The Isomorphism Conjectures in the torsionfree case (Lecture II)

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Hangzhou, July 2007
We have introduced $K_n(R)$ for $n \in \mathbb{Z}, n \leq 1$.

We have discussed the topological relevance of $K_0(RG)$ and the Whitehead group $Wh(G)$, e.g., the finiteness obstruction and the $s$-cobordism theorem.

We have stated the conjectures that $\tilde{K}_0(\mathbb{Z}G)$ and $Wh(G)$ vanish for torsionfree $G$.

We have presented the Bass-Heller-Swan decomposition and indicated some similarities between $K_n(RG)$ and group homology.

Cliffhanger

Question (K-theory of group rings and group homology)

Is there a relation between $K_n(RG)$ and the group homology of $G$?
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Question (K-theory of group rings and group homology)

Is there a relation between $K_n(RG)$ and the group homology of $G$?
We introduce **spectra** and how they yield **homology theories**.

We state the **Farrell-Jones-Conjecture** and the **Baum-Connes Conjecture** for torsionfree groups.

We discuss applications of these conjectures such as the **Kaplansky Conjecture** and the **Borel Conjecture**.

We explain that the formulations for torsionfree groups cannot extend to arbitrary groups.
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Homology theories and spectra

**Definition (Spectrum)**

A *spectrum* 

\[ E = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\} \]

is a sequence of pointed spaces \( \{E(n) \mid n \in \mathbb{Z}\} \) together with pointed maps called *structure maps*

\[ \sigma(n) : E(n) \wedge S^1 \to E(n + 1). \]

A *map of spectra* 

\[ f : E \to E' \]

is a sequence of maps \( f(n) : E(n) \to E'(n) \) which are compatible with the structure maps \( \sigma(n) \), i.e., 

\[ f(n + 1) \circ \sigma(n) = \sigma'(n) \circ (f(n) \wedge \text{id}_{S^1}) \]

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Given two pointed spaces \( X = (X, x_0) \) and \( Y = (Y, y_0) \), their one-point-union and their smash product are defined to be the pointed spaces

\[
X \vee Y := \{ (x, y_0) \mid x \in X \} \cup \{ (x_0, y) \mid y \in Y \} \subseteq X \times Y;
\]

\[
X \wedge Y := (X \times Y)/(X \vee Y).
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We have \( S^{n+1} \cong S^n \wedge S^1 \).

The sphere spectrum \( S \) has as \( n \)-th space \( S^n \) and as \( n \)-th structure map the homeomorphism \( S^n \wedge S^1 \cong S^{n+1} \).

Let \( X \) be a pointed space. Its suspension spectrum \( \Sigma^\infty X \) is given by the sequence of spaces \( \{ X \wedge S^n \mid n \geq 0 \} \) with the homeomorphism \( (X \wedge S^n) \wedge S^1 \cong X \wedge S^{n+1} \) as structure maps. We have \( S = \Sigma^\infty S^0 \).
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Definition (\(\Omega\)-spectrum)

Given a spectrum \(E\), we can consider instead of the structure map \(\sigma(n) : E(n) \wedge S^1 \to E(n + 1)\) its adjoint

\[\sigma'(n) : E(n) \to \Omega E(n + 1) = \text{map}(S^1, E(n + 1)).\]

We call \(E\) an \(\Omega\)-spectrum if each map \(\sigma'(n)\) is a weak homotopy equivalence.
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Definition (Homotopy groups of a spectrum)

Given a spectrum $E$, define for $n \in \mathbb{Z}$ its $n$-th homotopy group

$$\pi_n(E) := \colim_{k \to \infty} \pi_{k+n}(E(k))$$

to be the abelian group which is given by the colimit over the directed system indexed by $\mathbb{Z}$ with $k$-th structure map

$$\pi_{k+n}(E(k)) \xrightarrow{\sigma'(k)} \pi_{k+n}(\Omega E(k+1)) = \pi_{k+n+1}(E(k+1)).$$

- Notice that a spectrum can have in contrast to a space non-trivial negative homotopy groups.
- If $E$ is an $\Omega$-spectrum, then $\pi_n(E) = \pi_n(E(0))$ for all $n \geq 0$. 
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Eilenberg-MacLane spectrum

Let $A$ be an abelian group. The $n$-th Eilenberg-MacLane space $EM(A, n)$ associated to $A$ for $n \geq 0$ is a CW-complex with $\pi_m(EM(A, n)) = A$ for $m = n$ and $\pi_m(EM(A, n)) = \{0\}$ for $m \neq n$. The associated Eilenberg-MacLane spectrum $H(A)$ has as $n$-th space $EM(A, n)$ and as $n$-th structure map a homotopy equivalence $EM(A, n) \rightarrow \Omega EM(A, n + 1)$.

