Hyperbolic groups with spheres as boundary

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Conjecture (Gromov (1994))

Let $G$ be a hyperbolic group whose boundary is a sphere $S^{n-1}$. Then there is a closed aspherical manifold $M$ with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger (2011))

The Conjecture is true for $n \geq 6$.

We also deal with the questions:

- When is a Poincaré duality group the fundamental group of an aspherical closed manifold?
- When is an aspherical closed manifold a product?
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- When is an aspherical closed manifold a product?
A $\delta$-hyperbolic space $X$ is a geodesic space whose geodesic triangles are all $\delta$-thin. A geodesic space is called hyperbolic if it is $\delta$-hyperbolic for some $\delta > 0$.

- A geodesic space with bounded diameter is hyperbolic.
- A tree is 0-hyperbolic.
- A simply connected complete Riemannian manifold $M$ with $\sec(M) \leq \kappa$ for some $\kappa < 0$ is hyperbolic.
- $\mathbb{R}^n$ is hyperbolic if and only if $n \leq 1$. 

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Definition (Boundary of a hyperbolic space)

Let $X$ be a hyperbolic space. Define its boundary $\partial X$ to be the set of equivalence classes of geodesic rays. Put

$$\overline{X} := X \sqcup \partial X.$$ 

Two geodesic rays $c_1, c_2 : [0, \infty) \to X$ are called equivalent if there exists $C > 0$ satisfying $d_X(c_1(t), c_2(t)) \leq C$ for $t \in [0, \infty)$. 

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Lemma

There is a topology on $\overline{X}$ with the properties:

- $\overline{X}$ is compact and metrizable;
- The subspace topology $X \subseteq \overline{X}$ is the given one;
- $X$ is open and dense in $\overline{X}$.

Let $M$ be a simply connected complete Riemannian manifold $M$ with $\sec(M) \leq \kappa$ for some $\kappa < 0$. Then $M$ is hyperbolic and $\partial M = S^{\dim(M) - 1}$. 
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Definition (Quasi-isometry)

A map \( f : X \rightarrow Y \) of metric spaces is called a quasi-isometry if there exist real numbers \( \lambda, C > 0 \) satisfying:

- The inequality
  \[
  \lambda^{-1} \cdot d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \lambda \cdot d_X(x_1, x_2) + C
  \]
  holds for all \( x_1, x_2 \in X \);
- For every \( y \) in \( Y \) there exists \( x \in X \) with \( d_Y(f(x), y) < C \).
Lemma (Švarc-Milnor Lemma)

Let $X$ be a geodesic space. Suppose that $G$ acts properly, cocompactly and isometrically on $X$. Choose a base point $x \in X$. Then the map

$$f : G \to X, \quad g \mapsto gx$$

is a quasiisometry.

Lemma (Quasi-isometry invariance of the Cayley graph)

The quasi-isometry type of the Cayley graph of a finitely generated group is independent of the choice of a finite set of generators.
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The property “hyperbolic” is a quasi-isometry invariant of geodesic spaces.

Lemma (Quasi-isometry invariance of the boundary)

A quasi-isometry \( f : X_1 \to X_2 \) of hyperbolic spaces induces a homeomorphism

\[ \partial X_1 \xrightarrow{\simeq} \partial X_2. \]

Definition (Hyperbolic group)

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Define the boundary \( \partial G \) of a hyperbolic group to be the boundary of its Cayley graph.
A group $G$ is hyperbolic if and only if it acts properly, cocompactly and isometrically on a hyperbolic space. In this case $\partial G = \partial X$.

Let $M$ be a closed Riemannian manifold with $\sec(M) < 0$. Then $\pi_1(M)$ is hyperbolic with $S^{\dim(M) - 1}$ as boundary.

If $G$ is virtually torsionfree and hyperbolic, then $\vcd(G) = \dim(\partial G) + 1$.

If the boundary of a hyperbolic groups contains an open subset homeomorphic to $\mathbb{R}^n$, then the boundary is homeomorphic to $S^n$.

