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Hangzhou, July 2007
We have introduced the Farrell-Jones Conjecture and the Baum-Connes Conjecture for torsion-free groups:

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\begin{align*}
H_n(BG; K_R) & \xrightarrow{\text{R}} K_n(RG); \\
H_n(BG; L_R^{(-\infty)}) & \xrightarrow{\text{R}} L_n^{(-\infty)}(RG); \\
K_n(BG) & \xrightarrow{\text{R}} K_n(C^*_r(G)).
\end{align*}
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We have discussed applications of these conjectures such as to the Kaplansky Conjecture and the Borel Conjecture.
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We have discussed applications of these conjectures such as to the Kaplansky Conjecture and the Borel Conjecture.
Question (Classifying spaces for families)

Is there a version $E_{\mathcal{F}}(G)$ of the classifying space $EG$ which takes the structure of the family of finite subgroups or other families $\mathcal{F}$ of subgroups into account and can be used for a general formulation of the Farrell-Jones Conjecture?
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**Question (Classifying spaces for families)**

Is there a version $E_{\mathcal{F}}(G)$ of the classifying space $EG$ which takes the structure of the family of finite subgroups or other families $\mathcal{F}$ of subgroups into account and can be used for a general formulation of the Farrell-Jones Conjecture?
We introduce the notion of the classifying space of a family $\mathcal{F}$ of subgroups $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$.

In the case, where $\mathcal{F}$ is the family $\text{COM}$ of compact subgroups, we present some nice geometric models for $E_{\mathcal{F}}(G)$ and explain $E_{\mathcal{F}}(G) \simeq J_{\mathcal{F}}(G)$.

We discuss finiteness properties of these classifying spaces.
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Definition (\textit{G-CW-complex})

A \textit{G-CW-complex} \(X\) is a \(G\)-space together with a \(G\)-invariant filtration

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\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X
\]

such that \(X\) carries the \textit{colimit topology} with respect to this filtration, and \(X_n\) is obtained from \(X_{n-1}\) for each \(n \geq 0\) by attaching equivariant \(n\)-dimensional cells, i.e., there exists a \(G\)-pushout

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\begin{array}{ccc}
\bigsqcup_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\bigsqcup_{i \in I_n} q_i^n} & X_{n-1} \\
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Group means locally compact Hausdorff topological group with a countable basis for its topology, unless explicitly stated differently.

**Example (Simplicial actions)**

Let $X$ be a simplicial complex. Suppose that $G$ acts simplicially on $X$. Then $G$ acts simplicially also on the barycentric subdivision $X'$, and all isotropy groups are open and closed. The $G$-space $X'$ inherits the structure of a $G$-$CW$-complex.

**Example (Smooth actions)**

Let $G$ be a Lie group acting properly and smoothly on a smooth manifold $M$. Then $M$ inherits the structure of $G$-$CW$-complex.
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**Definition (Proper G-action)**

A $G$-space $X$ is called *proper* if for each pair of points $x$ and $y$ in $X$ there are open neighborhoods $V_x$ of $x$ and $W_y$ of $y$ in $X$ such that the closure of the subset $\{g \in G \mid gV_x \cap W_y \neq \emptyset\}$ of $G$ is compact.

**Lemma**

- A proper $G$-space has always compact isotropy groups.
- A $G$-CW-complex $X$ is proper if and only if all its isotropy groups are compact.
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Definition (Family of subgroups)

A family $\mathcal{F}$ of subgroups of $G$ is a set of (closed) subgroups of $G$ which is closed under conjugation and finite intersections.

Examples for $\mathcal{F}$ are:

- $TR = \{\text{trivial subgroup}\}$;
- $FIN = \{\text{finite subgroups}\}$;
- $VCYC = \{\text{virtually cyclic subgroups}\}$;
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Definition (Classifying $G$-$CW$-complex for a family of subgroups)

Let $\mathcal{F}$ be a family of subgroups of $G$. A model for the *classifying $G$-$CW$-complex for the family $\mathcal{F}$* is a $G$-$CW$-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any $G$-$CW$-complex $Y$, whose isotropy groups belong to $\mathcal{F}$, there is up to $G$-homotopy precisely one $G$-map $Y \to E_{\mathcal{F}}(G)$.

