The stable Cannon Conjecture

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The main conjectures

**Definition (Finite Poincaré complex)**

A (connected) finite \( n \)-dimensional \( CW \)-complex \( X \) is a finite \( n \)-dimensional Poincaré complex if there is \([X] \in H_n(X; \mathbb{Z}^w)\) such that the induced \( \mathbb{Z}\pi \)-chain map

\[
- \cap [X] : C^{n-*}(\tilde{X}) \to C_*(\tilde{X})
\]

is a \( \mathbb{Z}\pi \)-chain homotopy equivalence.

**Theorem (Closed manifolds are Poincaré complexes)**

A closed \( n \)-dimensional manifold \( M \) is a finite \( n \)-dimensional Poincaré complex with \( w = w_1(X) \).
**Definition (Poincaré duality group)**

A Poincaré duality group $G$ of dimension $n$ is a finitely presented group satisfying:

1. $G$ is of type FP;
2. $H^i(G; \mathbb{Z}G) \cong\begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

**Theorem (Wall)**

*If $G$ is a $d$-dimensional Poincaré duality group for $d \geq 3$ and $\tilde{K}_0(\mathbb{Z}G) = 0$, then there is a model for $BG$ which is a finite Poincaré complex of dimension $d$.***
Corollary

If $M$ is a closed aspherical manifold of dimension $d$, then $\pi_1(X)$ is a $d$-dimensional Poincaré duality group.

Theorem (Hadamard)

If $M$ is a closed smooth Riemannian manifold whose section curvature is negative, then $\pi_1(M)$ is a torsionfree hyperbolic group with $\partial G = S^{n-1}$.

Theorem (Bieri-Eckmann, Linnell)

Every 2-dimensional Poincaré duality group is the fundamental group of a closed surface.
Conjecture (Gromov)

Let $G$ be a torsionfree hyperbolic group whose boundary is a sphere $S^{n-1}$. Then there is a closed aspherical manifold $M$ with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger)

Gromov’s Conjecture is true for $n \geq 6$.

Conjecture (Wall)

Every Poincaré duality group is the fundamental group of an aspherical closed manifold.

Conjecture (Cannon’s Conjecture in the torsionfree case)

A torsionfree hyperbolic group $G$ has $S^2$ as boundary if and only if it is the fundamental group of a closed hyperbolic 3-manifold.
**Theorem** *(Cannon-Cooper, Eskin-Fisher-Whyte, Kapovich-Leeb)*

A Poincaré duality group $G$ of dimension 3 is the fundamental group of an aspherical closed 3-manifold if and only if it is quasiisometric to the fundamental group of an aspherical closed 3-manifold.

- A closed 3-manifold is a **Seifert manifold** if it admits a finite covering $\overline{M} \to M$ such that there exists a $S^1$-principal bundle $S^1 \to \overline{M} \to S$ for some closed orientable surface $S$.

**Theorem** *(Bowditch)*

If a Poincaré duality group of dimension 3 contains an infinite normal cyclic subgroup, then it is the fundamental group of a closed Seifert 3-manifold.
Theorem (Bestvina)

Let $G$ be a hyperbolic 3-dimensional Poincaré duality group. Then its boundary is homeomorphic to $S^2$.

Theorem (Bestvina-Mess)

Let $G$ be an infinite torsionfree hyperbolic group which is prime, not infinite cyclic, and the fundamental group of a closed 3-manifold $M$. Then $M$ is hyperbolic and $G$ satisfies the Cannon’s Conjecture.

