Topological Rigidity

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Conjecture (Borel Conjecture)

Two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

- Explain this conjecture. Put it into a general context. Report on its status.

Theorem (Bartels-Lück (2008))

The Borel Conjecture is true if the fundamental group is hyperbolic or CAT(0).
Some basic notions

Definition (Homeomorphism)

A **homeomorphism** \( f : X \to Y \) between topological spaces is a (continuous) map such that there exists a (continuous) map \( g : Y \to X \) with \( g \circ f = \text{id}_X \) and \( f \circ g = \text{id}_Y \).
Definition (**Manifold**)  

An *n-dimensional manifold* $M$ is a topological space which is locally homeomorphic to $\mathbb{R}^n$, i.e., for every point there is an open neighborhood which is homeomorphic to $\mathbb{R}^n$.  
It is called *closed* if it is compact.
A 2-dimensional orientable closed manifold is homeomorphic to the standard surface of genus $g$ for precisely one $g$. 
Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth map and $y \in \mathbb{R}^n$ be a regular value. Then the preimage $f^{-1}(y)$ is a manifold.
An example is the *n-dimensional sphere*

\[
S^n = \left\{ (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \left| \sum_{i=0}^{n} x_i^2 = 1 \right. \right\}
\]
Real or complex projective spaces are manifolds.
The product of an $m$- and an $n$-dimensional manifold is a $(m + n)$-dimensional manifold.
The connected sum $A \# B$ of two $n$-dimensional manifolds $A$ and $B$ is again one.
Definition (Homotopy)

Two maps $f_0, f_1 : X \to Y$ between topological spaces are called homotopic $f_0 \simeq f_1$, if and only if there is homotopy between them, i.e., a map

$$h : X \times [0, 1] \to Y$$

satisfying $h(x, 0) = f_0(x)$ and $h(x, 1) = f_1(x)$ for all $x \in X$. 
**Definition (Homotopy equivalence)**

A *homotopy equivalence* \( f : X \to Y \) between topological spaces is a map such that there exists a map \( g : Y \to X \) with \( g \circ f \simeq \text{id}_X \) and \( f \circ g \simeq \text{id}_Y \).

- A homeomorphism is a homotopy equivalence.
- The converse is not true in general. For instance \( \mathbb{R}^n \) and \( \mathbb{R}^m \) are homotopy equivalent for all \( m, n \), but they are homeomorphic if and only if \( m = n \).
**Definition (Contractible space)**

A space is called *contractible* if the projection $X \to \{\bullet\}$ to the one-point-space is a homotopy equivalence.

- A space is contractible if and only if the identity is homotopic to a constant map.
- Any convex or star-shaped subset of $\mathbb{R}^n$ is contractible. In particular $\mathbb{R}^n$ is contractible.
- $S^n$ is not contractible.
- A closed $n$-dimensional manifold is contractible if and only if it consists of one point.
Definition (Fundamental group)

Let \((X, x)\) be a pointed space. Its \emph{fundamental group} 
\(\pi_1(X, x)\) has as elements pointed homotopy classes of loops with base point \(x\), i.e., pointed maps 
\((S^1, 1) \to (X, x)\).

The multiplication is given by concatenation of loops. The unit is given by the constant loop. The inverse is given by running around in a loop in the opposite direction.
• If $X$ is path-connected, the fundamental group is independent of the choice of the base point up to group isomorphism.

• If two spaces are homotopy equivalent, their fundamental groups are isomorphic.

• Sending $n \in \mathbb{Z}$ to the loop $S^1 \to S^1$, $z \mapsto z^n$ induces an isomorphism

$$\mathbb{Z} \xrightarrow{\cong} \pi_1(S^1).$$
For $n \geq 2$ the $n$-dimensional sphere $S^n$ is simply-connected, i.e., it is path-connected and its fundamental group is trivial.

Fix $n \geq 4$ and a finitely presented group $G$. Then there exists an $n$-dimensional closed manifold $M$ with $\pi_1(M) \cong G$. 
Let $X$ be a path-connected space. Its universal covering $\tilde{X} \to X$ is the unique covering for which the total space is simply-connected.

Its group of deck transformations can be identified with $\pi_1(X)$. In particular we rediscover $X$ from $\tilde{X}$ by

$$X = \tilde{X}/\pi_1(X).$$

The universal covering of $S^1$ is

$$\mathbb{R} \to S^1, \quad r \mapsto e^{2\pi ir}.$$
Definition (Aspherical)

A path connected space is called \textit{aspherical} if the total space of its universal covering is contractible.

- A path-connected space is aspherical if and only if all its higher homotopy groups vanish.
- The fundamental group of an aspherical closed manifold is torsionfree.
- $S^n$ is aspherical if and only if $n = 1$. 
The orientable closed surface of genus $g$ is aspherical if and only if $g \geq 1$.

An orientable closed 3-manifold is aspherical if and only if its fundamental group is torsionfree, prime and not isomorphic to $\mathbb{Z}$.

A closed Riemannian manifold with non-positive sectional curvature is aspherical.
Let $L$ be a connected Lie group. Let $K \subseteq L$ be a maximal compact subgroup. Let $G \subseteq L$ be a torsionfree discrete subgroup. Then the double coset space

$$M := G\backslash L/K$$

is an aspherical manifold.

A simply connected closed manifold is aspherical if and only if it consists of one point.

Slogan: A “random” closed manifold is expected to be aspherical.
Definition (Topologically rigid)

A closed topological manifold \( M \) is called \textit{topologically rigid} if any homotopy equivalence \( N \to M \) with some manifold \( N \) as source and \( M \) as target is homotopic to a homeomorphism.