We abbreviate $EG := E_{COM}(G)$ and call it the *universal $G$-$CW$-complex for proper $G$-actions*. We also write $EG = E_{TR}(G)$. 
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Theorem \textbf{(Homotopy characterization of $E_{\mathcal{F}}(G)$)}

Let $\mathcal{F}$ be a family of subgroups.

- There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;
- Two models for $E_{\mathcal{F}}(G)$ are $G$-homotopy equivalent;
- A $G$-CW-complex $X$ is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the $H$-fixed point set $X^H$ is weakly contractible.
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- A $G$-CW-complex $X$ is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the $H$-fixed point set $X^H$ is weakly contractible.
Theorem (Homotopy characterization of $E_{\mathcal{F}}(G)$)

Let $\mathcal{F}$ be a family of subgroups.

- There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;
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A model for $E_{\mathcal{A}\mathcal{L}L}(G)$ is $G/G$;

$EG \to BG := G\backslash EG$ is the universal $G$-principal bundle for $G$-CW-complexes.

**Example (Infinite dihedral group)**

- Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 \ast \mathbb{Z}/2$ be the infinite dihedral group.
- A model for $ED_\infty$ is the universal covering of $RP^\infty \vee RP^\infty$.
- A model for $ED_\infty$ is $\mathbb{R}$ with the obvious $D_\infty$-action.

**Lemma**

If $G$ is totally disconnected, then $E_{CO\mathcal{M}OP}(G) = EG$. 
A model for $E_{\text{ALL}}(G)$ is $G/G$;

$EG \to BG := G\backslash EG$ is the **universal** $G$-principal bundle for $G$-CW-complexes.

**Example (Infinite dihedral group)**

- Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 \ast \mathbb{Z}/2$ be the infinite dihedral group.
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**Lemma**

If $G$ is totally disconnected, then $E_{\text{COMOP}}(G) = EG$. 

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A model for $E_{\mathcal{ALC}}(G)$ is $G/G$;

$EG \to BG := G\backslash EG$ is the universal $G$-principal bundle for $G$-CW-complexes.

**Example (Infinite dihedral group)**

- Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 \star \mathbb{Z}/2$ be the infinite dihedral group.
- A model for $ED_\infty$ is the universal covering of $\mathbb{RP}^\infty \lor \mathbb{RP}^\infty$.
- A model for $ED_\infty$ is $\mathbb{R}$ with the obvious $D_\infty$-action.

**Lemma**

If $G$ is totally disconnected, then $E_{\mathcal{COMOP}}(G) = EG$. 
A model for $E_{\text{ALL}}(G)$ is $G/G$;

$EG \to BG := G\backslash EG$ is the universal $G$-principal bundle for $G$-CW-complexes.

**Example (Infinite dihedral group)**

- Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 \ast \mathbb{Z}/2$ be the infinite dihedral group.
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**Lemma**

If $G$ is totally disconnected, then $E_{\text{COMOP}}(G) = EG$. 
A model for $\mathcal{E}_{\mathcal{A}LL}(G)$ is $G/G$;

$EG \to BG := G \backslash EG$ is the universal $G$-principal bundle for $G$-CW-complexes.

**Example (Infinite dihedral group)**

- Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 \star \mathbb{Z}/2$ be the infinite dihedral group. 
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**Lemma**

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**Example (Infinite dihedral group)**

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**Lemma**

If $G$ is totally disconnected, then $E_{\mathcal{COMOP}}(G) = EG$. 
A model for $E_{\mathcal{A} \mathcal{L} \mathcal{L}}(G)$ is $G/G$;

$EG \to BG := G \setminus EG$ is the universal $G$-principal bundle for $G$-$CW$-complexes.

**Example (Infinite dihedral group)**

- Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 \rtimes \mathbb{Z}/2$ be the infinite dihedral group.
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- A model for $ED_\infty$ is $\mathbb{R}$ with the obvious $D_\infty$-action.

