Topological Rigidity of Aspherical Manifolds

Wolfgang Lück
Münster
Germany
email lueck@math.uni-muenster.de
http://www.math.uni-muenster.de/u/lueck/

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Present a list of prominent conjectures such as the one due to Borel, Farrell-Jones, Kaplansky and Novikov.

State our main theorem which is joint work with Bartels. It says that these conjectures are true for an interesting class of groups including word-hyperbolic groups and CAT(0)-groups.

Discuss consequences and open cases.
The Borel Conjecture

Definition (Topologically rigid)
A closed topological manifold $M$ is called topologically rigid if any homotopy equivalence $N \to M$ with some manifold $N$ as source and $M$ as target is homotopic to a homeomorphism.

- The Poincaré Conjecture in dimension $n$ is equivalent to the statement that $S^n$ is topologically rigid.
Theorem (Kreck-Lück (2006))

- Suppose that $k + d \neq 3$. Then $S^k \times S^d$ is topologically rigid if and only if both $k$ and $d$ are odd.

- If Thurston’s Geometrization Conjecture is true, then every closed 3-manifold with torsionfree fundamental group is topologically rigid.

- Let $M$ and $N$ be closed manifolds of the same dimension $n \geq 5$ such that neither $\pi_1(M)$ nor $\pi_1(N)$ contains elements of order 2. If both $M$ and $N$ are topologically rigid, then the same is true for their connected sum $M \# N$. 
Theorem (Chang-Weinberger (2003))

Let \( M^{4k+3} \) be a closed oriented smooth manifold for \( k \geq 1 \) whose fundamental group has torsion. Then \( M \) is not topologically rigid.

- Hence in most cases the fundamental group of a topologically rigid manifold is torsionfree.
Definition \textbf{(Aspherical manifold)}

A manifold \( M \) is called \textit{aspherical} if \( \pi_n(M) = 0 \) for \( n \geq 2 \), or, equivalently, \( \tilde{M} \) is contractible.

- If \( M \) is a closed smooth Riemannian manifold with non-positive sectional curvature, then it is aspherical.
- Let \( L \) be a connected Lie group, \( K \subset L \) a maximal compact Lie group and \( G \subset L \) a discrete torsionfree group. Then \( G\backslash L/K \) is an aspherical closed smooth manifold.
Conjecture (Borel Conjecture)

The Borel Conjecture for $G$ predicts that a closed aspherical manifold $M$ with $\pi_1(M) \cong G$ is topologically rigid.

- Two aspherical manifolds are homotopy equivalent if and only if their fundamental groups are isomorphic.
- The Borel Conjecture predicts that two aspherical manifolds have isomorphic fundamental groups if and only if they are homeomorphic.
The Borel Conjecture can be viewed as the topological version of **Mostow rigidity**.

One version of Mostow rigidity says that any homotopy equivalence between hyperbolic closed Riemannian manifolds is homotopic to an isometric diffeomorphism.

In particular they are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.
The Borel Conjecture becomes definitely false if one replaces homeomorphism by diffeomorphism.

For instance, there are smooth manifolds $M$ which are homeomorphic to $T^n$ but not diffeomorphic to $T^n$. 
Other prominent Conjectures

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group $G$ and an integral domain $R$ that $0$ and $1$ are the only idempotents in $RG$.

Conjecture (Reduced projective class group)

If $R$ is a principal ideal domain and $G$ is torsionfree, then $\tilde{K}_0(RG) = 0$. 
The vanishing of $\tilde{K}_0(RG)$ is equivalent to the statement that any finitely generated projective $RG$-module $P$ is stably free, i.e., there are $m, n \geq 0$ with $P \oplus RG^m \cong RG^n$;

Let $G$ be a finitely presented group. The vanishing of $\tilde{K}_0(\mathbb{Z}G)$ is equivalent to the geometric statement that any finitely dominated space $X$ with $G \cong \pi_1(X)$ is homotopy equivalent to a finite $CW$-complex.

The last conjecture implies the Conjecture due to Serre that a group of type FP is already of type FF.
**Conjecture (Whitehead group)**

If $G$ is torsionfree, then the Whitehead group $\text{Wh}(G)$ vanishes.

