)$ for $n \in \mathbb{Z}$. 
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Conjecture \((L\text{-theoretic Farrell-Jones Conjecture for torsionfree groups})\)

The \(L\text{-theoretic Farrell-Jones Conjecture}\) with coefficients in the ring with involution \(R\) for the torsionfree group \(G\) predicts that the assembly map

\[
H_n(BG; L_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(RG)
\]

is bijective for all \(n \in \mathbb{Z}\).

- \(L_n^{\langle -\infty \rangle}(RG)\) is the algebraic \(L\)-theory of \(RG\) with decoration \(\langle -\infty \rangle\);
- \(L_R^{\langle -\infty \rangle}\) is the algebraic \(L\)-theory spectrum of \(R\) with decoration \(\langle -\infty \rangle\);
- \(H_n(pt; L_R^{\langle -\infty \rangle}) \cong \pi_n(L_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(R)\) for \(n \in \mathbb{Z}\).
Conjecture (*L-theoretic Farrell-Jones Conjecture for torsionfree groups*)

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is bijective for all $n \in \mathbb{Z}$.

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- $L^{\langle -\infty \rangle}_R$ is the algebraic $L$-theory spectrum of $R$ with decoration $\langle -\infty \rangle$;
- $H_n(pt; L^{\langle -\infty \rangle}_R) \cong \pi_n(L^{\langle -\infty \rangle}_R) \cong L_n^{\langle -\infty \rangle}(R)$ for $n \in \mathbb{Z}$. 

Wolfgang Lück (Bonn, Germany)
Conjecture (\textit{L-theoretic Farrell-Jones Conjecture for torsionfree groups})

The \textit{L-theoretic Farrell-Jones Conjecture} with coefficients in the ring with involution \( R \) for the torsionfree group \( G \) predicts that the \textit{assembly map}

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\begin{itemize}
  \item \( L_n^{\langle -\infty \rangle}(RG) \) is the algebraic \( L \)-theory of \( RG \) with decoration \( \langle -\infty \rangle \);
  \item \( L_R^{\langle -\infty \rangle} \) is the algebraic \( L \)-theory spectrum of \( R \) with decoration \( \langle -\infty \rangle \);
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Consequences of the Isomorphism Conjectures for torsionfree groups

- Let \( \mathcal{FJ}_K(R) \) and \( \mathcal{FJ}_L(R) \) respectively be the class of groups which satisfy the \( K \)-theoretic and \( L \)-theoretic respectively Farrell-Jones Conjecture for the coefficient ring \( R \).
- Let \( BC \) be the class of groups which satisfy the Baum-Connes Conjecture.
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Let \( \mathcal{BC} \) be the class of groups which satisfy the Baum-Connes Conjecture.
Lemma

Suppose that $R$ is a regular ring, $G$ is torsionfree and $G \in \mathcal{FJ}_K(R)$. Then

- $K_n(RG) = 0$ for $n \leq -1$;
- The change of rings map $K_0(R) \to K_0(RG)$ is bijective. In particular $\tilde{K}_0(RG)$ is trivial if and only if $\tilde{K}_0(R)$ is trivial.

Lemma

Suppose that $G$ is torsionfree and $G \in \mathcal{FJ}_K(\mathbb{Z})$. Then the Whitehead group $\text{Wh}(G)$ is trivial.

Proof.

The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; K_R)$ whose $E^2$-term is given by

$$E_{p,q}^2 = H_p(BG, K_q(R)).$$
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In particular we get for a torsionfree group $G \in \mathcal{FJ}_K(\mathbb{Z})$:

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\text{Wh}(G) = 0$;
- Every finitely dominated $CW$-complex $X$ with $G = \pi_1(X)$ is homotopy equivalent to a finite $CW$-complex;
- Every compact $h$-cobordism $W$ of dimension $\geq 6$ with $\pi_1(W) \cong G$ is trivial.
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Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group $G$ and an integral domain $R$ that $0$ and $1$ are the only idempotents in $RG$.

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture, Bartels-L.-Reich(2007))

Let $F$ be a field and let $G$ be a torsionfree group with $G \in \mathcal{FJ}_K(F)$. Then $0$ and $1$ are the only idempotents in $FG$. 
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Let $F$ be a field and let $G$ be a torsion-free group with $G \in \mathcal{FJ}_K(F)$. Then 0 and 1 are the only idempotents in $FG$. 
Proof.

Let \( p \) be an idempotent in \( FG \). We want to show \( p \in \{0, 1\} \).

Denote by \( \epsilon : FG \to F \) the augmentation homomorphism sending \( \sum_{g \in G} r_g \cdot g \) to \( \sum_{g \in G} r_g \). Obviously \( \epsilon(p) \in F \) is 0 or 1. Hence it suffices to show \( p = 0 \) under the assumption that \( \epsilon(p) = 0 \).

Let \( (p) \subseteq FG \) be the ideal generated by \( p \) which is a finitely generated projective \( FG \)-module.