**Lemma**

*If $G$ is totally disconnected, then $E_{\mathcal{C} \mathcal{O} \mathcal{M} \mathcal{O} \mathcal{P}}(G) = EG$.***
Definition (**\(F\)-numerable **\(G\)**-space)

A **\(F\)**-**\(G\)**-space is a **\(G\)**-space, for which there exists an open covering \(\{U_i \mid i \in I\}\) by **\(G\)**-subspaces satisfying:

- For each \(i \in I\) there exists a **\(G\)**-map \(U_i \rightarrow G/G_i\) for some **\(G\)**\(i\) \(\in\) **\(F\)**;
- There is a locally finite partition of unity \(\{e_i \mid i \in I\}\) subordinate to \(\{U_i \mid i \in I\}\) by **\(G\)**-invariant functions \(e_i : X \rightarrow [0, 1]\).

- Notice that we do not demand that the isotropy groups of a **\(F\)**-**\(G\)**-space belong to **\(F\)**.
- If \(f : X \rightarrow Y\) is a **\(G\)**-map and **\(Y\)** is **\(F\)**-numerable, then **\(X\)** is also **\(F\)**-numerable.
- A **\(G\)**-**\(CW\)**-complex is **\(F\)**-numerable if and only if each isotropy group appears as a subgroup of an element in **\(F\)**.
Definition \((\mathcal{F}\text{-numerable } G\text{-space})\)

A \(\mathcal{F}\text{-numerable } G\text{-space}\) is a \(G\text{-space}\), for which there exists an open covering \(\{U_i \mid i \in I\}\) by \(G\)-subspaces satisfying:

- For each \(i \in I\) there exists a \(G\)-map \(U_i \to G/G_i\) for some \(G_i \in \mathcal{F}\);
- There is a locally finite partition of unity \(\{e_i \mid i \in I\}\) subordinate to \(\{U_i \mid i \in I\}\) by \(G\)-invariant functions \(e_i : X \to [0,1]\).

- Notice that we do not demand that the isotropy groups of a \(\mathcal{F}\)-numerable \(G\)-space belong to \(\mathcal{F}\).
- If \(f : X \to Y\) is a \(G\)-map and \(Y\) is \(\mathcal{F}\)-numerable, then \(X\) is also \(\mathcal{F}\)-numerable.
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If \(f : X \to Y\) is a \(G\)-map and \(Y\) is \(\mathcal{F}\)-numerable, then \(X\) is also \(\mathcal{F}\)-numerable.

A \(G\)-CW-complex is \(\mathcal{F}\)-numerable if and only if each isotropy group appears as a subgroup of an element in \(\mathcal{F}\).
A $\mathcal{F}$-numerable $G$-space is a $G$-space, for which there exists an open covering $\{U_i \mid i \in I\}$ by $G$-subspaces satisfying:

- For each $i \in I$ there exists a $G$-map $U_i \to G/G_i$ for some $G_i \in \mathcal{F}$;
- There is a locally finite partition of unity $\{e_i \mid i \in I\}$ subordinate to $\{U_i \mid i \in I\}$ by $G$-invariant functions $e_i: X \to [0, 1]$.

Notice that we do not demand that the isotropy groups of a $\mathcal{F}$-numerable $G$-space belong to $\mathcal{F}$.

If $f: X \to Y$ is a $G$-map and $Y$ is $\mathcal{F}$-numerable, then $X$ is also $\mathcal{F}$-numerable.

A $G$-CW-complex is $\mathcal{F}$-numerable if and only if each isotropy group appears as a subgroup of an element in $\mathcal{F}$.
Definition (\(\mathcal{F}\)-numerable \(G\)-space)

A \(\mathcal{F}\)-numerable \(G\)-space is a \(G\)-space, for which there exists an open covering \(\{U_i \mid i \in I\}\) by \(G\)-subspaces satisfying:

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If \(f : X \rightarrow Y\) is a \(G\)-map and \(Y\) is \(\mathcal{F}\)-numerable, then \(X\) is also \(\mathcal{F}\)-numerable.

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Definition (**F**-numerable **G**-space)

A **F**-numerable **G**-space is a **G**-space, for which there exists an open covering \( \{ U_i \mid i \in I \} \) by **G**-subspaces satisfying:

- For each \( i \in I \) there exists a **G**-map \( U_i \to G/G_i \) for some \( G_i \in F \);
- There is a locally finite partition of unity \( \{ e_i \mid i \in I \} \) subordinate to \( \{ U_i \mid i \in I \} \) by **G**-invariant functions \( e_i : X \to [0, 1] \).

