Topological rigidity for non-aspherical manifolds

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**Conjecture (Borel Conjecture)**

Let $M$ and $N$ be closed aspherical topological manifolds. Then every homotopy equivalence $M \to N$ is homotopic to a homeomorphism.

**Conjecture ($n$-dimensional Poincaré Conjecture)**

Let $M$ be a closed topological manifold. Then every homotopy equivalence $M \to S^n$ is homotopic to a homeomorphism.
Manifold will always mean connected oriented closed topological manifold.

**Definition (Borel-manifold)**

A manifold $M$ is called a *Borel manifold* if for any orientation preserving homotopy equivalence $f : N \to M$ of manifolds there exists an orientation preserving homeomorphism $h : N \to M$ such that $f$ and $h$ induce the same map on the fundamental groups up to conjugation. It is called a *strong Borel manifold* if every orientation preserving homotopy equivalence $f : N \to M$ of manifolds is homotopic to a homeomorphism $h : N \to M$. 
The Borel Conjecture is equivalent to the statement that every aspherical manifold is strongly Borel.

If $M$ is aspherical, then: Borel $\iff$ Strongly Borel.

The $n$-dimensional Poincaré Conjecture is equivalent to the statement that $S^n$ is strongly Borel.

Both conjectures become false in the smooth category.

Question: Which manifolds are (strongly) Borel?

Slogan: Interpolation between the Borel and the Poincaré Conjecture.
- If \( \dim(M) \leq 2 \), then \( M \) is strongly Borel.
- The Lens space \( L(7, 1, 1) \) is not Borel.
- The following assertions are equivalent:
  - \( S^1 \times S^2 \) is strongly Borel;
  - \( S^1 \times S^2 \) is Borel;
  - The 3-dimensional Poincaré Conjecture is true.

Idea of proof: Suppose \( S^1 \times S^2 \) is Borel. Then:

\[
M \cong S^3 \Rightarrow M \# (S^1 \times S^2) \cong S^3 \# (S^1 \times S^2) \cong S^1 \times S^3
\]

\[
\Rightarrow M \# (S^1 \times S^2) \cong S^3 \# (S^1 \times S^2) \Rightarrow M \cong S^3.
\]

The other direction uses the prime decomposition and the characterization of \( S^1 \times S^2 \) as the only non-irreducible prime 3-manifold with infinite \( \pi_1 \).
Theorem (Dimension 3)

Suppose that Thurston’s Geometrization Conjecture for irreducible 3-manifolds with infinite fundamental group and the 3-dimensional Poincaré Conjecture are true. Then every 3-manifold with torsionfree fundamental group is a strong Borel manifold.

- The main input in the proof are Waldhausen’s rigidity results for Haken manifolds.
- Conclusion: If $\pi_1(M)$ is torsionfree, then $\pi_1(M)$ determines the homeomorphism type.
Example (Examples in dimension 4)

- Let $M$ be a 4-manifold with Spin structure such that its fundamental group is finite cyclic. Then $M$ is Borel. This follows from a classification result of Hambleton-Kreck.
- If $M$ is simply connected and Borel, then it has a Spin structure. This follows results from the star operation $M \mapsto \ast M$.
- $T^4$ and $S^1 \times S^3$ are strongly Borel.
- $S^2 \times S^2$ is Borel but not strongly Borel.
Theorem (Connected sums)

Let $M$ and $N$ be manifolds of the same dimension $n \geq 5$ such that neither $\pi_1(M)$ nor $\pi_1(N)$ contains elements of order 2 or that $n = 0, 3 \mod 4$.

If both $M$ and $N$ are (strongly) Borel, then the same is true for their connected sum $M \# N$.