- Fix $n \geq 6$. The vanishing of $\text{Wh}(G)$ is equivalent to the following geometric statement: Every compact $n$-dimensional $h$-cobordism $W$ with $G \cong \pi_1(W)$ is trivial.
Conjecture (Novikov Conjecture)

The *Novikov Conjecture* for $G$ predicts for a closed oriented manifold $M$ together with a map $f : M \to BG$ that for any $x \in H^*(BG)$ the *higher signature*

$$\text{sign}_x(M, f) := \langle \mathcal{L}(M) \cup f^* x, [M] \rangle$$

is an oriented homotopy invariant of $(M, f)$. 
Definition (Poincaré duality group)

A group is called a \textit{Poincaré duality group of dimension} \( n \) if it is of type \( \text{FP} \) and

\[
H^i(G; \mathbb{Z}G) \cong \begin{cases} 
\{0\} & \text{for } i \neq n; \\
\mathbb{Z} & \text{for } i = n.
\end{cases}
\]
Conjecture (Poincaré duality groups)

Let $G$ be a finitely presented Poincaré duality group. Then there is a closed ANR-homology manifold with $\pi_1(M) \cong G$.

- One may also hope that $M$ can be chosen to be a closed manifold.
- But then one runs into Quinn’s resolutions obstruction.
The Farrell-Jones Conjecture and its consequences

Conjecture (\textit{K}-theoretic Farrell-Jones Conjecture for regular rings and torsion free groups)

The \textit{K}-theoretic Farrell-Jones Conjecture with coefficients in the regular ring \( R \) for the torsion free group \( G \) predicts that the assembly map

\[ H_n(BG; K_R) \rightarrow K_n(RG) \]

is bijective for all \( n \in \mathbb{Z} \).
There is an $L$-theoretic version of the Farrell-Jones Conjecture.

Both the $K$-theoretic and the $L$-theoretic Farrell-Jones Conjecture can be formulated for arbitrary groups $G$ and arbitrary rings $R$ allowing also a $G$-twist on $R$. 
Theorem (The Farrell-Jones Conjecture implies (nearly) everything)

If $G$ satisfies both the $K$-theoretic and $L$-theoretic Farrell-Jones Conjecture (for any additive $G$-category as coefficients), then all the conjectures mentioned above follow for $G$, i.e., for the Borel Conjecture (for $\dim \geq 5$), Kaplansky Conjecture, Vanishing of $\tilde{K}_0(RG)$ and $\text{Wh}(G)$, Novikov Conjecture (for $\dim \geq 5$), Serre’s Conjecture, Conjecture about Poincaré duality groups, and other conjecture as well.
We want to explain this for the Borel Conjecture.

**Definition (Structure set)**

The *structure set* $S^{\text{top}}(M)$ of a manifold $M$ consists of equivalence classes of homotopy equivalences $N \to M$ with a manifold $N$ as source. Two such homotopy equivalences $f_0: N_0 \to M$ and $f_1: N_1 \to M$ are equivalent if there exists a homeomorphism $g: N_0 \to N_1$ with $f_1 \circ g \simeq f_0$.

**Theorem**

A closed manifold $M$ is topologically rigid if and only if $S^{\text{top}}(M)$ consists of one element.
Theorem (Algebraic surgery sequence Ranicki (1992))

There is an exact sequence of abelian groups called algebraic surgery exact sequence for an $n$-dimensional closed manifold $M$

$$
\ldots \xrightarrow{\sigma_{n+1}} H_{n+1}(M; \mathbb{L}\langle 1 \rangle) \xrightarrow{A_{n+1}} L_{n+1}(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_{n+1}} S^{\text{top}}(M) \xrightarrow{\sigma_n} H_n(M; \mathbb{L}\langle 1 \rangle) \xrightarrow{A_n} L_n(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial_n} \ldots
$$