A subgroup of a hyperbolic group is either virtually cyclic or contains $\mathbb{Z} \ast \mathbb{Z}$ as subgroup. In particular $\mathbb{Z}^2$ is not a subgroup of a hyperbolic group.
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A subgroup of a hyperbolic group is either virtually cyclic or contains $\mathbb{Z} \ast \mathbb{Z}$ as subgroup. In particular $\mathbb{Z}^2$ is not a subgroup of a hyperbolic group.
• A free product of two hyperbolic groups is again hyperbolic.
• A direct product of two finitely generated groups is hyperbolic if and only if one of the two groups is finite and the other is hyperbolic.
• The Rips complex of a hyperbolic group $G$ is a cocompact model for its classifying space $E_G$ for proper actions. This implies that there is a model of finite type for $BG$ and hence that $G$ is finitely presented and that there are only finitely many conjugacy classes of finite subgroups.
• A finitely generated torsion group is hyperbolic if and only if it is finite.
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A hyperbolic group has $S^1$ as boundary if and only if it is a Fuchsian group.

### Conjecture (Cannon’s Conjecture)

A hyperbolic group $G$ has $S^2$ as boundary if and only if it acts properly, cocompactly and isometrically on $\mathbb{H}^3$.

### Theorem (Bestvina-Mess (1991))

Let $G$ be an infinite hyperbolic group which is the fundamental group of a closed irreducible 3-manifold $M$. Then $M$ is hyperbolic and $G$ satisfies Cannon’s Conjecture.

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Gromov's Conjecture in low dimensions


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ANR-homology manifolds

Definition (Absolute neighborhood retract (ANR))

A topological space $X$ is called absolute neighborhood retract (ANR) if it is normal and for every normal space $Z$, which contains $X$ as a closed subset, there exists an open neighborhood $U$ of $X$ in $Z$ together with a retraction of $U$ onto $X$. 
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Definition (Homology ANR-manifold)

A homology ANR-manifold $X$ is an ANR satisfying:

- $X$ has a countable basis for its topology;
- The topological dimension of $X$ is finite;
- $X$ is locally compact;
- for every $x \in X$ we have for the singular homology

$$H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

If $X$ is additionally compact, it is called a closed ANR-homology manifold.

There is also the notion of a compact ANR-homology manifold with boundary.
Every closed topological manifold is a closed ANR-homology manifold.

Let $M$ be homology sphere with non-trivial fundamental group. Then its suspension $\Sigma M$ is a closed ANR-homology manifold but not a topological manifold.
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• Let $M$ be homology sphere with non-trivial fundamental group. Then its suspension $\Sigma M$ is a closed ANR-homology manifold but not a topological manifold.
**Definition (Disjoint Disk Property (DDP))**

A homology ANR-manifold $M$ has the **disjoint disk property (DDP)**, if for any $\epsilon > 0$ and maps $f, g : D^2 \to M$, there are maps $f', g' : D^2 \to M$ so that $f'$ is $\epsilon$-close to $f$, $g'$ is $\epsilon$-close to $g$ and $f'(D^2) \cap g'(D^2) = \emptyset$.

- A topological manifold of dimension $\geq 5$ is a closed ANR-homology manifold, which has the DDP by transversality.
A homology ANR-manifold $M$ has the disjoint disk property (DDP), if for any $\epsilon > 0$ and maps $f, g : D^2 \to M$, there are maps $f', g' : D^2 \to M$ so that $f'$ is $\epsilon$-close to $f$, $g'$ is $\epsilon$-close to $g$ and $f'(D^2) \cap g'(D^2) = \emptyset$.

- A topological manifold of dimension $\geq 5$ is a closed ANR-homology manifold, which has the DDP by transversality.
Definition (Poincaré duality group)

A Poincaré duality group $G$ of dimension $n$ is a finitely presented group satisfying:

- $G$ is of type FP;
- $H^i(G; \mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Lemma

Let $X$ be a closed aspherical ANR-homology manifold of dimension $n$. Then its fundamental group is a Poincaré duality group of dimension $n$. 

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Theorem (Poincaré duality groups and ANR-homology manifolds
Bartels-Lück-Weinberger (2011))

Let $G$ be a torsionfree group. Suppose that its satisfies the $K$- and
$L$-theoretic Farrell-Jones Conjecture. Consider $n \geq 6$.

Then the following statements are equivalent:

1. $G$ is a Poincaré duality group of dimension $n$;
2. There exists a closed aspherical $n$-dimensional ANR-homology
   manifold $M$ with $\pi_1(M) \cong G$;
3. There exists a closed aspherical $n$-dimensional ANR-homology
   manifold $M$ with $\pi_1(M) \cong G$ which has the DDP.

