VANISHING OF NIL-TERMS AND NEGATIVE $K$-THEORY FOR ADDITIVE CATEGORIES

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Abstract. We extend notions such as Noetherian, regular, or regular coherent for rings to additive categories. We show that well-known properties for rings carry over to additive categories. For instance, the negative $K$-groups and all twisted Nil-groups vanish for an additive category if it is regular. Moreover, the additive category of twisted finite Laurent series associated to any automorphism of a Noetherian or regular additive category is again Noetherian or regular.

Introduction

The goal of this paper is to find conditions ensuring that for an additive category $A$ its algebraic $K$-groups $K_n(A)$ vanish for $n \in \mathbb{Z}, n \leq -1$ and that for every automorphism $\Phi: A \xrightarrow{\cong} A$ the twisted Nil-terms $NK_n(A_\Phi[t])$ vanish for all $n \in \mathbb{Z}$. The vanishing of the twisted Nil-terms together with the twisted Bass-Heller-Swan decomposition, see Theorem 3.2, leads to the long exact Wang sequence

\[
\ldots \xrightarrow{\partial_{n+1}} K_n(A) \xrightarrow{id-K_n(\Phi)} K_n(A) \xrightarrow{j} K_n(A_\Phi[t, t^{-1}]) \xrightarrow{\partial_n} K_{n-1}(A) \xrightarrow{id-K_{n-1}(\Phi)} K_{n-1}(A) \xrightarrow{k_{n-1}(j)} K_{n-1}(A_\Phi[t, t^{-1}]) \xrightarrow{\partial_{n-1}} \ldots
\]

where $A_\Phi[t, t^{-1}]$ is the additive category given by $\Phi$-twisted finite Laurent series over $A$, see Definition 1.3, and $j: A \to A_\Phi[t, t^{-1}]$ is the inclusion.

The special case of finitely generated projective modules over a ring.

Consider a ring $R$ with an automorphism $\phi: R \xrightarrow{\cong} R$. Let $A = R\text{-MOD}_{fgp}$ be the additive category of finitely generated projective $R$-modules. Then the following assertions are well-known to be true. The algebraic $K$-group $K_n(R) = K_n(A)$ vanishes for $n \leq -1$ if $R$ is regular, and $NK_n(R_\phi[t, t^{-1}]) = NK_n(A_\phi[t, t^{-1}])$ vanishes for $n \in \mathbb{Z}$ for the automorphism $\Phi: A \to A$ induced by $\phi$, if $R$ is regular coherent. If $R$ is regular, then $R_\phi[t, t^{-1}]$ is regular.

Main results. We extend all of this from rings to additive categories. We introduce for any additive category $A$ the notions Noetherian, regular coherent, and regular, and show that the analogue results are true. Namely, $K_n(A)$ vanishes for $n \in \mathbb{Z}, n \leq -1$ if $A$ is regular, see Corollary 11.2. For every automorphism $\Phi: A \xrightarrow{\cong} A$ the twisted Nil-terms $NK_n(A_\phi[t])$ vanish for all $n \in \mathbb{Z}$ if $A$ is regular coherent, see Subsection 6.3. If $A$ is regular, then $A_\phi[t, t^{-1}]$ is regular, see Theorem 9.1.

These new notions for additive categories are defined in terms of the Yoneda-embedding which is also used in the proof of the various claims. One can also give intrinsic definitions of these notions. For instance, we call a sequence $A_0 \xrightarrow{f_0} \ldots$
$A_1 \xrightarrow{f_1} A_2$ in $A$ exact at $A_1$, if $f_1 \circ f_0 = 0$ and for every object $A$ and morphism $g: A \to A_1$ with $f_1 \circ g = 0$ there exists a morphism $\overline{g}: A \to A_0$ with $f_0 \circ \overline{g} = g$, see Definition 1.9. We show in Lemma 5.6 (iv) that an idempotent complete additive category $A$ is regular coherent if and only if for every morphism $f_1: A_1 \to A_0$ we can find a sequence of finite length in $A$

$$0 \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0$$

which is exact at $A_i$ for $i = 1, 2, \ldots, n$. An intrinsic definition of Noetherian is given in Lemma 5.8. As for rings, we call $A$ regular if and only if $A$ is Noetherian and regular coherent.

We will also introduce the notion of $l$-uniformly regular coherent which in contrast to the other notions behaves well under directed unions, sums, and products of additive categories over arbitrary index sets, see Section 11.