In order to prove the Cannon Conjecture, it suffices to show for a hyperbolic group $G$, whose boundary is $S^2$, that it is quasiisometric to the fundamental group of some aspherical closed 3-manifold.
Theorem

Let $G$ be the fundamental group of an aspherical oriented closed 3-manifold. Then $G$ satisfies:

- $G$ is residually finite and Hopfian.
- All its $L^2$-Betti numbers $b_n^{(2)}(G)$ vanish;
- Its deficiency is 0. In particular it possesses a presentation with the same number of generators and relations.
- Suppose that $M$ is hyperbolic. Then $G$ is virtually compact special and linear over $\mathbb{Z}$. It contains a subgroup of finite index $G'$ which can be written as an extension $1 \to \pi_1(S) \to G \to \mathbb{Z} \to 1$ for some closed orientable surface $S$.

Recall that any finitely presented groups occurs as the fundamental group of a closed $d$-dimensional smooth manifold for every $d \geq 4$. 
Theorem (Bestvina-Mess)

A torsionfree hyperbolic $G$ is a Poincaré duality group of dimension $n$ if and only if its boundary $\partial G$ and $S^{n-1}$ have the same Čech cohomology.

Theorem

If the boundary of a hyperbolic group contains an open subset homeomorphic to $\mathbb{R}^n$, then the boundary is homeomorphic to $S^n$. 

The main results

Theorem (Ferry-Lück-Weinberger, (preprint, 2018), Vanishing of the surgery obstruction)

Let $G$ be a hyperbolic 3-dimensional Poincaré duality group. Then there is a normal map of degree one (in the sense of surgery theory)

$$\begin{align*}
TM \oplus \mathbb{R}^a & \xrightarrow{\bar{f}} \xi \\
\downarrow & \\
M & \xrightarrow{f} BG
\end{align*}$$

satisfying

1. The space $BG$ is a finite 3-dimensional CW-complex;
2. The map $H_n(f, \mathbb{Z}) : H_n(M; \mathbb{Z}) \cong H_n(BG; \mathbb{Z})$ is bijective for all $n \geq 0$;
3. The simple algebraic surgery obstruction $\sigma(f, \bar{f}) \in L^s_3(\mathbb{Z}G)$ vanishes.
Let $G$ be a hyperbolic 3-dimensional Poincaré duality group. Let $N$ be any smooth, PL or topological manifold respectively which is closed and whose dimension is $\geq 2$.

Then there is a closed smooth, PL or topological manifold $M$ and a normal map of degree one

$$TM \oplus \mathbb{R}^a \xrightarrow{f} \xi \times TN$$

$$\downarrow \quad \downarrow$$

$$M \xrightarrow{f} BG \times N$$

such that the map $f$ is a simple homotopy equivalence.
Obviously the last two theorems follow from the stable Cannon Conjecture.

By the product formula for surgery theory the second last theorem implies the last theorem.

The manifold $M$ appearing in the last theorem is unique up to homeomorphism by the Borel Conjecture, provided that $\pi_1(N)$ satisfies the Farrell-Jones Conjecture.

If we take $N = T^k$ for some $k \geq 2$, then the Cannon Conjecture is equivalent to the statement that this $M$ is homeomorphic to $M' \times T^k$ for some closed 3-manifold $M'$. 
Let $X$ be a connected oriented finite 3-dimensional Poincaré complex. Then there are an integer $a \geq 0$ and a vector bundle $\xi$ over $BG$ and a normal map of degree one.
Proof.

Notice that by the Pontrjagin-Thom construction this claim is equivalent to the existence of a vector bundle reduction of the stable Spivak normal spherical fibration.

Recall that this is a \((k-1)\)-spherical fibration \(p: E \to X\) together with a map \(c: S^{n+k} \to \text{Th}(p)\) such that the Hurewicz homomorphism \(\pi_{n+k}(\text{Th}(p)) \to H_{n+k}(\text{Th}(p))\) sends \([c]\) to a generator of the infinite cyclic group \(H_{n+k}(\text{Th}(p))\).

Stable vector bundles over \(X\) are classified by the first and second Stiefel-Whitney class \(w_1(\xi)\) and \(w_2(\xi)\) in \(H^*(X; \mathbb{Z}/2)\).