If the first statements holds, then the homology ANR-manifold $M$
appearing above is unique up to s-cobordism of ANR-homology manifolds.
The proof of the result above relies on

- Surgery theory as developed by Browder, Novikov, Sullivan, Wall for smooth manifolds and its extension to topological manifolds using the work of Kirby-Siebenmann.
- The algebraic surgery theory of Ranicki.
- The surgery theory for ANR-manifolds due to Bryant-Ferry-Mio-Weinberger and basic ideas of Quinn.
- The Farrell-Jones Conjecture.
The Farrell-Jones Conjecture

Conjecture (K-theoretic Farrell-Jones Conjecture for torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring $R$ for the torsionfree group $G$ predicts that the assembly map

$$H_n(BG; K_R) \rightarrow K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- There is also a version for $L$-theory.
- The most general version called Full Farrell-Jones Conjecture makes sense for all groups and all possible coefficient rings and twistings and extensions with finite groups as quotient.
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- There is also a version for $L$-theory.
- The most general version called \textit{Full Farrell-Jones Conjecture} makes sense for all groups and all possible coefficient rings and twistings and extensions with finite groups as quotient.
Theorem (Bartels, Echterhoff, Farrell, Lück, Reich, Roushon, Rüping, Wegner, Wu)

Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:

- Hyperbolic groups belong to $\mathcal{FJ}$;
- CAT(0)-groups belong to $\mathcal{FJ}$;
- Virtually poly-cyclic groups belong to $\mathcal{FJ}$;
- Solvable groups belong to $\mathcal{FJ}$;
- Cocompact lattices in almost connected Lie groups belong to $\mathcal{FJ}$;
- All 3-manifold groups belong to $\mathcal{FJ}$;
- If $R$ is a ring whose underlying abelian group is finitely generated free, then $SL_n(R)$ and $GL_n(R)$ belong to $\mathcal{FJ}$ for all $n \geq 2$;
- All arithmetic groups belong to $\mathcal{FJ}$.
- All Baumslag-Solitar groups belong to $\mathcal{FJ}$.
Moreover, $\mathcal{FJ}$ has the following inheritance properties:

- If $G_1$ and $G_2$ belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 \ast G_2$ belong to $\mathcal{FJ}$;
- If $H$ is a subgroup of $G$ and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;
- If $H \subseteq G$ is a subgroup of $G$ with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;
- Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\text{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;
Theorem (Bestvina-Mess (1991))

A hyperbolic $G$ is a Poincaré duality group of dimension $n$ if and only if its boundary and $S^{n-1}$ have the same Čech cohomology.

Corollary

Let $G$ be a torsionfree word-hyperbolic group. Let $n \geq 6$.

Then the following statements are equivalent:

1. The boundary $\partial G$ has the integral Čech cohomology of $S^{n-1}$;
2. $G$ is a Poincaré duality group of dimension $n$;
3. There exists a closed aspherical $n$-dimensional ANR-homology manifold $M$ with $\pi_1(M) \cong G$;
4. There exists a closed aspherical $n$-dimensional ANR-homology manifold $M$ with $\pi_1(M) \cong G$ which has the DDP.

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Quinn’s resolution obstruction

Theorem (Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:

- if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;
- $i(M \times N) = i(M) \cdot i(N)$;
- Let $M$ be a homology ANR-manifold of dimension $\geq 5$. Then $M$ is a topological manifold if and only if $M$ has the DDP and $\iota(M) = 1$. 
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Theorem (Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:

- if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;
- $i(M \times N) = i(M) \cdot i(N)$;
- Let $M$ be a homology ANR-manifold of dimension $\geq 5$. Then $M$ is a topological manifold if and only if $M$ has the DDP and $\iota(M) = 1$. 

Wolfgang Lück (Bonn)

Hyperbolic groups with spheres as boundary

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Question

Does the Quinn obstruction always vanishes for aspherical closed homology ANR-manifolds?

- If the answer is yes, we can replace “closed ANR-homology manifold” by “closed topological manifold” in the theorem above.
- In general the Quinn obstruction is not a homotopy invariant but it is a homotopy invariant for aspherical closed ANR-homology manifolds, provided that the Farrell-Jones Conjecture holds.
- However, some experts expect the answer no.
- I am not an expert and hope that the answer is yes.
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Theorem (Quasi-isometry invariance of Quinn’s resolution obstruction
Bartels-Lück-Weinberger (2011))

Let $G_1$ and $G_2$ be torsionfree hyperbolic groups.