**Future applications to the algebraic $K$-theory of Hecke algebras of totally disconnected groups.** Our motivation comes from the theorem that for the Hecke algebra $H(K)$ of a compact totally disconnected group $K$ we have $K_n(A(K)) = 0$ for $n \leq -1$ and for any automorphism $\phi: K \to K$ we get for the semi-direct product $G = K \rtimes \phi \mathbb{Z}$ a long exact Wang sequence

$$
\cdots \xrightarrow{\partial_{n+1}} K_n(H(K)) \xrightarrow{\text{id} - K_n(H(\phi))} K_n(H(K)) \xrightarrow{\partial_n} K_{n-1}(H(K))
$$

$$
\xrightarrow{\text{id} - K_{n-1}(H(\phi))} K_{n-1}(H(K)) \xrightarrow{K_{n-1}(j)} K_{n-1}(H(G)) \xrightarrow{\partial_{n-1}} \cdots .
$$

Its proof is based on the material of this paper and will be given in another paper. Our ultimate goal is to prove the Farrell-Jones Conjecture for the algebraic $K$-theory of the Hecke algebra of a totally disconnected group $G$ in the most interesting case that $G$ is a reductive $p$-adic group. The Wang sequence (0.2) is a special case of this conjecture and will play a key role in its proof for reductive $p$-adic groups.

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2. The algebraic $K$-theory of $\mathbb{Z}$-categories
1. **Z-categories, additive categories and idempotent completions**

1.1. **Z-categories.** A Z-category is a category \( A \) such that for every two objects \( A \) and \( A' \) in \( A \) the set of morphism \( \text{mor}_A(A, A') \) has the structure of a \( \mathbb{Z} \)-module for which composition is a \( \mathbb{Z} \)-bilinear map. Given a ring \( R \), we denote by \( \mathbb{Z} \)-category \( R \) the \( \mathbb{Z} \)-category with precisely one object whose \( \mathbb{Z} \)-module of endomorphisms is given by \( R \) with its \( \mathbb{Z} \)-module structure and composition is given by the multiplication in \( R \).

1.2. **Additive categories.** An additive category is a \( \mathbb{Z} \)-category such that for any two objects \( A_1 \) and \( A_2 \) there is a model for their direct sum, i.e., an object \( A \) together with morphisms \( i_k : A_k \to A \) for \( k = 1, 2 \) such that for every object \( B \) in \( A \) the \( \mathbb{Z} \)-map

\[
\text{mor}_A(A, B) \cong \text{mor}_A(A_1, B) \times \text{mor}_A(A_2, B), \quad f \mapsto (f \circ i_0, f \circ i_1)
\]

is bijective.

Given a ring \( R \), the category \( \text{R-MOD}_{fgf} \) of finitely generated free left \( R \)-modules carries an obvious structure of an additive category.

An equivalence \( F : A \to A' \) of \( \mathbb{Z} \)-categories or of additive categories respectively is a functor of \( \mathbb{Z} \)-categories or of additive categories respectively such that for all objects \( A_1, A_2 \) in \( A \) the induced map \( F_{A_1, A_2} : \text{mor}_A(A_1, A_2) \xrightarrow{\cong} \text{mor}_{A'}(F(A_1), F(A_2)) \) sending \( f \) to \( F(f) \) is bijective, and for any object \( A' \) in \( A' \) there exists an object \( A \) in \( A \) such that \( F(A) \) and \( A' \) are isomorphic in \( A' \). This is equivalent to the existence of a functor \( F : A' \to A \) of \( \mathbb{Z} \)-categories or of additive categories respectively such that both composites \( F \circ F' \) and \( F' \circ F \) are naturally equivalent as such functors to the identity functors.

Given a \( \mathbb{Z} \)-category, let \( A_{\oplus} \) be the associated additive category whose objects are finite tuples of objects in \( A \) and whose morphisms are given by matrices of
morphisms in $\mathcal{A}$ (of the right size) and the direct sum is given by concatenation of tuples and the block sum of matrices, see for instance [3] Section 1.3.

Let $R$ be a ring. Then we can consider the additive category $R\text{-}_\mathcal{A}$ of right $\mathcal{A}$-modules. We get an equivalence of additive categories

$$\theta_{R\text{-}_\mathcal{A}}: R\text{-}_\mathcal{A} \cong R\text{-}\text{MOD}_{\text{fgp}}$$

is an equivalence of additive categories. Note that $R\text{-}_\mathcal{A}$ is small, in contrast to $R\text{-}\text{MOD}_{\text{fgp}}$.

1.3. Idempotent completion. Given an additive category $\mathcal{A}$, its idempotent completion $\text{Idem}(\mathcal{A})$ is defined to be the following additive category. Objects are morphisms $p: A \to A$ in $\mathcal{A}$ satisfying $p \circ p = p$. A morphism $f$ from $p_1: A_1 \to A_2$ to $p_2: A_2 \to A_2$ is a morphism $f: A_1 \to A_2$ in $\mathcal{A}$ satisfying $p_2 \circ f \circ p_1 = f$. The structure of an additive category on $\mathcal{A}$ induces the structure of an additive category on $\text{Idem}(\mathcal{A})$ in the obvious way. The identity of an object $(A, p)$ is given by the morphism $p: (A, p) \to (A, p)$. A functor of additive categories $F: \mathcal{A} \to \mathcal{A}'$ induces a functor $\text{Idem}(F): \text{Idem}(\mathcal{A}) \to \text{Idem}(\mathcal{A}')$ of additive categories by sending $(A, p)$ to $(F(A), F(p))$.