- Notice that we do not demand that the isotropy groups of a **F**-numerable **G**-space belong to **F**.
- If \( f : X \to Y \) is a **G**-map and \( Y \) is **F**-numerable, then \( X \) is also **F**-numerable.
- A **G-CW**-complex is **F**-numerable if and only if each isotropy group appears as a subgroup of an element in **F**.
A $\mathcal{F}$-numerable $G$-space is a $G$-space, for which there exists an open covering $\{U_i \mid i \in I\}$ by $G$-subspaces satisfying:

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Notice that we do not demand that the isotropy groups of a $\mathcal{F}$-numerable $G$-space belong to $\mathcal{F}$.

If $f : X \to Y$ is a $G$-map and $Y$ is $\mathcal{F}$-numerable, then $X$ is also $\mathcal{F}$-numerable.

A $G$-$CW$-complex is $\mathcal{F}$-numerable if and only if each isotropy group appears as a subgroup of an element in $\mathcal{F}$.
There is also a version $J_{\mathcal{F}}(G)$ of a classifying space for $\mathcal{F}$-numerable $G$-spaces.

It is characterized by the property that $J_{\mathcal{F}}(G)$ is $\mathcal{F}$-numerable and for every $\mathcal{F}$-numerable $G$-space $Y$ there is up to $G$-homotopy precisely one $G$-map $Y \to J_{\mathcal{F}}(G)$.

We abbreviate $JG = J_{COM}(G)$ and $JG = J_{TR}(G)$.

$JG \to G \backslash JG$ is the universal $G$-principal bundle for numerable free proper $G$-spaces.
There is also a version $J_{\mathcal{F}}(G)$ of a classifying space for $\mathcal{F}$-numerable $G$-spaces.

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$J G \to G \backslash J G$ is the universal $G$-principal bundle for numerable free proper $G$-spaces.
There is also a version $J_F(G)$ of a classifying space for $\mathcal{F}$-numerable $G$-spaces.

It is characterized by the property that $J_F(G)$ is $\mathcal{F}$-numerable and for every $\mathcal{F}$-numerable $G$-space $Y$ there is up to $G$-homotopy precisely one $G$-map $Y \to J_F(G)$.

We abbreviate $J_G = J_{COM}(G)$ and $J_G = J_{TR}(G)$.

$J_G \to G \backslash J_G$ is the universal $G$-principal bundle for numerable free proper $G$-spaces.
There is also a version $J_{\mathcal{F}}(G)$ of a classifying space for $\mathcal{F}$-numerable $G$-spaces.

It is characterized by the property that $J_{\mathcal{F}}(G)$ is $\mathcal{F}$-numerable and for every $\mathcal{F}$-numerable $G$-space $Y$ there is up to $G$-homotopy precisely one $G$-map $Y \to J_{\mathcal{F}}(G)$.

We abbreviate $\underline{J}G = J_{\text{COM}}(G)$ and $JG = J_{\text{TR}}(G)$.

$JG \to G\backslash JG$ is the universal $G$-principal bundle for numerable free proper $G$-spaces.
Theorem (Comparison of $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$, L. (2005))

There is up to $G$-homotopy precisely one $G$-map

$$\phi: E_{\mathcal{F}}(G) \to J_{\mathcal{F}}(G);$$

It is a $G$-homotopy equivalence if one of the following conditions is satisfied:
- Each element in $\mathcal{F}$ is open and closed;
- $G$ is discrete;
- $\mathcal{F}$ is $COM$;

Let $G$ be totally disconnected. Then $EG \to JG$ is a $G$-homotopy equivalence if and only if $G$ is discrete.
**Theorem (Comparison of $E_{\mathcal{F}}(G)$ and $J_{\mathcal{F}}(G)$, L. (2005))**

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- Let $G$ be totally disconnected. Then $EG \to JG$ is a $G$-homotopy equivalence if and only if $G$ is discrete.
We want to illustrate that the space $EG = JG$ often has very nice geometric models and appear naturally in many interesting situations.

Let $C_0(G)$ be the Banach space of complex valued functions of $G$ vanishing at infinity with the supremum-norm. The group $G$ acts isometrically on $C_0(G)$ by $(g \cdot f)(x) := f(g^{-1}x)$ for $f \in C_0(G)$ and $g, x \in G$.

Let $PC_0(G)$ be the subspace of $C_0(G)$ consisting of functions $f$ such that $f$ is not identically zero and has non-negative real numbers as values.