- The proof is based on Cappell’s work on splitting obstructions and of UNIL-groups and recent improvements by Banagl, Connolly, Davis, Ranicki.
Theorem (Products of two spheres)

Suppose that \( k + d \neq 3 \). Then \( S^k \times S^d \) is a strong Borel manifold if and only if both \( k \) and \( d \) are odd;

Suppose \( k, d > 1 \) and \( k + d \geq 4 \). Then the manifold \( S^k \times S^d \) is Borel if and only if the following conditions are satisfied:

1. Neither \( k \) nor \( d \) is divisible by 4;
2. If \( k = 2 \mod 4 \), then there is a map \( g_k: S^k \times S^d \to S^d \) such that its Arf invariant \( \text{Arf}(g_k) \) is non-trivial and its restriction to \( \text{pt} \times S^d \) is an orientation preserving homotopy equivalence \( \text{pt} \times S^d \to S^d \);
3. The same condition with the role of \( k \) and \( d \) interchanged.
The condition (2) appearing in the last Theorem implies that the Arf invariant homomorphism

$$\text{Arf}_k : \Omega_k^{fr} \rightarrow \mathbb{Z}/2$$

is surjective. This is the famous \textit{Arf-invariant-one-problem}. 
Definition (structure set)

The *structure set* $S^{\text{top}}(M)$ of a manifold $M$ consists of equivalence classes of orientation preserving homotopy equivalences $N \to M$ with a manifold $N$ as source. Two such homotopy equivalences $f_0 : N_0 \to M$ and $f_1 : N_1 \to M$ are equivalent if there exists a homeomorphism $g : N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

- Let $\text{ho-aut}_{\pi_1}(M)$ be the group of homotopy classes of self equivalences inducing the identity on $\pi_1$ up to conjugation.
- It acts on the structure set by composition.
Theorem (Surgery criterion for Borel manifolds)

- A manifold $M$ is a strong Borel manifold if and only if $S^{\text{top}}(M)$ consists of one element;
- A manifold $M$ is a Borel manifold if and only if $S^{\text{top}}(M)/\text{ho-aut}_{\pi}(M)$ consists of one element.
Theorem (Ranicki)

There is an exact sequence of abelian groups called \textit{algebraic surgery exact sequence}

\[ \cdots \xrightarrow{\sigma_{n+1}} H_{n+1}(M; L\langle 1 \rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} \]
\[ S^{\text{top}}(M) \xrightarrow{\sigma_n} H_n(M; L\langle 1 \rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \cdots \]

It can be identified with the classical geometric surgery sequence due to Sullivan and Wall in high dimensions.

- \( M \) is strongly Borel \( \iff \) \( A_{n+1} \) is surjective and \( A_n \) is injective.
- The Farrell-Jones Conjecture predicts for torsionfree \( \pi \) that

\[ H_n(B\pi; L) \xrightarrow{A_n} L_n(\mathbb{Z}\pi) \]

is bijective for \( n \in \mathbb{Z} \).
Example

- Consider $M = S^k \times S^d$ for $k + d \geq 4$.
- Then $\pi_1(M)$ is trivial and the assembly map can be identified with

$$H_m(S^k \times S^d; L\langle 1 \rangle) \to H_m(\text{pt}; L\langle 1 \rangle).$$

- $S^{\text{top}}(S^k \times S^d) \cong H_{k+d}(S^k \times S^d, \text{pt}; L\langle 1 \rangle) \cong L_d(\mathbb{Z}) \oplus L_k(\mathbb{Z})$.
- Hence $S^k \times S^d$ is strongly Borel if and only if $k$ and $d$ are odd.
- To prove that $S^k \times S^d$ is Borel, one has to construct enough selfhomotopy equivalences of $S^k \times S^d$. 
Theorem (A necessary homological criterion for being Borel)

Let $M$ be a Borel manifold and let $c : M \to B\pi$ be the classifying map. Then for every $i \geq 1$ with $\mathcal{L}(M)_i = 0$ the map

$$c_* : H_{n-4i}(M; \mathbb{Q}) \to H_{n-4i}(B\pi; \mathbb{Q})$$

is injective.