There is an obvious embedding

$$\eta(\mathcal{A}): \mathcal{A} \to \text{Idem}(\mathcal{A})$$

sending an object $A$ to $\text{id}_A: A \to A$ and a morphism $f: A \to B$ to the morphism given by $f$ again. An additive category $\mathcal{A}$ is called idempotent complete if $\eta(\mathcal{A}): \mathcal{A} \to \text{Idem}(\mathcal{A})$ is an equivalence of additive categories, or, equivalently, if for every idempotent $p: A \to A$ in $\mathcal{A}$ there exists objects $B$ and $C$ and an isomorphism $f: A \cong B \oplus C$ in $\mathcal{A}$ such that $f \circ p \circ f^{-1}: B \oplus C \to B \oplus C$ is given by

$$\begin{pmatrix} \text{id}_B & 0 \\ 0 & 0 \end{pmatrix}.$$  

The idempotent completion $\text{Idem}(\mathcal{A})$ of an additive category $\mathcal{A}$ is idempotent complete.

For a ring $R$, let $R\text{-}\text{MOD}_{\text{fgp}}$ be the additive category of finitely generated projective $R$-modules. We get an equivalence of additive categories $\text{Idem}(R\text{-}\text{MOD}_{\text{fgp}}) \cong R\text{-}\text{MOD}_{\text{fgp}}$ by sending an object $(F, p)$ to $\text{im}(p)$. It and the functor of (1.1) induce an equivalence of additive categories

$$\theta_{R \text{-MOD}_{\text{fgp}}}: \text{Idem}(R\text{-}_\mathcal{A}) \cong R\text{-}\text{MOD}_{\text{fgp}}$$

Notice that $\text{Idem}(R\text{-}_\mathcal{A})$ is small, in contrast to $R\text{-}\text{MOD}_{\text{fgp}}$.

1.4. Twisted finite Laurent category. Let $\mathcal{A}$ be an additive category. Let $\Phi: \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories.

**Definition 1.3** (Twisted finite Laurent category $\mathcal{A}_\Phi[t, t^{-1}]$). Define the $\Phi$-twisted finite Laurent category $\mathcal{A}_\Phi[t, t^{-1}]$ as follows. It has the same objects as $\mathcal{A}$. Given two objects $A$ and $B$, a morphism $f: A \to B$ in $\mathcal{A}_\Phi[t, t^{-1}]$ is a formal sum $f = \sum_{i \in \mathbb{Z}} f_i \cdot t^i$, where $f_i: \Phi^i(A) \to B$ is a morphism in $\mathcal{A}$ from $\Phi^i(A)$ to $B$ and only finitely many of the morphisms $f_i$ are non-trivial. If $g = \sum_{j \in \mathbb{Z}} g_j \cdot t^j$ is a morphism in $\mathcal{A}_\Phi[t, t^{-1}]$ from $B$ to $C$, we define the composite $g \circ f: A \to C$ by

$$g \circ f := \sum_{k \in \mathbb{Z}} \left( \sum_{i+j=k} g_j \circ \Phi^i(f_i) \right) \cdot t^k.$$  

The direct sum and the structure of a $\mathbb{Z}$-module on the set of morphism from $A$ to $B$ in $\mathcal{A}_\Phi[t, t^{-1}]$ are defined in the obvious way using the corresponding structures of $\mathcal{A}$.

We sometimes also write $\mathcal{A}_\Phi[\mathbb{Z}]$ instead of $\mathcal{A}_\Phi[t, t^{-1}]$.  

Example 1.4. Let $R$ be a ring with an automorphism $\phi: R \xrightarrow{\cong} R$ of rings. Let $R_\phi[t, t^{-1}]$ be the ring of $\phi$-twisted finite Laurent series with coefficients in $R$. We obtain from $\phi$ an automorphism $\Phi: R \xrightarrow{\cong} R$ of $\mathbb{Z}$-categories. There is an obvious isomorphism of $\mathbb{Z}$-categories

$$R_\phi[t, t^{-1}] \cong R_\phi[t, t^{-1}].$$

We obtain equivalences of additive categories

$$(R_\phi)_\phi[t, t^{-1}] \cong R_\phi[t, t^{-1}]$$

$\text{Idem}((R_\phi)_\phi[t, t^{-1}]) \cong R_\phi[t, t^{-1}]$$

Definition 1.6 (P_\phi[t] and P_\phi[t^{-1}]). Let $P_\phi[t]$ and $P_\phi[t^{-1}]$ respectively be the additive subcategory of $A_\phi[t, t^{-1}]$ whose set of objects is the set of objects in $A$ and whose morphism from $A$ to $B$ are given by finite formal Laurent series $\sum_{i\in\mathbb{Z}} f_i \cdot t^i$ with $f_i = 0$ for $i < 0$ and $i > 0$ respectively.