**Theorem (Operator theoretic model, Abels (1978))**

The $G$-space $PC_0(G)$ is a model for $JG$. 
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Special models for $EG$

- We want to illustrate that the space $EG = JG$ often has very nice geometric models and appear naturally in many interesting situations.

- Let $C_0(G)$ be the Banach space of complex valued functions of $G$ vanishing at infinity with the supremum-norm. The group $G$ acts isometrically on $C_0(G)$ by $(g \cdot f)(x) := f(g^{-1}x)$ for $f \in C_0(G)$ and $g, x \in G$.

  Let $PC_0(G)$ be the subspace of $C_0(G)$ consisting of functions $f$ such that $f$ is not identically zero and has non-negative real numbers as values.

**Theorem (Operator theoretic model, Abels (1978))**

The $G$-space $PC_0(G)$ is a model for $JG$. 
Theorem

Let $G$ be discrete. A model for $JG$ is the space

$$X_G = \left\{ f : G \to [0, 1] \mid f \text{ has finite support, } \sum_{g \in G} f(g) = 1 \right\}$$

with the topology coming from the supremum norm.

Theorem (Simplicial Model)

Let $G$ be discrete. Let $P_\infty(G)$ be the geometric realization of the simplicial set whose $k$-simplices consist of $(k + 1)$-tupels $(g_0, g_1, \ldots, g_k)$ of elements $g_i$ in $G$. This is a model for $EG$. 

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Theorem

Let $G$ be discrete. A model for $\mathbb{J}G$ is the space

$$X_G = \left\{ f : G \to [0,1] \left| \begin{array}{c} f \text{ has finite support, } \\ \sum_{g \in G} f(g) = 1 \end{array} \right. \right\}$$

with the topology coming from the supremum norm.

Theorem (Simplicial Model)

Let $G$ be discrete. Let $P_\infty(G)$ be the geometric realization of the simplicial set whose $k$-simplices consist of $(k + 1)$-tupels $(g_0, g_1, \ldots, g_k)$ of elements $g_i$ in $G$. This is a model for $E \mathbb{G}$. 
Theorem

Let $G$ be discrete. A model for $\mathcal{J}G$ is the space

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Theorem

Let $G$ be discrete. A model for $JG$ is the space

$$X_G = \left\{ f : G \to [0, 1] \, \bigg| \, f \text{ has finite support, } \sum_{g \in G} f(g) = 1 \right\}$$

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Theorem

Let $G$ be discrete. A model for $J_G$ is the space

\[ X_G = \left\{ f : G \to [0, 1] \ \middle|\ f \text{ has finite support, } \sum_{g \in G} f(g) = 1 \right\} \]

with the topology coming from the supremum norm.

Theorem (Simplicial Model)

Let $G$ be discrete. Let $P_\infty(G)$ be the geometric realization of the simplicial set whose $k$-simplices consist of $(k + 1)$-tupels $(g_0, g_1, \ldots, g_k)$ of elements $g_i$ in $G$. This is a model for $E_G$. 
The spaces $X_G$ and $P_\infty(G)$ have the same underlying sets but in general they have different topologies.

The identity map induces a $G$-map $P_\infty(G) \to X_G$ which is a $G$-homotopy equivalence, but in general not a $G$-homeomorphism.
The spaces $X_G$ and $P_\infty(G)$ have the same underlying sets but in general they have different topologies.

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Theorem (Almost connected groups, Abels (1978).)

Suppose that $G$ is almost connected, i.e., the group $G/G^0$ is compact for $G^0$ the component of the identity element. Then $G$ contains a maximal compact subgroup $K$ which is unique up to conjugation, and the $G$-space $G/K$ is a model for $\mathbb{J}_G$.

As a special case we get:

Theorem (Discrete subgroups of almost connected Lie groups)

Let $L$ be a Lie group with finitely many path components. Then $L$ contains a maximal compact subgroup $K$ which is unique up to conjugation, and the $L$-space $L/K$ is a model for $\mathbb{E}_L$. If $G \subseteq L$ is a discrete subgroup of $L$, then $L/K$ with the obvious left $G$-action is a finite dimensional $G$-CW-model for $\mathbb{E}G$. 
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Theorem (Actions on CAT(0)-spaces)

Let $G$ be a (locally compact Hausdorff) topological group. Let $X$ be a proper $G$-CW-complex. Suppose that $X$ has the structure of a complete simply connected CAT(0)-space for which $G$ acts by isometries.