- This criterion is obviously empty for aspherical manifolds.
- Input in the proof: The image of $[f : N \to M]$ under the map

$$S^{\text{top}}(M) \xrightarrow{\sigma_n} H_n(M; \mathbb{L}\langle 1 \rangle) \to \bigoplus_{i \geq 1} H_{4i+n}(M; \mathbb{Q})$$

is

$$f_* (\mathcal{L}(N) \cap [N]) - \mathcal{L}(M) \cap [M].$$
Theorem (Sphere bundles over surfaces)

Let $K$ be $S^1$ or a 2-dimensional manifold different from $S^2$. Let $S^d \to E \to K$ be a fiber bundle over $K$ for $d \geq 3$. Then $E$ is a Borel manifold. It is a strong Borel manifold if and only if $K = S^1$. 

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Theorem (Sphere bundles over 3-manifolds)

Let $K$ be an aspherical 3-dimensional manifold. Suppose that the Farrell-Jones Conjecture holds for $\pi_1(K)$. Let $S^d \to E \overset{p}{\to} K$ be a fiber bundle over $K$ with orientable $E$ such that $d \geq 4$ or such that $d = 2, 3$ and there is a map $i : K \to E$ with $p \circ i \simeq \text{id}_K$. Then

- $E$ is strongly Borel if and only if $H_1(K; \mathbb{Z}/2) = 0$;
- If $d = 3 \mod 4$ and $d \geq 7$, then $K \times S^d$ is Borel;
- If $d = 0 \mod 4$ and $d \geq 8$ and $H_1(K; \mathbb{Z}/2) \neq 0$, then $K \times S^d$ is not Borel.
Theorem (Chang-Weinberger)

Let $M^{4k+3}$ be a manifold for $k \geq 1$ whose fundamental group has torsion.
Then there are infinitely many pairwise not homeomorphic smooth manifolds which are homotopy equivalent to $M$ but not homeomorphic to $M$. In particular $M$ is not Borel.
Theorem (Homology spheres)

Let $M$ be a manifold of dimension $n \geq 5$ with fundamental group $\pi = \pi_1(M)$.

- Let $M$ be an integral homology sphere. Then $M$ is a strong simple Borel manifold if and only if

$$L^s_{n+1}(\mathbb{Z}) \xrightarrow{\cong} L^s_{n+1}(\mathbb{Z}_\pi).$$

- Suppose that $M$ is a rational homology sphere and Borel. Suppose that $\pi$ satisfies the Novikov Conjecture. Then

$$H_{n+1-4i}(B\pi; \mathbb{Q}) = 0$$

for $i \geq 1$ and $n + 1 - 4i \neq 0$. 

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Theorem (Another construction of strongly Borel manifolds)

Start with a strongly Borel manifold $M$ of dimension $n \geq 5$. Choose an embedding $S^1 \times D^{n-1} \to M$ which induces an injection on $\pi_1$. Choose a high dimensional knot $K \subseteq S^n$ with complement $X$ such that the inclusion $\partial X \cong S^1 \times S^{n-2} \to X$ induces an isomorphism on $\pi_1$. Put

$$M' = M - (S^1 \times D^{n-1}) \cup S^1 \times S^{n-2} \ X.$$ 

Then $M'$ is strongly Borel.

- If $M$ is aspherical, then $M'$ is in general not aspherical.
Problem (Classification of certain low-dimensional manifolds)

Classify up to orientation preserving homotopy equivalence, homeomorphism (or diffeomorphism in the smooth case) all manifolds in dimension $1 \leq k < n \leq 6$ satisfying:

- $\pi = \pi_1(M)$ is isomorphic to $\pi_1(K)$ for a manifold $K$ of dimension $k \leq 2$.
- $\pi_2(M)$ vanishes.

- The case $\pi = \{1\}$ was already solved by Wall;
- In dimension $\leq 5$ we give a complete answer in terms of the second Stiefel Whitney class.
- In dimension 6 we give in the Spin case a complete answer in terms of the equivariant intersection pairing of the universal covering.
- Such a manifold is never strongly Borel but always Borel.
In nearly all examples of Borel manifolds we have constructed — what we call — a *generalized topological space form*, i.e., manifolds $M$, whose universal covering $\tilde{M}$ is contractible or homotopy-equivalent to a wedge of $k$-spheres $S^k$ for some $2 \leq k < \infty$. 

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