2. The Algebraic $K$-Theory of $\mathbb{Z}$-Categories

Let $A$ be an additive category. One can interprete it as an exact category in the sense of Quillen or as a category with cofibrations and weak equivalence in the sense of Waldhausen and obtains the connective algebraic $K$-theory spectrum $K(A)$ by the constructions due to Quillen [6] or Waldhausen [9]. A construction of the non-connective $K$-theory spectrum $K^\infty(A)$ of an additive category can be found for instance in [3] or [5].

Definition 2.1 (Algebraic $K$-theory of $\mathbb{Z}$-categories). We will define the algebraic $K$-theory spectrum $K^\infty(A)$ of the $\mathbb{Z}$-category $A$ to be the non-connective algebraic $K$-theory spectrum of the additive category $A_\mathbb{Z}$. Define for $n \in \mathbb{Z}$

$$K_n(A) := \pi_n(K^\infty(A)).$$

The connective algebraic $K$-theory spectrum $K(A)$ is defined to be the connective algebraic $K$-theory spectrum of the additive category $A_\mathbb{Z}$.

If $A$ is an additive category and $i(A)$ is the underlying $\mathbb{Z}$-category, then there is a canonical equivalence of additive categories $i(A)_\mathbb{Z} \to A$. Hence there are canonical weak homotopy equivalences $K(i(A)) \to K(A)$ and $K^\infty(i(A)) \to K^\infty(A)$.

A functor $F: A \to A'$ of $\mathbb{Z}$-categories induces a map of spectra

$$K^\infty(F): K^\infty(A) \to K^\infty(A').$$

We call a full additive subcategory $A$ of $A'$ cofinal if for any object $A'$ in $A'$ there is an object $A$ in $A$ together with morphisms $i: A' \to A$ and $r: A' \to A$ satisfying $r \circ i = \text{id}$.

Lemma 2.3. Let $I: A \to A'$ be the inclusion of a full cofinal additive subcategory.

(i) The induced map

$$\pi_n(K(I)): \pi_n(K(A)) \to \pi_n(K(A'))$$

is bijective for $n \geq 1$;

(ii) The induced map

$$K^\infty(I): K^\infty(A) \to K^\infty(A')$$

is a weak homotopy equivalence.

Proof. (i) This is proved for $A' = \text{Idem}(A)$ in [8] Theorem A.9.1.]. Now the general case follows from the observation that $\text{Idem}(A) \to \text{Idem}(A')$ is an equivalence of additive categories.

(ii) This follows from assertion (i) and [3] Corollary 3.7. \qed
3. The Bass-Heller-Swan decomposition for additive categories

Denote by Add-Cat the category of additive categories. Let us consider the group $\mathbb{Z}$ as a groupoid with one object and denote by $\text{Add-Cat}^\mathbb{Z}$ the category of functors $\mathbb{Z} \to \text{Add-Cat}$, with natural transformations as morphisms. Note that an object of this category is a pair $(\mathcal{A}, \Phi)$ consisting of an additive category together with an automorphism $\Phi: \mathcal{A} \xrightarrow{\cong} \mathcal{A}$ of additive categories. We recall from [3] Theorem 0.1 and Theorem 0.4 using the notation of this paper here and in the sequel:

**Theorem 3.1** (The Bass-Heller-Swan decomposition for non-connective $K$-theory of additive categories). Let $\Phi: \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories.

(i) There exists a weak homotopy equivalence of spectra, natural in $(\mathcal{A}, \Phi)$,

$$a^\infty \vee b^\infty \vee b^\infty : T_{K(\Phi^{-1})} \vee NK^\infty(\mathcal{A}[t]) \vee NK^\infty(\mathcal{A}[t^{-1}]) \xrightarrow{\cong} K^\infty(\mathcal{A}[t, t^{-1}])$$

where $T_{K(\Phi^{-1})}$ is the mapping torus of $K^\infty(\Phi^{-1}): K^\infty(\mathcal{A}) \to K^\infty(\mathcal{A})$ and $NK^\infty(\mathcal{A}[t])$ is the homotopy fiber of the map $K^\infty(\mathcal{A}[t^{-1}]) \to K^\infty(\mathcal{A})$ given by evaluation $t = 0$;

(ii) There exist a functor $E^\infty: \text{Add-Cat}^\mathbb{Z} \to \text{Spectra}$ and weak homotopy equivalences of spectra, natural in $(\mathcal{A}, \Phi)$,

$$\Omega NK^\infty(\mathcal{A}[t]) \xleftarrow{\sim} E^\infty(\mathcal{A}, \Phi);$$

$$K^\infty(\mathcal{A}) \vee E^\infty(\mathcal{A}, \Phi) \xrightarrow{\cong} K^\infty_{\text{Nil}}(\mathcal{A}, \Phi),$$

where $K^\infty_{\text{Nil}}(\mathcal{A}, \Phi)$ is the non-connective $K$-theory of a certain Nil-category $\text{Nil}(\mathcal{A}, \Phi)$.