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Theorem (Affine buildings)

Let $G$ be a totally disconnected group. Suppose that $G$ acts on the affine building $\Sigma$ by simplicial automorphisms such that each isotropy group is compact. Then $\Sigma$ is a model for both $J_{\text{COMOP}}(G)$ and $\underline{\text{G}}$ and the barycentric subdivision $\Sigma'$ is a model for both $E_{\text{COMOP}}(G)$ and $\underline{\text{EG}}$.

- An important example is the case of a reductive $p$-adic algebraic group $G$ and its associated affine Bruhat-Tits building $\beta(G)$. Then $\beta(G)$ is a model for $\underline{\text{G}}$ and $\beta(G)'$ is a model for $\underline{\text{EG}}$ by the previous result.
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The **Rips complex** $P_d(G, S)$ of a group $G$ with a symmetric finite set $S$ of generators for a natural number $d$ is the geometric realization of the simplicial set whose set of $k$-simplices consists of $(k + 1)$-tuples $(g_0, g_1, \ldots g_k)$ of pairwise distinct elements $g_i \in G$ satisfying $d_S(g_i, g_j) \leq d$ for all $i, j \in \{0, 1, \ldots, k\}$.

The obvious $G$-action by simplicial automorphisms on $P_d(G, S)$ induces a $G$-action by simplicial automorphisms on the barycentric subdivision $P_d(G, S)'$.

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**Theorem (Rips complex, Meintrup-Schick (2002))**

Let $G$ be a discrete group with a finite symmetric set of generators. Suppose that $(G, S)$ is $\delta$-hyperbolic for the real number $\delta \geq 0$. Let $d$ be a natural number with $d \geq 16\delta + 8$. Then the barycentric subdivision of the Rips complex $P_d(G, S)'$ is a finite $G$-CW-model for $\underline{E}G$. 
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• Arithmetic groups in a semisimple connected linear $\mathbb{Q}$-algebraic group possess finite models for $EG$.

• Namely, let $G(\mathbb{R})$ be the $\mathbb{R}$-points of a semisimple $\mathbb{Q}$-group $G(\mathbb{Q})$ and let $K \subseteq G(\mathbb{R})$ be a maximal compact subgroup.

• If $A \subseteq G(\mathbb{Q})$ is an arithmetic group, then $G(\mathbb{R})/K$ with the left $A$-action is a model for $EA$ as already explained above.

• The $A$-space $G(\mathbb{R})/K$ is not necessarily cocompact.

**Theorem (Borel-Serre compactification)**

The Borel-Serre compactification of $G(\mathbb{R})/K$ is a finite $A$-CW-model for $EA$.

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Let $\Gamma_{g,r}$ be the mapping class group of an orientable compact surface $F$ of genus $g$ with $s$ punctures and $r$ boundary components.

We will always assume that $2g + s + r > 2$, or, equivalently, that the Euler characteristic of the punctured surface $F$ is negative.

It is well-known that the associated Teichmüller space $\mathcal{T}_{g,r}^s$ is a contractible space on which $\Gamma_{g,r}^s$ acts properly.

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The $\Gamma_{g,r}^s$-space $\mathcal{T}_{g,r}^s$ is a model for $E\Gamma_{g,r}^s$. 
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Let $\Gamma_{g,r}^s$ be the mapping class group of an orientable compact surface $F$ of genus $g$ with $s$ punctures and $r$ boundary components. We will always assume that $2g + s + r > 2$, or, equivalently, that the Euler characteristic of the punctured surface $F$ is negative.

It is well-known that the associated Teichmüller space $T_{g,r}^s$ is a contractible space on which $\Gamma_{g,r}^s$ acts properly.

**Theorem (Teichmüller space)**

The $\Gamma_{g,r}^s$-space $T_{g,r}^s$ is a model for $E\Gamma_{g,r}^s$. 
Let $F_n$ be the free group of rank $n$.

Denote by $\text{Out}(F_n)$ the group of outer automorphisms of $F_n$, i.e., the quotient of the group of all automorphisms of $F_n$ by the normal subgroup of inner automorphisms.