Let \(\xi\) be a \(k\)-dimensional vector bundle over \(X\) such that \(w_1(\xi) = w_1(X)\) and \(w_2(\xi) = w_1(\xi) \cup w_1(\xi)\) holds.
Proof (continued).

- A spectral sequence argument applied to $\Omega_3(X, w_1(X))$ shows that there is a closed 3-manifold $M$ together with a map $f : M \to X$ of degree one such that $f^* w_1(X) = w_1(M)$.

- Then $w_1(f^* \xi) = w_1(M)$ and the Wu formula implies $w_2(M) = w_1(f^* \xi) \cup w_1(f^* \xi)$.

- Hence $f^* \xi$ is stably isomorphic to the stable tangent bundle of $M$ and hence there is a collaps map $c' : S^{3+k} \to \text{Th}(f^* \xi)$ such that the Hurewicz homomorphism $\pi_{n+k}(\text{Th}(f^* \xi)) \to H_{n+k}(\text{Th}(f^* \xi))$ sends $[c']$ to a generator of the infinite cyclic group $H_{n+k}(\text{Th}(p))$. 
Proof (continued).

Now define \( c := \text{Th}(\bar{f}) \circ c' \), where \((\bar{f}, f)\) is the bundle map from \( f^*\xi \) to \( \xi \) given by the pullback construction. Then the Hurewicz homomorphism \( \pi_{n+k}(\text{Th}(\xi)) \to H_{n+k}(\text{Th}(\xi)) \) sends \([c]\) to a generator of the infinite cyclic group \( H_{n+k}(\text{Th}(p)) \).

By the uniqueness of the stable Spivak fibration \( \xi \) is a vector bundle reduction of the Spivak normal fibration.
The total surgery obstruction

- We have to find one normal map of degree one

\[ TM \oplus \mathbb{R}^a \xrightarrow{\bar{f}} \xi \]
\[ \downarrow \quad \downarrow \]
\[ \tilde{M} \xrightarrow{f} X \]

whose simple surgery obstruction \( \sigma^s(f, \bar{f}) \in L^s_3(\mathbb{Z}G) \) vanishes.

- Consider an aspherical finite \( n \)-dimensional Poincaré complex \( X \) such that \( \pi_1(X) \) is a Farrell-Jones group, i.e., satisfies both the \( K \)-theoretic and the \( L \)-theoretic Farrell-Jones Conjecture with coefficients in additive categories, and \( \mathcal{N}(X) \) is non-empty. (For simplicity we assume \( w_1(X) = 0 \) in the sequel.)
Recall that the simple surgery obstruction defines a map

$$\sigma^s : \mathcal{N}(X) \to L_n^s(\mathbb{Z}G)$$

Fix a normal map $\left( f_0, \overline{f_0} \right)$.

Then there is a commutative diagram

\[
\begin{array}{ccc}
\mathcal{N}(X) & \xrightarrow{\sigma^s(-,-) - \sigma^s(f_0, \overline{f_0})} & L_n^s(\mathbb{Z}G) \\
\downarrow s_0 & & \downarrow \text{asmb}_n^s(X) \\
H_n(X; L^s_{\mathbb{Z}} \langle 1 \rangle) & \xrightarrow{H_n^G(\text{id}_X;i)} & H_n(X; L^s_{\mathbb{Z}}) \\
\end{array}
\]

whose vertical arrows are bijections thanks to the Farrell-Jones Conjecture and the upper arrow sends the class of $(f, \overline{f})$ to the difference $\sigma^s(f, \overline{f}) - \sigma^s(f, \overline{f_0})$ of simple surgery obstructions.
An easy spectral sequence argument yields a short exact sequence

\[ 0 \to H_n(X; L^s_{\mathbb{Z}} \langle 1 \rangle) \xrightarrow{H_n(\text{id}_X; i)} H_n(X; L^s_{\mathbb{Z}}) \xrightarrow{\lambda^s_n(X)} L_0(\mathbb{Z}). \]