Algebraic $K$-theory spectrum

For a ring $R$ there is the algebraic $K$-theory spectrum $K_R$ with the property

$$\pi_n(K_R) = K_n(R) \quad \text{for } n \in \mathbb{Z}.$$
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Algebraic $L$-theory spectrum

For a ring with involution $R$ there is the algebraic $L$-theory spectrum $L^{\langle -\infty \rangle}_R$ with the property

$$\pi_n(L^{\langle -\infty \rangle}_R) = L^{\langle -\infty \rangle}_n(R) \quad \text{for } n \in \mathbb{Z}.$$ 

Topological $K$-theory spectrum

By Bott periodicity there is a homotopy equivalence

$$\beta : BU \times \mathbb{Z} \xrightarrow{\sim} \Omega^2(BU \times \mathbb{Z}).$$

The topological $K$-theory spectrum $K^{\text{top}}$ has in even degrees $BU \times \mathbb{Z}$ and in odd degrees $\Omega(BU \times \mathbb{Z})$. The structure maps are given in even degrees by the map $\beta$ and in odd degrees by the identity $\text{id} : \Omega(BU \times \mathbb{Z}) \to \Omega(BU \times \mathbb{Z})$. 

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The topological $K$-theory spectrum $K^{\text{top}}$ has in even degrees $BU \times \mathbb{Z}$ and in odd degrees $\Omega(BU \times \mathbb{Z})$. The structure maps are given in even degrees by the map $\beta$ and in odd degrees by the identity $\text{id} : \Omega(BU \times \mathbb{Z}) \to \Omega(BU \times \mathbb{Z})$. 
Algebraic $L$-theory spectrum

For a ring with involution $R$ there is the algebraic $L$-theory spectrum $L^{\langle -\infty \rangle}_R$ with the property

$$\pi_n(L^{\langle -\infty \rangle}_R) = L^{\langle -\infty \rangle}_n(R) \quad \text{for } n \in \mathbb{Z}.$$ 

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Definition (Homology theory)

Let $\Lambda$ be a commutative ring, for instance $\mathbb{Z}$ or $\mathbb{Q}$. A homology theory $\mathcal{H}_*$ with values in $\Lambda$-modules is a covariant functor from the category of $CW$-pairs to the category of $\mathbb{Z}$-graded $\Lambda$-modules together with natural transformations

$$\partial_n(X, A): \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- Homotopy invariance
- Long exact sequence of a pair
- Excision

If $(X, A)$ is a $CW$-pair and $f: A \to B$ is a cellular map, then

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- **Disjoint union axiom**

\[ \bigoplus_{i \in I} \mathcal{H}_n(X_i) \xrightarrow{\cong} \mathcal{H}_n \left( \bigsqcup_{i \in I} X_i \right). \]

Definition (Smash product)

Let \( E \) be a spectrum and \( X \) be a pointed space. Define the **smash product** \( X \land E \) to be the spectrum whose \( n \)-th space is \( X \land E(n) \) and whose \( n \)-th structure map is

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The homology theory associated to the sphere spectrum $S$ is stable homotopy $\pi^S_\ast(X)$. The groups $\pi^S_n(pt)$ are finite abelian groups for $n \neq 0$ by a result of Serre (1953). Their structure is only known for small $n$.

Example (Singular homology theory with coefficients)

The homology theory associated to the Eilenberg-MacLane spectrum $H(A)$ is singular homology with coefficients in $A$.

Example (Topological $K$-homology)

The homology theory associated to the topological $K$-theory spectrum $K^{\text{top}}$ is $K$-homology $K_\ast(X)$. We have

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The Isomorphism Conjectures for torsionfree groups

Conjecture \((K\text{-theoretic Farrell-Jones Conjecture for torsionfree groups})\)

The \textit{K-theoretic Farrell-Jones Conjecture} with coefficients in the regular ring \(R\) for the torsionfree group \(G\) predicts that the assembly map

\[
H_n(BG; K_R) \rightarrow K_n(RG)
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is bijective for all \(n \in \mathbb{Z}\).