It can be identified with the classical geometric surgery sequence due to Browder, Novikov, Sullivan and Wall in high dimensions.
\begin{itemize}
\item $S^{\text{top}}(M)$ consist of one element if and only if $A_{n+1}$ is surjective and $A_n$ is injective.
\item $H_k(M; L\langle 1 \rangle) \rightarrow L_k(\mathbb{Z}G)$ is bijective for $k \geq n + 1$ and injective for $k = n$ if $M = BG$ and both the $K$-theoretic and $L$-theoretic Farrell-Jones Conjectures hold for $G = \pi_1(M)$ and $R = \mathbb{Z}$.
\end{itemize}
The status of the Farrell-Jones Conjecture

**Theorem (Main Theorem Bartels-Lück (2008))**

Let $\mathcal{FJ}$ be the class of groups for which both the $K$-theoretic and the $L$-theoretic Farrell-Jones Conjectures holds (in his most general form, namely with coefficients in any additive $G$-category) has the following properties:

- Hyperbolic group and virtually nilpotent groups belongs to $\mathcal{FJ}$;
- If $G_1$ and $G_2$ belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 \ast G_2$ belong to $\mathcal{FJ}$;

Wolfgang Lück (Münster, Germany)
If $H$ is a subgroup of $G$ and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;

Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\text{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;

If we demand on the $K$-theory version only that the assembly map is 1-connected and keep the full $L$-theory version, then the properties above remain valid and the class $\mathcal{FJ}$ contains also all CAT(0)-groups.
Limit groups in the sense of Zela are CAT(0)-groups (Alibegovic-Bestvina (2005)).

There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.

One example is the construction of groups with expanders due to Gromov. These yield counterexamples to the Baum-Connes Conjecture with coefficients (see Higson-Lafforgue-Skandalis (2002)).
However, our results show that these groups do satisfy the Farrell-Jones Conjecture in its most general form and hence also the other conjectures mentioned above.

Bartels-Echterhoff-Lück (2007) show that the Bost Conjecture with coefficients in $C^*$-algebras is true for colimits of hyperbolic groups. Thus the failure of the Baum-Connes Conjecture with coefficients comes from the fact that the change of rings map

$$K_0(\mathcal{A} \rtimes_{\ell^1} G) \rightarrow K_0(\mathcal{A} \rtimes_{C^*_r} G)$$

is not bijective for all $G$-$C^*$-algebras $\mathcal{A}$. 
Mike Davis (1983) has constructed exotic closed aspherical manifolds using hyperbolization techniques. For instance there are examples which do not admit a triangulation or whose universal covering is not homeomorphic to Euclidean space.

However, in all cases the universal coverings are CAT(0)-spaces and hence the fundamental groups are CAT(0)-groups.

Hence by our main theorem they satisfy the Farrell-Jones Conjecture and hence the Borel Conjecture in dimension $\geq 5$. 
There are still many interesting groups for which the Farrell-Jones Conjecture in its most general form is open. Examples are:

- Amenable groups;
- $SL_n(\mathbb{Z})$ for $n \geq 3$;
- Mapping class groups;
- $\text{Out}(F_n)$;
- Thompson groups.

If one looks for a counterexample, there seems to be no good candidates which do not fall under our main theorems.
Computational aspects

**Theorem (The $K$- and $L$-theory of torsionfree hyperbolic groups)**

Let $G$ be a torsionfree hyperbolic group and let $R$ be a ring. Then we get isomorphisms

$$H_n(BG; K_R) \bigoplus \left( \bigoplus_{(C), C \subseteq G, C \neq 1, C \text{ maximal cyclic}} NK_n(R) \right) \overset{\cong}{\longrightarrow} K_n(RG)$$

and

$$H_n(BG; L_R^{(-\infty)}) \overset{\cong}{\longrightarrow} L_n^{(-\infty)}(RG);$$
Theorem (Bartels-Lück-Weinberger (in progress))

Let $G$ be a torsionfree hyperbolic group and let $n$ be an integer $\geq 5$. Then the following statements are equivalent:

- The boundary $\partial G$ is homeomorphic to $S^{n-1}$;
- There is a closed aspherical topological manifold $M$ such that $G \cong \pi_1(M)$, its universal covering $\tilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\tilde{M}$ by $\partial G$ is homeomorphic to $D^n$. 