**Theorem 3.2** (Fundamental sequence of $K$-groups). Let $\mathcal{A}$ be an additive category. Then there exists for $n \in \mathbb{Z}$ a split exact sequence, natural in $\mathcal{A}$

$$0 \to K_n(\mathcal{A}) \xrightarrow{(k_+, \oplus-)(k_-)} K_n(\mathcal{A}[t]) \oplus K_n(\mathcal{A}[t^{-1}]) \xrightarrow{(l_+), \oplus-(l_-),} K_n(\mathcal{A}[t, t^{-1}]) \xrightarrow{\delta_n} K_{n-1}(\mathcal{A}) \to 0,$$

where $(k_+)$, $(k_-)$, $(l_+)$, and $(l_-)$ are induced by the obvious inclusions $k_+$, $k_-$, $l_+$, and $l_-$ and $\delta_n$ is the composite of the inverse of the (untwisted) Bass-Heller-Swan isomorphism

$$K_n(\mathcal{A}) \oplus K_{n-1}(\mathcal{A}) \oplus NK_n(\mathcal{A}[t]) \oplus NK_n(\mathcal{A}[t^{-1}]) \xrightarrow{\cong} K_n(\mathcal{A}[t, t^{-1}]),$$

see Theorem 2.3 with the projection onto the summand $K_{n-1}(\mathcal{A})$.

**Proof.** This follows directly from the untwisted version of Theorem 3.1. \qed

There is also a version for the connective $K$-theory spectrum $\mathbf{K}$. Denote by $\text{Add-Cat}_\text{ic} \subset \text{Add-Cat}$ the full subcategory of idempotent complete additive categories.

**Theorem 3.4** (The Bass-Heller-Swan decomposition for connective $K$-theory of additive categories). Let $\mathcal{A}$ be an additive category which is idempotent complete. Let $\Phi: \mathcal{A} \to \mathcal{A}$ be an automorphism of additive categories.

(i) Then there is a weak equivalence of spectra, natural in $(\mathcal{A}, \Phi)$,

$$a \vee b_+ \vee b_- : T_{K(\Phi^{-1})} \vee NK(\mathcal{A}[t]) \vee NK(\mathcal{A}[t^{-1}]) \xrightarrow{\cong} K(\mathcal{A}[t, t^{-1}])$$

where $T_{K(\Phi^{-1})}$ is the mapping torus of $K(\Phi^{-1}): K(\mathcal{A}) \to K(\mathcal{A})$ and $NK(\mathcal{A}[t])$ is the homotopy fiber of the map $K(\mathcal{A}[t^{-1}]) \to K(\mathcal{A})$ given by evaluation $t = 0$;
(ii) There exist a functor $E: (\text{Add}-\text{Cat}_{\omega})^2 \to \text{Spectra}$ and weak homotopy equivalences of spectra, natural in $(A, \Phi)$,

$$\Omega NK(A, [t]) \subseteq E(A, \Phi);$$

$$K(A) \vee E(A, \Phi) \xrightarrow{\cong} K(\text{Nil}(A, \Phi)),$$

where $K(\text{Nil}(A, \Phi))$ is the connective $K$-theory of a certain Nil-category $\text{Nil}(A, \Phi)$.

The purpose of the following sections is to find properties of $\mathcal{A}$, which imply for any automorphism $\Phi$ the vanishing of the Nil-terms above and are hopefully inherited by the passage from $\mathcal{A}$ to $\mathcal{A}[t, t^{-1}]$.

4. $Z\mathcal{A}$-modules and the Yoneda Embedding

4.1. Basics about $Z\mathcal{A}$-modules. Let $\mathcal{A}$ be a $Z$-category. We denote by $Z\mathcal{A}$-MOD and $\text{MOD-}Z\mathcal{A}$ respectively the abelian category of covariant or contravariant respectively functors of $Z$-categories $\mathcal{A}$ to $Z$-MOD. The abelian structure comes from the abelian structure in $Z$-MOD. For instance, a sequence $F_0 \xrightarrow{f_1} F_1 \xrightarrow{f_2} F_2$ in $\text{MOD-}Z\mathcal{A}$ is declared to be exact if for each object $A \in \mathcal{A}$ the evaluation at $A$ yields an exact sequence of $Z$-modules $F_0(A) \xrightarrow{T_1(A)} F_1(A) \xrightarrow{T_2(A)} F_2(A)$. The cokernel and kernel of a morphism $T: F_0 \to F_1$ are defined by taking for each object $A \in \mathcal{A}$ the kernel or cokernel of the morphism $T(A): F_0(A) \to F_1(A)$ in $\text{MOD-}Z\mathcal{A}$.