- \(K_n(RG)\) is the algebraic \(K\)-theory of the group ring \(RG\);
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- \(BG\) is the classifying space of the group \(G\), i.e., the base space of the universal \(G\)-principal \(G\)-bundle \(G \rightarrow EG \rightarrow BG\). Equivalently, \(BG = EM(G, 1)\). The space \(BG\) is unique up to homotopy.
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Conjecture \((L\)-theoretic Farrell-Jones Conjecture for torsionfree groups\)

The \(L\)-theoretic Farrell-Jones Conjecture with coefficients in the ring with involution \(R\) for the torsionfree group \(G\) predicts that the assembly map

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is bijective for all \(n \in \mathbb{Z}\).

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- \(\mathbb{L}_R^{\langle -\infty \rangle}\) is the algebraic \(L\)-theory spectrum of \(R\) with decoration \(\langle -\infty \rangle\);
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The *$L$-theoretic Farrell-Jones Conjecture* with coefficients in the ring with involution $R$ for the torsionfree group $G$ predicts that the *assembly map*

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**Conjecture (Baum-Connes Conjecture for torsionfree groups)**

The Baum-Connes Conjecture for the torsionfree group predicts that the assembly map

\[ K_n(BG) \to K_n(C^*_r(G)) \]

is bijective for all \( n \in \mathbb{Z} \).

- \( K_n(BG) \) is the topological \( K \)-homology of \( BG \), where \( K_*(-) = H_*(-; K_{\text{top}}) \) for \( K_{\text{top}} \) the topological \( K \)-theory spectrum.
- \( K_n(C^*_r(G)) \) is the topological \( K \)-theory of the reduced complex group \( C^* \)-algebra \( C^*_r(G) \) of \( G \) which is the closure in the norm topology of \( \mathbb{C}G \) considered as subalgebra of \( B(l^2(G)) \).
- There is also a real version of the Baum-Connes Conjecture

\[ KO_n(BG) \to K_n(C^*_r(G; \mathbb{R})) \].
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Conjecture (Baum-Connes Conjecture for torsionfree groups)

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Consequences of the Isomorphism Conjectures for torsionfree groups

In order to illustrate the depth of the Farrell-Jones Conjecture and the Baum-Connes Conjecture, we present some conclusions which are interesting in their own right.

Let $\mathcal{FJ}_K(R)$ and $\mathcal{FJ}_L(R)$ respectively be the class of groups which satisfy the $K$-theoretic and $L$-theoretic respectively Farrell-Jones Conjecture for the coefficient ring $R$.

Let $\mathcal{BC}$ be the class of groups which satisfy the Baum-Connes Conjecture.
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Let $BC$ be the class of groups which satisfy the Baum-Connes Conjecture.
Lemma

Let $R$ be a regular ring. Suppose that $G$ is torsionfree and $G \in \mathcal{FJ}_K(R)$. Then

- $K_n(RG) = 0$ for $n \leq -1$;
- The change of rings map $K_0(R) \to K_0(RG)$ is bijective. In particular $\tilde{K}_0(RG)$ is trivial if and only if $\tilde{K}_0(R)$ is trivial.

Lemma

Suppose that $G$ is torsionfree and $G \in \mathcal{FJ}_K(\mathbb{Z})$. Then the Whitehead group $\text{Wh}(G)$ is trivial.

Proof.

The idea of the proof is to study the Atiyah-Hirzebruch spectral sequence converging to $H_n(BG; K_R)$ whose $E^2$-term is given by

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Proof (continued).

- Since $R$ is regular by assumption, we get $K_q(R) = 0$ for $q \leq -1$.
- Hence the edge homomorphism yields an isomorphism

$$K_0(R) = H_0(\text{pt}, K_0(R)) \xrightarrow{\cong} H_0(BG; K_R) \cong K_0(RG).$$

- We have $K_0(\mathbb{Z}) = \mathbb{Z}$ and $K_1(\mathbb{Z}) = \{\pm 1\}$. We get an exact sequence

$$0 \rightarrow H_0(BG; K_1(\mathbb{Z})) = \{\pm 1\} \rightarrow H_1(BG; K_\mathbb{Z}) \cong K_1(\mathbb{Z}G) \rightarrow H_1(BG; K_0(\mathbb{Z})) = G/[G, G] \rightarrow 0.$$

- This implies $\text{Wh}(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\} \cong 0$. 
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- This implies $\text{Wh}(G) := K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\} \cong 0$. 
In particular we get for a torsionfree group $G \in \mathcal{FK}(\mathbb{Z})$:

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\tilde{K}_0(\mathbb{Z}G) = 0$;
- $\text{Wh}(G) = 0$;
- Every finitely dominated $CW$-complex $X$ with $G = \pi_1(X)$ is homotopy equivalent to a finite $CW$-complex;
- Every compact $h$-cobordism $W$ of dimension $\geq 6$ with $\pi_1(W) \cong G$ is trivial;
- If $G$ belongs to $\mathcal{FK}(\mathbb{Z})$, then it is of type FF if and only if it is of type FP (Serre’s problem).
In particular we get for a torsionfree group $G \in \mathcal{FJ}_K(\mathbb{Z})$:

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
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In particular we get for a torsionfree group \( G \in \mathcal{FJ}_K(\mathbb{Z}) \):

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Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group $G$ and an integral domain $R$ that $0$ and $1$ are the only idempotents in $RG$.