Consider the composite

\[ \mu^s_n(X) : \mathcal{N}(X) \xrightarrow{\sigma^s} L^s_n(\mathbb{Z}G, w) \xrightarrow{\text{asmb}^s_n(X)^{-1}} H_n(X; L^s_{\mathbb{Z}}) \xrightarrow{\lambda^s_n(X)} L_0(\mathbb{Z}). \]
We conclude that there is precisely one element, called the total surgery obstruction,

\[ s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z} \]

such that for any element \([\bar{f}, \bar{f}]\) in \(\mathcal{N}(X)\) its image under \(\mu_n^s(X)\) is \(s(X)\).

**Theorem (Total surgery obstruction)**

- There exists a normal map of degree one \((\bar{f}, \bar{f})\) with target \(X\) and vanishing simple surgery obstruction \(\sigma^s(\bar{f}, \bar{f}) \in L_n^s(\mathbb{Z}G)\) if and only if \(s(X) \in L_0(\mathbb{Z}) \cong \mathbb{Z}\) vanishes.
- The total surgery obstruction is a homotopy invariant of \(X\) and hence depends only on \(G\).
Definition (Homology ANR-manifold)

A homology ANR-manifold $X$ is an ANR satisfying:

- $X$ has a countable basis for its topology;
- The topological dimension of $X$ is finite;
- $X$ is locally compact;
- for every $x \in X$ we have for the singular homology

\[
H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 
0 & i \neq n; \\
\mathbb{Z} & i = n.
\end{cases}
\]

If $X$ is additionally compact, it is called a closed ANR-homology manifold.
Every closed topological manifold is a closed ANR-homology manifold.

Let $M$ be homology sphere with non-trivial fundamental group. Then its suspension $\Sigma M$ is a closed ANR-homology manifold but not a topological manifold.
Quinn’s resolution obstruction

**Theorem (Quinn (1987))**

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:

- If $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;

- $\iota(M \times N) = \iota(M) \cdot \iota(N)$;

- Let $M$ be a homology ANR-manifold of dimension $\geq 5$. Then $M$ is a topological manifold if and only if $\iota(M) = 1$.

- The Quinn obstruction and the total surgery obstruction are related for an aspherical closed ANR-homology manifold $M$ of dimension $\geq 5$ by

  $$\iota(M) = 8 \cdot s(X) + 1.$$
Proof of the Theorem about the vanishing of the surgery obstruction

We have to show for the aspherical finite 3-dimensional Poincaré complex $X$ that its total surgery obstruction vanishes.

The total surgery obstruction satisfies a product formula

\[ s(X \times Y) = s(X) + s(Y). \]

This implies

\[ s(X \times T^3) = s(X). \]

Hence it suffices to show that $s(X \times T^3)$ vanishes.
Proof (continued).

- There exists an aspherical closed ANR-homology manifold $M$ and a homotopy equivalence to $f : M \to X \times T^3$.

- There is a $\mathbb{Z}$-compactification $\tilde{X}$ of $\tilde{X}$ by the boundary $\partial G = S^2$.

- One then constructs an appropriate $\mathbb{Z}$-compactification $\tilde{M}$ of $\tilde{M}$ so that we get a ANR-homology manifold $\tilde{M}$ whose boundary is a topological manifold and whose interior is $\tilde{M}$.

- By adding a collar to $\tilde{M}$ one obtains a ANR-homology manifold $Y$ which contains $\tilde{M}$ as an open subset and contains an open subset $U$ which is homeomorphic to $\mathbb{R}^6$. 
Proof (continued).

Hence we get

$$8s(X \times T^3) + 1 = 8s(M) + 1 = i(M) = i(\tilde{M})$$

$$= i(Y) = i(U) = i(\mathbb{R}^6) = 1.$$ 

This implies $s(X \times T^3) = 0$ and hence $s(X) = 0$. 