In the sequel $Z\mathcal{A}$-module means contravariant $Z\mathcal{A}$-module unless specified explicitly differently.

Given an object $A$ in $\mathcal{A}$ we obtain an object $\text{mor}_\mathcal{A}(?, A)$ in $\text{MOD-}Z\mathcal{A}$ by assigning to an object $B$ the $Z$-module $\text{mor}_\mathcal{A}(B, A)$ and to a morphism $g: B_0 \to B_1$ the $Z$-homomorphism $g^*: \text{mor}_\mathcal{A}(B_1, A) \to \text{mor}_\mathcal{A}(B_0, A)$ given by precomposition with $g$.

The elementary proof of the following lemma is left to the reader.

Lemma 4.1 (Yoneda Lemma). For each object $A$ in $\mathcal{A}$ and each object $M$ in $\text{MOD-}Z\mathcal{A}$ we obtain an isomorphism of $Z$-modules

$$\text{mor}_{\text{MOD-}Z\mathcal{A}}(\text{mor}_\mathcal{A}(?, A), M(\cdot)) \xrightarrow{\cong} M(A), \quad T \mapsto T(A)(\text{id}_A).$$

We call a $Z\mathcal{A}$-module $M$ free if it is isomorphic as $Z\mathcal{A}$-module to $\bigoplus_J \text{mor}_\mathcal{A}(?, A_i)$ for some collection of objects $\{A_i \mid i \in I\}$ in $\mathcal{A}$ for some index set $I$. A $Z\mathcal{A}$-module $M$ is called projective if for any epimorphism $p: N_0 \to N_1$ of $Z\mathcal{A}$-modules and morphism $f: M \to N_1$ there is a morphism $\bar{f}: M \to N_0$ with $p\bar{f} = f$. A $Z\mathcal{A}$-module $M$ is finitely generated if there exists a collection of objects $\{A_j \mid j \in J\}$ in $\mathcal{A}$ for some finite index set $J$ and an epimorphism of $Z\mathcal{A}$-modules $\bigoplus_{j \in J} \text{mor}_\mathcal{A}(?, A_j) \to M$. Equivalently, $M$ is finitely generated if there exists a finite collection of objects $\{A_j \mid j \in J\}$ in $\mathcal{A}$ together with elements $x_j \in M(A_j)$ such that for any object $A$ and any $x \in M(A)$ there are morphisms $\varphi: A \to A_j$ such that $x = \sum_J M(\varphi_j)(x_j)$.

(The $x_j$ are the images of $\text{id}_{A_j}$ under the above epimorphism.) Given a collection of objects $\{A_i \mid i \in I\}$ in $\mathcal{A}$ for some index set $I$, the free $Z\mathcal{A}$-module $\bigoplus_I \text{mor}_\mathcal{A}(?, A_i)$ is finitely generated if and only if $I$ is finite. A $Z\mathcal{A}$-module $M$ is finitely presented if there are finitely generated free $Z\mathcal{A}$-modules $F_1$ and $F_0$ and an exact sequence $F_1 \to F_0 \to M \to 0$. We say that a $Z\mathcal{A}$-module has projective dimension $\leq d$, denoted by $\text{pd}_{Z\mathcal{A}}(M) \leq d$, for a natural number $d$ if there exists an exact sequence $0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ such that each $Z\mathcal{A}$-module $P_i$ is projective. If we replace projective by free, we get an equivalent definition if $d \geq 1$. We call a $Z\mathcal{A}$-module of type $FL$ or of type $FP$ respectively if there exists an exact sequence of finite length $0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ such
that each $\mathbb{Z}A$-module $F_i$ is finitely generated free or finitely generated projective respectively.

**Remark 4.2.** Note the setting in this paper is different from the one appearing in [1], since here a $\mathbb{Z}A$-module $M$ satisfies $M(f + g) = M(f) + M(g)$ for two morphisms $f, g: A \to B$ which is not required in [1]. Nevertheless many of the arguments in [1] carry over to the setting of this paper because of the Yoneda Lemma [1] which replaces the corresponding Yoneda Lemma in [1] Subsection 9.16 on page 167.

However, the next result has no analogue in the setting of [1].

**Lemma 4.3.** Let $A$ be an additive category. For a $\mathbb{Z}A$-module $M$ and objects $A_1, A_2, \ldots, A_n$ we obtain a natural isomorphism

$$
\bigoplus_{i=1}^n M(\text{pr}_i): \bigoplus_{i=1}^n M(A_i) \xrightarrow{\cong} M\left(\bigoplus_{i=1}^n A_i\right)
$$

where $\text{pr}_j: \bigoplus_{i=1}^n A_i \to A_j$ is the canonical projection for $j = 1, 2, \ldots, n$.