Since \( G \in \mathcal{J}_K(F) \), we can conclude that

\[
i_* : K_0(F) \otimes \mathbb{Q} \to K_0(FG) \otimes \mathbb{Q}
\]

is surjective.

Hence we can find a finitely generated projective \( F \)-module \( P \) and integers \( k, m, n \geq 0 \) satisfying

\[
(p)^k \oplus FG^m \cong_{FG} i_*(P) \oplus FG^n.
\]
Proof (continued).

If we now apply \( i_* \circ \epsilon_* \) and use \( \epsilon \circ i = \text{id} \), \( i_* \circ \epsilon_*(FG^l) \cong FG^l \) and \( \epsilon(p) = 0 \) we obtain

\[
FG^m \cong i_*(P) \oplus FG^n.
\]

Inserting this in the first equation yields

\[
(p)^k \oplus i_*(P) \oplus FG^n \cong i_*(P) \oplus FG^n.
\]

Our assumptions on \( F \) and \( G \) imply that \( FG \) is stably finite, i.e., if \( A \) and \( B \) are square matrices over \( FG \) with \( AB = I \), then \( BA = I \).

This implies \( (p)^k = 0 \) and hence \( p = 0 \).
**Conjecture (Novikov Conjecture)**

The Novikov Conjecture for $G$ predicts for a closed oriented manifold $M$ together with a map $f: M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\text{sign}_x(M, f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of $(M, f)$, i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g: M_0 \to M_1$ and homotopy equivalence $f_i: M_i \to BG$ with $f_1 \circ g \simeq f_2$ we have

$$\text{sign}_x(M_0, f_0) = \text{sign}_x(M_1, f_1).$$

**Theorem (Baum-Connes Conjecture and the Farrell-Jones Conjecture imply the Novikov Conjecture)**

The Novikov Conjecture is true if the assembly map appearing in the Baum-Connes Conjecture or in the L-theoretic Farrell-Jones Conjecture are rationally injective.
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The Novikov Conjecture predicts for a homotopy equivalence \( f: M \to N \) of closed aspherical manifolds

\[ f_*(\mathcal{L}(M)) = \mathcal{L}(N). \]

This is surprising since this is not true in general and in many case one could detect that two specific closed homotopy equivalent manifolds cannot be diffeomorphic by the failure of this equality to be true.

A deep theorem of Novikov (1965) predicts that \( f_*(\mathcal{L}(M)) = \mathcal{L}(N) \) holds for a homeomorphism of closed manifolds.

Hence an explanation why the Novikov Conjecture may be true for closed aspherical manifolds is due to the next conjecture.
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Hence an explanation why the Novikov Conjecture may be true for closed aspherical manifolds is due to the next conjecture.
The Borel Conjecture for $G$ predicts for two closed aspherical manifolds $M$ and $N$ with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that $M$ and $N$ are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity.
  A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension $\geq 3$ is homotopic to an isometric diffeomorphism.

- The Borel Conjecture is not true in the smooth category by results of Farrell-Jones(1989).

- There are also non-aspherical manifolds which are topologically rigid in the sense of the Borel Conjecture (see Kreck-L. (2005)).
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If the $K$- and $L$-theoretic Farrell-Jones Conjecture hold for $G$ in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension $\geq 5$ and in dimension 4 if $G$ is good in the sense of Freedman.

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What happens for groups with torsion?

- The versions of the Farrell-Jones Conjecture and the Baum-Connes Conjecture above become false for finite groups unless the group is trivial.
- For instance the version of the Baum-Connes Conjecture above would predict for a finite group $G$
  \[ K_0(BG) \cong K_0(C^*_r(G)) \cong R_C(G). \]
  However, $K_0(BG) \otimes \mathbb{Q} \cong Q K_0(pt) \otimes \mathbb{Q} \cong Q$ and $R_C(G) \otimes \mathbb{Q} \cong Q$ holds if and only if $G$ is trivial.
- Next we formulate a general version.
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Definition (Family of subgroups)

A family $\mathcal{F}$ of subgroups of $G$ is a set of (closed) subgroups of $G$ which is closed under conjugation and finite intersections.

Examples for $\mathcal{F}$ are:

- $TR = \{\text{trivial subgroup}\}$;
- $FIN = \{\text{finite subgroups}\}$;
- $VCYC = \{\text{virtually cyclic subgroups}\}$;
- $ALL = \{\text{all subgroups}\}$.
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Definition (Classifying $G$-$CW$-complex for a family of subgroups)

Let $\mathcal{F}$ be a family of subgroups of $G$. A model for the classifying $G$-$CW$-complex for the family $\mathcal{F}$ is a $G$-$CW$-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any $G$-$CW$-complex $Y$, whose isotropy groups belong to $\mathcal{F}$, there is up to $G$-homotopy precisely one $G$-map $Y \to E_{\mathcal{F}}(G)$.

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Theorem \textbf{(Homotopy characterization of $E_{\mathcal{F}}(G)$)}

Let $\mathcal{F}$ be a family of subgroups.