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture, Bartels-L.-Reich(2007))

Let $F$ be a skew-field and let $G$ be a group with $G \in \mathcal{FJ}_K(F)$. Suppose that one of the following conditions is satisfied:

- $F$ is commutative and has characteristic zero and $G$ is torsionfree;
- $G$ is torsionfree and sofic, e.g., residually amenable;
- The characteristic of $F$ is $p$, all finite subgroups of $G$ are $p$-groups and $G$ is sofic.

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- $F$ is commutative and has characteristic zero and $G$ is torsionfree;
- $G$ is torsionfree and sofic, e.g., residually amenable;
- The characteristic of $F$ is $p$, all finite subgroups of $G$ are $p$-groups and $G$ is sofic.

Then $0$ and $1$ are the only idempotents in $FG$. 
Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group $G$ and an integral domain $R$ that $0$ and $1$ are the only idempotents in $RG$.

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Proof.

Let $p$ be an idempotent in $FG$. We want to show $p \in \{0, 1\}$.

Denote by $\epsilon : FG \to F$ the augmentation homomorphism sending $\sum_{g \in G} r_g \cdot g$ to $\sum_{g \in G} r_g$. Obviously $\epsilon(p) \in F$ is 0 or 1. Hence it suffices to show $p = 0$ under the assumption that $\epsilon(p) = 0$.

Let $(p) \subseteq FG$ be the ideal generated by $p$ which is a finitely generated projective $FG$-module.

Since $G \in \mathcal{F}_K(F)$, we can conclude that

$$i_* : K_0(F) \otimes \mathbb{Q} \to K_0(FG) \otimes \mathbb{Q}$$

is surjective.

Hence we can find a finitely generated projective $F$-module $P$ and integers $k, m, n \geq 0$ satisfying

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Inserting this in the first equation yields

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Our assumptions on $F$ and $G$ imply that $FG$ is stably finite, i.e., if $A$ and $B$ are square matrices over $FG$ with $AB = I$, then $BA = I$. This implies $(p)^k = 0$ and hence $p = 0$. 

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Theorem (The Baum-Connes Conjecture and the Kaplansky Conjecture)

Let $G$ be a torsionfree group with $G \in \mathcal{B}C$. Then 0 and 1 are the only idempotents in $\mathbb{C}G$.

**Proof.**

- There is a trace map
  \[ \text{tr}: C^*_r(G) \to \mathbb{C} \]
  which sends $f \in C^*_r(G) \subseteq \mathcal{B}(l^2(G))$ to $\langle f(e), e \rangle_{l^2(G)}$.

- The $L^2$-index theorem due to Atiyah (1976) shows that the composite
  \[ K_0(BG) \to K_0(C^*_r(G)) \xrightarrow{\text{tr}} \mathbb{C} \]
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**Theorem (The Baum-Connes Conjecture and the Kaplansky Conjecture)**

Let $G$ be a torsionfree group with $G \in BC$. Then 0 and 1 are the only idempotents in $\mathbb{C}G$.

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- Hence $G \in \mathcal{B}C$ implies $\text{tr}(p) \in \mathbb{Z}$.
- Since $\text{tr}(1) = 1$, $\text{tr}(0) = 0$, $0 \leq p \leq 1$ and $p^2 = p$, we get $\text{tr}(p) \in \mathbb{R}$ and $0 \leq \text{tr}(p) \leq 1$.
- We conclude $\text{tr}(0) = \text{tr}(p)$ or $\text{tr}(1) = \text{tr}(p)$.
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Conjecture (Borel Conjecture)

The Borel Conjecture for $G$ predicts for two closed aspherical manifolds $M$ and $N$ with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \rightarrow N$ is homotopic to a homeomorphism and in particular that $M$ and $N$ are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity. A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension $\geq 3$ is homotopic to an isometric diffeomorphism.
- The Borel Conjecture is not true in the smooth category by results of Farrell-Jones(1989).
- There are also non-aspherical manifolds which are topologically rigid in the sense of the Borel Conjecture (see Kreck-L. (2005)).
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Theorem (The Farrell-Jones Conjecture and the Borel Conjecture)

If the $K$- and $L$-theoretic Farrell-Jones Conjecture hold for $G$ in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension $\geq 5$ and in dimension $4$ if $G$ is good in the sense of Freedman.