- Let $G_1$ and $G_2$ be quasi-isometric. Then $G_1$ is a Poincaré duality group of dimension $n$ if and only $G_2$ is;

- Let $M_i$ be an aspherical closed ANR-homology manifold with $\pi_1(M_i) \cong G_i$ for $i = 1, 2$. If $\partial G_1$ and $\partial G_2$ are homeomorphic, then the Quinn obstructions of $M_1$ and $M_2$ agree;

- Let $G_1$ and $G_2$ be quasi-isometric. Then there exists an aspherical closed topological manifold $M_1$ with $\pi_1(M_1) = G_1$ if and only if there exists an aspherical closed topological manifold $M_2$ with $\pi_1(M_2) = G_2$. 
Theorem (Hyperbolic groups with spheres as boundary
Bartels-Lück-Weinberger (2011))

Let $G$ be a torsionfree hyperbolic group and let $n$ be an integer $\geq 6$. Then the following statements are equivalent:

1. The boundary $\partial G$ is homeomorphic to $S^{n-1}$;
2. There is a closed aspherical topological manifold $M$ such that $G \cong \pi_1(M)$, its universal covering $\tilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\tilde{M}$ by $\partial G$ is homeomorphic to $D^n$.

If the first statement is true, the manifold appearing above is unique up to homeomorphism.
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Exotic Examples

By hyperbolization techniques due to Charney, Davis, Januskiewicz one can find the following
examples:

Examples (Exotic universal coverings)

Given \( n \geq 5 \), there are aspherical closed topological manifolds \( M \) of
dimension \( n \) with hyperbolic fundamental group \( G = \pi_1(M) \) satisfying:

- The universal covering \( \tilde{M} \) is not homeomorphic to \( \mathbb{R}^n \) and \( \partial G \) is not
  homeomorphic to \( S^{n-1} \).
- \( M \) is smooth and \( \tilde{M} \) is homeomorphic to \( \mathbb{R}^n \) but \( \partial G \) is not \( S^{n-1} \).
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- $M$ is smooth and $\tilde{M}$ is homeomorphic to $\mathbb{R}^n$ but $\partial G$ is not $S^{n-1}$. 
Example (No smooth structures)

For every $k \geq 2$ there exists a torsionfree hyperbolic group $G$ with $\partial G \cong S^{4k-1}$ such that there is no aspherical closed smooth manifold $M$ with $\pi_1(M) \cong G$. In particular $G$ is not the fundamental group of a closed smooth Riemannian manifold with $\text{sec}(M) < 0$.

Example (No triangulation)

For any $n \geq 6$ there exists a hyperbolic group which is the fundamental group of an aspherical topological manifold but not the fundamental group of an aspherical triangulable topological manifold.
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Direct product decompositions of aspherical closed manifolds

**Theorem (Product decomposition Lück (2010))**

Let $M$ be a closed aspherical manifold of dimension $n$ with $n \neq 3, 4$ with fundamental group $G = \pi_1(M)$ together with a product decomposition

$$p_1 \times p_2 : G \cong G_1 \times G_2.$$

Suppose that $G$ satisfy the Farrell-Jones Conjecture and that the cohomological dimension of $G_1$ and $G_2$ is different from 3, 4 and 5.
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Then

1. There are topological closed aspherical manifolds $M_1$ and $M_2$ together with maps $f_i : M \to M_i$ for $i = 1, 2$ such that

   $$f = f_1 \times f_2 : M \to M_1 \times M_2$$

   is a homeomorphism and $\pi_1(f_i) = p_i$.

2. The decomposition above is unique up to homeomorphism.
Problems

Can one give an example of a hyperbolic group (with torsion) whose boundary is a sphere, such that the group does not act properly discontinuously on some contractible manifold?

Let $p: M \to N$ be a map of aspherical closed manifolds whose homotopy fiber is homotopy equivalent to a connected $CW$-complex of finite type. When is $p$ homotopy equivalent to the projection of a locally trivial fiber bundle with a connected closed aspherical topological manifold as typical fiber?
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