- There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;
- Two models for $E_{\mathcal{F}}(G)$ are $G$-homotopy equivalent;
- A $G$-CW-complex $X$ is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the $H$-fixed point set $X^H$ is weakly contractible.
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A model for $E_{\text{ALL}}(G)$ is $G/G$;

$EG \to BG := G\backslash EG$ is the universal $G$-principal bundle for $G$-$CW$-complexes.

**Example (Infinite dihedral group)**

- Let $D_\infty = \mathbb{Z} \times \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$ be the infinite dihedral group.
- A model for $ED_\infty$ is the universal covering of $\mathbb{R}P^\infty \vee \mathbb{R}P^\infty$.
- A model for $ED_\infty$ is $\mathbb{R}$ with the obvious $D_\infty$-action.
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The general formulation of the Isomorphism Conjectures

Conjecture (K-theoretic Farrell-Jones-Conjecture)

The $K$-theoretic Farrell-Jones Conjecture with coefficients in $R$ for the group $G$ predicts that the assembly map

$$H_n^G(E_{\text{VCyc}}(G), K_R) \rightarrow H_n^G(pt, K_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- $H_n^G(\ast, K_R)$ is a $G$-homology theory defined for $G$-$CW$-complexes which satisfies $H_n^G(G/H, K_R) \cong K_n(RH)$ for all subgroups $H \subseteq G$;
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Conjecture (**L-theoretic Farrell-Jones-Conjecture**)

The **L-theoretic Farrell-Jones Conjecture** with coefficients in \( R \) for the group \( G \) predicts that the assembly map

\[
H_n^G \left( E_{\text{VCYC}}(G), L_R^{(-\infty)} \right) \rightarrow H_n^G (pt, L_R^{(-\infty)}) = L_n^{(-\infty)}(RG)
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is bijective for all \( n \in \mathbb{Z} \).

- \( H_n^G (\_ , L_R^{(-\infty)}) \) is a \( G \)-homology theory defined for \( G \)-CW-complexes which satisfies \( H_n^G (G/H, L_R^{(-\infty)}) \cong L_n^{(-\infty)}(RH) \) for all subgroups \( H \subseteq G \);
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Conjecture (*L*-theoretic Farrell-Jones-Conjecture)

The *L*-theoretic Farrell-Jones Conjecture with coefficients in *R* for the group *G* predicts that the assembly map

\[ H^n_G(E_{\mathcal{VCyc}}(G), \mathbb{L}_{\mathbb{R}}^{<-\infty}) \to H^n_G(pt, \mathbb{L}_{\mathbb{R}}^{<-\infty}) = \mathbb{L}_{\mathbb{R}}^{<-\infty}(RG) \]

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There are more general versions of the Farrell-Jones Conjecture, where one allows twisted coefficients which can actually be additive $G$-categories. In the sequel we refer to this general version.

**Theorem (Main Theorem)**


Let $\mathcal{FJ}$ be the class of groups for which both the $K$-theoretic and the $L$-theoretic Farrell-Jones Conjectures holds. It has the following properties:

- Hyperbolic groups belong to $\mathcal{FJ}$;
- If $G_1$ and $G_2$ belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 \ast G_2$ belong to $\mathcal{FJ}$;
- If $H$ is a subgroup of $G$ and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;
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- If $H$ is a subgroup of $G$ and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;
Theorem (continued)

- Let \( \{ G_i \mid i \in I \} \) be a directed system of groups (with not necessarily injective structure maps) such that \( G_i \in \mathcal{FJ} \) for \( i \in I \). Then \( \text{colim}_{i \in I} G_i \) belongs to \( \mathcal{FJ} \);
- CAT(0)-groups belong to \( \mathcal{FJ} \);
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- If \( R \) is a ring whose underlying abelian is finitely generated free, then \( \text{SL}_n(R) \) and \( \text{GL}_n(R) \) belong to \( \mathcal{FJ} \) for all \( n \geq 2 \);
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Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina).

There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.

One example is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients due to Higson-Lafforgue-Skandalis.

However, our results show that these groups do satisfy the Farrell-Jones Conjecture and hence also the other conjectures mentioned above.
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Davis-Januszkiewics have constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space. However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups and Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension \( \geq 5 \).

The Baum-Connes Conjecture is open for CAT(0)-groups, cocompact lattices in almost connected Lie groups and \( SL_n(\mathbb{Z}) \) for \( n \geq 3 \).
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Open problems

- What are candidates for groups or closed aspherical manifolds for which the conjectures due to Farrell-Jones, Novikov or Borel may be false?
- There are still many interesting groups for which the Farrell-Jones Conjecture is open.
- Examples are:
  - $\mathbb{Z}[1/p] \ltimes \mathbb{Z}$;
  - Solvable groups;
  - Amenable groups;
  - Mapping class groups;
  - $\text{Out}(F_n)$;
  - Thompson groups.
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There is an analogue of the Farrell-Jones Conjecture for the topological $K$-theory of group $C^*$-algebras, the Baum-Connes Conjecture. Can methods of proof be transferred from one setting to the other?

Question (Proofs)

How can one prove the Farrell-Jones Conjecture?
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**Question (Proofs)**

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