**Proof.** One easily checks using the fact that the functor $M$ is compatible with the $\mathbb{Z}$-module structures on the morphisms that the inverse is given

$$
M\left(\bigoplus_{i=1}^n A_i\right) \to \bigoplus_{i=1}^n M(A_i), \quad x \mapsto (M(k_i)(x)),
$$

where $k_j: A_j \to \bigoplus_{i=1}^n A_i$ is the inclusion of the $j$-the summand for $j = 1, 2, \ldots, n$. \qed

**Lemma 4.4.** Let $A$ be a $\mathbb{Z}$-category.

(i) Every free $\mathbb{Z}A$-module is projective;

(ii) Let $0 \to M \to M' \to M'' \to 0$ be an exact sequence of $\mathbb{Z}A$-modules. If both $M$ and $M''$ are free or projective respectively, then $M'$ is free or projective respectively;

(iii) Let $0 \to M \to M' \to M'' \to 0$ be an exact sequence of $\mathbb{Z}A$-modules. If two of the $\mathbb{Z}A$-modules $M$, $M'$ and $M''$ are of type FL or FP respectively, then all three are of type FL or FP respectively;

(iv) Let $C_\ast$ be a projective $\mathbb{Z}A$-chain complex i.e., a $\mathbb{Z}A$-chain complex all whose chain modules $C_n$ are projective. Then the following assertions are equivalent

(a) Consider a natural number $d$. Let $B_d(C_\ast)$ be the image of $c_{d+1}: C_{d+1} \to C_d$ and $j: B_d(C) \to C_d$ be the inclusion. There is a $\mathbb{Z}A$-submodule $C_d^+$ such that for the inclusion $i: C_d^+ \to C_d$ the map $i \oplus j: C_d^+ \oplus B_d(C_\ast) \to C_d$ is an isomorphism. Moreover, the following chain map from a $d$-dimensional projective $\mathbb{Z}A$-chain complex to $C_\ast$ is a $\mathbb{Z}A$-chain homotopy equivalence

\[
\begin{array}{cccccccc}
\cdots & 0 & 0 & C_d^+ & C_{d-1} & \cdots & 0 & C_0 \\
\cdots & c_{d+2} & c_{d+1} & 0 & c_{d-1} & \cdots & C_d & C_0 \\
\end{array}
\]

(b) $C_\ast$ is $\mathbb{Z}A$-chain homotopy equivalent to a $d$-dimensional projective $\mathbb{Z}A$-chain complex;

(c) $C_\ast$ is dominated by $d$-dimensional projective $\mathbb{Z}A$-chain complex $D_\ast$, i.e., there are $\mathbb{Z}A$-chain maps $i: C_\ast \to D_\ast$ and $r_\ast: D_\ast \to C_\ast$ satisfying $r_\ast \circ i_\ast \simeq \text{id}_{C_\ast}$.
Lemma 4.1. We conclude from \( (4.5) \) for a module an identification \( F \) sends a contravariant \( Z \) sequence associated to a short exact sequence of (co)chain complexes, since every \( Z \)-module is isomorphic to \( Z \)-module of the shape \( \text{mor}_\mathcal{A}(?, A) \) for an appropriate object \( A \) in \( \mathcal{A} \).

Proof. (i) This follows from the Yoneda Lemma 4.1.

(ii) This is obviously true.

(iii) The proof is analogous to the one of [1, Lemma 11.6 on page 216].

(iv) The proof is analogous to the one of [1, Proposition 11.10 on page 221].

(v) This follows from (iv) for the projective dimension using the long exact (co)homology sequence associated to a short exact sequence of (co)chain complexes, since every \( Z \)-module has a free resolution by the Yoneda Lemma 4.1.

(vi) This is obvious and hence the proof of Lemma 4.4 is finished.

If \( M \) and \( N \) are \( \mathcal{A} \)-modules, then \( \text{hom}_{\mathcal{A}}(M, N) \) is the \( Z \)-module of \( \mathcal{A} \)-homomorphisms \( M \to N \). Given a contravariant or covariant \( \mathcal{A} \)-module and a \( Z \)-module \( T \), then we obtain a covariant or contravariant \( \mathcal{A} \)-module \( \text{hom}_Z(M, T) \) by sending an object \( A \) to \( \text{hom}_Z(M(A), T) \). Given a contravariant \( \mathcal{A} \)-module \( M \) and covariant \( \mathcal{A} \)-module \( N \), their tensor product \( M \otimes_Z N \) is the \( Z \)-module given by \( \bigoplus_{A \in \text{ob}(\mathcal{A})} M(A) \otimes_Z N(A) / T \) where \( T \) is the \( Z \)-submodule of \( \bigoplus_{A \in \text{ob}(\mathcal{A})} M(A) \otimes_Z N(A) \) generated by elements of the form \( m f \otimes n - m \otimes f n \) for a morphism \( f : A \to B \) in \( \mathcal{A} \), \( m \in M(A) \) and \( n \in N(B) \), where \( m f := M(f)(m) \) and \( f n = N(f)(n) \). It is characterized by the property that for any \( Z \)-module \( T \), there are natural adjunction isomorphisms

\[
\text{hom}_Z(M \otimes_Z N, T) \cong \text{hom}_Z(M, \text{hom}_Z(N, T));
\]

\[
\text{hom}_Z(M \otimes_Z N, T) \cong \text{hom}_Z(N, \text{hom}_Z(M, T)).
\]