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Definition (Structure set)

The *structure set* $S^{top}(M)$ of a manifold $M$ consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold $N$ as source. Two such homotopy equivalences $f_0 : N_0 \to M$ and $f_1 : N_1 \to M$ are equivalent if there exists a homeomorphism $g : N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

Theorem

The Borel Conjecture holds for a closed manifold $M$ if and only if $S^{top}(M)$ consists of one element.
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Theorem (Ranicki (1992))

There is an exact sequence of abelian groups, called algebraic surgery exact sequence, for an n-dimensional closed manifold $M$:

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\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M; L\langle 1 \rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \\
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It can be identified with the classical geometric surgery sequence due to Sullivan and Wall in high dimensions.

- $S_{\text{top}}(M)$ consist of one element if and only if $A_{n+1}$ is surjective and $A_n$ is injective.
- $H_k(M; L\langle 1 \rangle) \rightarrow H_k(M; L)$ is bijective for $k \geq n + 1$ and injective for $k = n$. 

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S_{\text{top}}(M) \xrightarrow{\sigma_n} H_n(M; L\langle 1 \rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots
$$

It can be identified with the classical geometric surgery sequence due to *Sullivan and Wall* in high dimensions.

- $S_{\text{top}}(M)$ consist of one element if and only if $A_{n+1}$ is surjective and $A_n$ is injective.
- $H_k(M; L\langle 1 \rangle) \rightarrow H_k(M; L)$ is bijective for $k \geq n + 1$ and injective for $k = n$. 
What happens for groups with torsion?

- The versions of the Farrell-Jones Conjecture and the Baum-Connes Conjecture above become false for finite groups unless the group is trivial.
- For instance the version of the Baum-Connes Conjecture above would predict for a finite group $G$

$$K_0(BG) \cong K_0(C^*_r(G)) \cong R\mathbb{C}(G).$$

However, $K_0(BG) \otimes \mathbb{Q} \cong \mathbb{Q} K_0(pt) \otimes \mathbb{Q} \cong \mathbb{Q} \otimes \mathbb{Q}$ and $R\mathbb{C}(G) \otimes \mathbb{Q} \cong \mathbb{Q} \otimes \mathbb{Q}$ holds if and only if $G$ is trivial.
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In view of the Bass-Heller-Swan decomposition this is only possible if $NK_n(R)$ vanishes which is true for regular rings $R$ but not for general rings $R$.

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Assembly

For a field $F$ of characteristic zero and some groups $G$ one knows that there is an isomorphism

$$\text{colim}_{H \subseteq G} K_0(FH) \cong K_0(FG).$$

This indicates that one has at least to take into account the values for all finite subgroups to assemble $K_n(FG)$.

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The Bass-Heller-Swan decomposition shows that the $K$-theory of finite subgroups in degree $m \leq n$ can affect the $K$-theory in degree $n$ and that at least in the Farrell-Jones setting finite subgroups are not enough.
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\[ K_n(C^*_r(G \times \mathbb{Z})) \cong K_n(C^*_r(G)) \oplus K_{n-1}(C^*_r(G)). \]

Homological behaviour

There is still a lot of homological behaviour known for \( K_\ast(C^*_r(G)) \). For instance there exists a long exact Mayer-Vietoris sequence associated to amalgamated products \( G_1 *_{G_0} G_2 \) and a Wang-sequence associated to semi-direct products \( G \rtimes \mathbb{Z} \) by Pimsner-Voiculescu (1982).

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Question (Classifying spaces for families)

Is there a version $E_{\mathcal{F}}(G)$ of the classifying space $EG$ which takes the structure of the family of finite subgroups or other families $\mathcal{F}$ of subgroups into account and can be used for a general formulation of the Farrell-Jones Conjecture?

Question (Equivariant homology theories)

Can one define appropriate $G$-homology theories $\mathcal{H}_{*}^{G}$ that are in some sense computable and yield when applied to $E_{\mathcal{F}}(G)$ a term which potentially is isomorphic to the groups $K_{n}(RG)$, $L^{-\langle \infty \rangle}(RG)$ or $K_{n}(C_{r}^{*}(G))$?

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To be continued
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