Culler-Vogtmann (1996) have constructed a space $X_n$ called outer space on which $\text{Out}(F_n)$ acts with finite isotropy groups. It is analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface.

The space $X_n$ contains a spine $K_n$ which is an $\text{Out}(F_n)$-equivariant deformation retraction. This space $K_n$ is a simplicial complex of dimension $(2n - 3)$ on which the $\text{Out}(F_n)$-action is by simplicial automorphisms and cocompact.

**Theorem (Spine of outer space)**

The barycentric subdivision $K'_n$ is a finite $(2n - 3)$-dimensional model of $E\text{Out}(F_n)$. 
Let $F_n$ be the free group of rank $n$.

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Example \((SL_2(\mathbb{R}) \text{ and } SL_2(\mathbb{Z}))\)

- In order to illustrate some of the general statements above we consider the special example \(SL_2(\mathbb{R})\) and \(SL_2(\mathbb{Z})\).

- Let \(\mathbb{H}^2\) be the 2-dimensional hyperbolic space. The group \(SL_2(\mathbb{R})\) acts by isometric diffeomorphisms on the upper half-plane by Moebius transformations. This action is proper and transitive. The isotropy group of \(z = i\) is \(SO(2)\). Since \(\mathbb{H}^2\) is a simply-connected Riemannian manifold, whose sectional curvature is constant \(-1\), the \(SL_2(\mathbb{R})\)-space \(\mathbb{H}^2\) is a model for \(E_{SL_2(\mathbb{R})}\).

- The group \(SL_2(\mathbb{R})\) is a connected Lie group and \(SO(2) \subseteq SL_2(\mathbb{R})\) is a maximal compact subgroup. Hence \(SL_2(\mathbb{R})/SO(2)\) is a model for \(E_{SL_2(\mathbb{R})}\).

- Since the \(SL_2(\mathbb{R})\)-action on \(\mathbb{H}^2\) is transitive and \(SO(2)\) is the isotropy group at \(i \in \mathbb{H}^2\), we see that the \(SL_2(\mathbb{R})\)-manifolds \(SL_2(\mathbb{R})/SO(2)\) and \(\mathbb{H}^2\) are \(SL_2(\mathbb{R})\)-diffeomorphic.
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- In order to illustrate some of the general statements above we consider the special example \( \text{SL}_2(\mathbb{R}) \) and \( \text{SL}_2(\mathbb{Z}) \).

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- The group \( \text{SL}_2(\mathbb{R}) \) is a connected Lie group and \( \text{SO}(2) \subseteq \text{SL}_2(\mathbb{R}) \) is a maximal compact subgroup. Hence \( \text{SL}_2(\mathbb{R})/\text{SO}(2) \) is a model for \( \text{ESL}_2(\mathbb{R}) \).

- Since the \( \text{SL}_2(\mathbb{R}) \)-action on \( \mathbb{H}^2 \) is transitive and \( \text{SO}(2) \) is the isotropy group at \( i \in \mathbb{H}^2 \), we see that the \( \text{SL}_2(\mathbb{R}) \)-manifolds \( \text{SL}_2(\mathbb{R})/\text{SO}(2) \) and \( \mathbb{H}^2 \) are \( \text{SL}_2(\mathbb{R}) \)-diffeomorphic.
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- The group \(SL_2(\mathbb{R})\) is a connected Lie group and \(SO(2) \subseteq SL_2(\mathbb{R})\) is a maximal compact subgroup. Hence \(SL_2(\mathbb{R})/SO(2)\) is a model for \(ESL_2(\mathbb{R})\).
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Example (continued)

- Since $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$, the space $\mathbb{H}^2$ with the obvious $SL_2(\mathbb{Z})$-action is a model for $E SL_2(\mathbb{Z})$.

- The group $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product $\mathbb{Z}/4 \ast_{\mathbb{Z}/2} \mathbb{Z}/6$. This implies that there is a tree on which $SL_2(\mathbb{Z})$ acts with finite stabilizers. The tree has alternately two and three edges emanating from each vertex. This is a 1-dimensional model for $E SL_2(\mathbb{Z})$.