- The Poincaré Conjecture in dimension \( n \) is equivalent to the statement that \( S^n \) is topologically rigid.
Theorem (Kreck-Lück (to appear in 2009))

1. Suppose that $k + d \neq 3$. Then $S^k \times S^d$ is topologically rigid if and only if both $k$ and $d$ are odd.

2. Every closed 3-manifold with torsionfree fundamental group is topologically rigid.

3. Let $M$ and $N$ be closed manifolds of the same dimension $n \geq 5$ with torsionfree fundamental groups. If both $M$ and $N$ are topologically rigid, then the same is true for their connected sum $M \# N$. 
Theorem (Chang-Weinberger (2003))

Let $M^{4k+3}$ be a closed oriented smooth manifold for $k \geq 1$ whose fundamental group has torsion. Then $M$ is not topologically rigid.

- Hence in most cases the fundamental group of a topologically rigid manifold is torsionfree.
Conjecture (Borel Conjecture)

The Borel Conjecture for $G$ predicts that a closed aspherical manifold $M$ with $\pi_1(M) \cong G$ is topologically rigid.

- Two aspherical manifolds are homotopy equivalent if and only if their fundamental groups are isomorphic.
- The Borel Conjecture predicts that two aspherical manifolds have isomorphic fundamental groups if and only if they are homeomorphic.
The Borel Conjecture can be viewed as the topological version of **Mostow rigidity**.

One version of Mostow rigidity says that any homotopy equivalence between hyperbolic closed Riemannian manifolds is homotopic to an isometric diffeomorphism.

In particular they are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.
The Borel Conjecture becomes definitely false if one replaces homeomorphism by diffeomorphism.

For instance, there are smooth manifolds $M$ which are homeomorphic to $T^n$ but not diffeomorphic to $T^n$.

The Borel Conjecture is true in dimensions 1 and 2 by classical results. It is true in dimension 3 by Perelman’s proof of Thurston’s Geometrization Conjecture.
Other prominent Conjectures

**Conjecture (Kaplansky Conjecture)**

The **Kaplansky Conjecture** says for a torsionfree group $G$ and an integral domain $R$ that $0$ and $1$ are the only idempotents in $RG$.

**Conjecture (Reduced projective class group)**

If $R$ is a principal ideal domain and $G$ is torsionfree, then $\tilde{K}_0(RG) = 0$. 
Conjecture (Serre)

If $G$ is of type FP, then $G$ is already of type FF.

Conjecture (Whitehead group)

If $G$ is torsionfree, then the Whitehead group $\text{Wh}(G)$ vanishes.

Conjecture (Novikov Conjecture)

The Novikov Conjecture for $G$ predicts for a closed oriented manifold $M$ that its higher signatures over $BG$ are homotopy invariants.
Conjecture (K-theoretic Farrell-Jones Conjecture for regular rings and torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring $R$ for the torsionfree group $G$ predicts that the assembly map

$$H_n(BG; \mathcal{K}_R) \rightarrow K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$. 

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• There is an $L$-theoretic version of the Farrell-Jones Conjecture.

• Both the $K$-theoretic and the $L$-theoretic Farrell-Jones Conjecture can be formulated for arbitrary groups $G$ and arbitrary rings $R$ allowing also a $G$-twist on $R$. 
Theorem (The Farrell-Jones Conjecture implies (nearly) everything)

If $G$ satisfies both the $K$-theoretic and $L$-theoretic Farrell-Jones Conjecture (for any additive $G$-category as coefficients), then all the conjectures mentioned above (and further conjectures) will follow for $G$. 
The Borel Conjecture (for dim \( \geq 5 \)),
Kaplansky Conjecture (for \( R \) a field of characteristic zero),
Vanishing of \( \tilde{K}_0(RG) \) and \( \text{Wh}(G) \),
Serre’s Conjecture,
Novikov Conjecture (for dim \( \geq 5 \)),
other conjecture, e.g., the ones due to Bass and Moody, the one about Poincaré duality groups (for dim \( \geq 5 \)) and the one about the homotopy invariance of \( L^2 \)-torsion.
The status of the Farrell-Jones Conjecture

Theorem (Bartels-Lück (preprint will be available in the beginning of 2009))

Let $\mathcal{FJ}$ be the class of groups for which both the $K$-theoretic and the $L$-theoretic Farrell-Jones Conjectures holds (in his most general form, namely with coefficients in any additive $G$-category) has the following properties:

- Hyperbolic groups, CAT(0)-groups and virtually nilpotent groups belongs to $\mathcal{FJ}$;
Theorem (Continued)

- If $G_1$ and $G_2$ belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 \ast G_2$ belong to $\mathcal{FJ}$;
- If $H$ is a subgroup of $G$ and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;
- Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\text{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$.
Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina (2005)).

There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.

One example is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients (see Higson-Lafforgue-Skandalis (2002)).
However, our results show that these groups do satisfy the Farrell-Jones Conjecture in its most general form and hence also the other conjectures mentioned above.
- Mike Davis (1983) has constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.
- However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.
- Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension $\geq 5$. 
There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:

- Amenable groups;
- $\text{Sl}_n(\mathbb{Z})$ for $n \geq 3$;
- Mapping class groups;
- $\text{Out}(F_n)$;
- Thompson groups.

If one looks for a counterexample, there seems to be no good candidates which do not fall under our main theorems.
Theorem (Bartels-Lück-Weinberger (2009?))

Let $G$ be a torsionfree hyperbolic group and let $n$ be an integer $\geq 6$. Then the following statements are equivalent:

- The boundary $\partial G$ is homeomorphic to $S^{n-1}$;
- There is a closed aspherical topological manifold $M$ such that $G \cong \pi_1(M)$, its universal covering $\tilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\tilde{M}$ by $\partial G$ is homeomorphic to $D^n$. 