- The tree model and the other model given by $\mathbb{H}^2$ must be $SL_2(\mathbb{Z})$-homotopy equivalent. They can explicitly be related by the following construction.
Example (continued)

- Since $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{R})$, the space $\mathbb{H}^2$ with the obvious $SL_2(\mathbb{Z})$-action is a model for $E_{SL_2}(\mathbb{Z})$.

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- The tree model and the other model given by $\mathbb{H}^2$ must be $SL_2(\mathbb{Z})$-homotopy equivalent. They can explicitly be related by the following construction.
Example (continued)

- Divide the Poincaré disk into fundamental domains for the $SL_2(\mathbb{Z})$-action. Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior. Then the union of the edges, whose end points lie in the interior of the Poincaré disk, is a tree $T$ with $SL_2(\mathbb{Z})$-action which is the tree model above. The tree is a $SL_2(\mathbb{Z})$-equivariant deformation retraction of the Poincaré disk. A retraction is given by moving a point $p$ in the Poincaré disk along a geodesic starting at the vertex at infinity, which belongs to the triangle containing $p$, through $p$ to the first intersection point of this geodesic with $T$. 

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Example (continued)

- Divide the Poincaré disk into fundamental domains for the $SL_2(\mathbb{Z})$-action. Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior. Then the union of the edges, whose end points lie in the interior of the Poincaré disk, is a tree $T$ with $SL_2(\mathbb{Z})$-action which is the tree model above. The tree is a $SL_2(\mathbb{Z})$-equivariant deformation retraction of the Poincaré disk. A retraction is given by moving a point $p$ in the Poincaré disk along a geodesic starting at the vertex at infinity, which belongs to the triangle containing $p$, through $p$ to the first intersection point of this geodesic with $T$. 
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The tree $T$ above can be identified with the Bruhat-Tits building of $SL_2(\mathbb{Q}_p)$ and hence is a model for $E SL_2(\mathbb{Q}_p)$. Since $SL_2(\mathbb{Z})$ is a discrete subgroup of $SL_2(\mathbb{Q}_p)$, we get another reason why this tree is a model for $SL_2(\mathbb{Z})$. 
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Finiteness properties of the spaces $EG$ and $E_G$ have been intensively studied in the literature. We mention a few examples and results. For more information we refer to the lectures of Brown.

If $EG$ has a finite-dimensional model, the group $G$ must be torsionfree. There are often finite models for $E_G$ for groups $G$ with torsion.

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Theorem (Discrete subgroups of Lie groups)

Let $L$ be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup $K$. Let $G \subseteq L$ be a discrete subgroup of $L$. Then $L/K$ with the left $G$-action is a model for $E_G$.

Suppose additionally that $G$ is virtually torsionfree, i.e., contains a torsionfree subgroup $\Delta \subseteq G$ of finite index. Then we have for its virtual cohomological dimension

$$\text{vcd}(G) \leq \dim(L/K).$$

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Theorem (A criterion for 1-dimensional models for $BG$, Stallings (1968), Swan (1969))

Let $G$ be a discrete group. The following statements are equivalent:

- There exists a 1-dimensional model for $EG$;
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Then

$$\text{vcd}(G) \leq \dim(EG)$$

for any model for $EG$.

Let $l \geq 0$ be an integer such that for any chain of finite subgroups $H_0 \subsetneq H_1 \subsetneq \ldots \subsetneq H_r$ we have $r \leq l$.

Then there exists a model for $EG$ of dimension

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The following problem has been stated by Brown (1979) and has created a lot of activities.

**Problem**

For which discrete groups $G$, which are virtually torsionfree, does there exist a $G$-CW-model for $EG$ of dimension $vcd(G)$?

- The results above do give some evidence for a positive answer.
- However, Leary-Nucinkis (2003) have constructed groups, where the answer is negative.

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Let $X$ be a $CW$-complex. Then there exists a group $G$ with $X \simeq G\backslash EG$. 
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Can nice geometric models for $EG$ be used to compute the group homology and more general homology and cohomology theories of a group $G$?

Question ($K$-theory of group rings and group homology)

Is there a relation between $K_n(RG)$ and the group homology of $G$?

Question (Isomorphism Conjectures and classifying spaces of families)

Can classifying spaces of families be used to formulate a version of the Farrell-Jones Conjecture and the Baum-Connes Conjecture which may hold for all group $G$ and all rings?
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