Let \( F : \mathcal{A} \to \mathcal{B} \) be a functor of \( Z \)-categories. Then the restriction functor

\[
F^* : \text{MOD-}Z\mathcal{B} \to \text{MOD-}Z\mathcal{A}.
\]

is given by precomposition with \( F \). The induction functor

\[
F_* : \text{MOD-}Z\mathcal{A} \to \text{MOD-}Z\mathcal{B}.
\]

sends a contravariant \( \mathcal{A} \)-module \( M \) to \( M(?) \otimes_{\mathcal{A}} \text{mor}_\mathcal{B}(?, F(?)) \). We get for a \( Z\mathcal{B} \)-module an identification \( F^* N = \text{hom}_{Z\mathcal{B}}(\text{mor}_\mathcal{A}(?, F(?)), N(??)) \) from the Yoneda Lemma 4.1. We conclude from (4.5)

\[
\text{hom}_{Z\mathcal{B}}(F, M, N) \cong \text{hom}_{\mathcal{A}}(M, F^* N)
\]

for a \( \mathcal{A} \)-module \( M \) and a \( Z\mathcal{B} \)-module \( N \). The counit \( \beta(N) : F_* F^*(N) \to N \) of the adjunction \( F_* \) is the adjoint of \( \text{id}_{F_* N} \) and sends the equivalence class of \( x \otimes f \), with for \( x \in N(F(A)) \) and \( f \in \text{mor}_\mathcal{B}(B, F(A)) \) to \( xf = N(f)(x) \). The
unit \( \alpha(M) \colon M \to F^*F_\ast(M) \) is the adjoint of \( \text{id}_{F,M} \) and sends \( x \in M(A) \) to the equivalence class of \( x \otimes \text{id}_{F(A)} \).

The functor \( F^* \) is flat. The functor \( F_\ast \) is compatible with direct sums over arbitrary index sets, is right exact, see [10] Theorem 2.6.1. on page 51, and \( F_\ast \text{mor}_{\mathbb{Z}A}(?,C) \) is \( \mathbb{Z}B \)-isomorphic to \( \text{mor}_{\mathbb{Z}B}(?, F(C)) \). In particular \( F_\ast \) respect the properties finitely generated, free, and projective.

4.2. The Yoneda embedding. The Yoneda embedding is the following covariant functor

\[
\iota \colon \mathcal{A} \to \text{MOD-}\mathbb{Z}A.
\]

It sends an object \( A \) to \( \iota(A) = \text{mor}_{\mathcal{A}}(?,A) \) and a morphism \( f \colon A_0 \to A_1 \) to the transformation \( \iota(f) \colon \text{mor}_{\mathcal{A}}(?,A_0) \to \text{mor}_{\mathcal{A}}(?,A_1) \) given by composition with \( f \).

Let \( \text{MOD-}\mathbb{Z}A \Box \) be the full subcategory of \( \text{MOD-}\mathbb{Z}A \) consisting of \( \mathbb{Z}A \)-modules \( \text{mor}_{\mathcal{A}}(?,A) \) for any object \( A \) in \( \mathcal{A} \). Let \( \text{MOD-}\mathbb{Z}A_{\text{fg}} \) be the full subcategory of \( \text{MOD-}\mathbb{Z}A \) consisting of finitely generated free \( \mathbb{Z}A \)-modules.

Definition 4.9. Let \( \mathcal{A} \) be a \( \mathbb{Z} \)-category. We call a sequence \( A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \) in \( \mathcal{A} \) exact at \( A_1 \), if \( f_1 \circ f_0 = 0 \) and for every object \( A \) and morphism \( g \colon A \to A_1 \) with \( f_1 \circ g = 0 \) there exists a morphism \( \overline{g} \colon A \to A_0 \) with \( f_0 \circ \overline{g} = g \).

Lemma 4.10. If \( \mathcal{A} \) is a \( \mathbb{Z} \)-category, the Yoneda embedding \( \iota \Box \) induces an equivalence of \( \mathbb{Z} \)-categories denoted by the same symbol

\[
\iota \colon \mathcal{A} \to \text{MOD-}\mathbb{Z}A_{\Box}.
\]

If \( \mathcal{A} \) is an additive category, the Yoneda embedding \( \iota \Box \) induces an equivalence of additive categories denoted by the same symbol

\[
\iota \colon \mathcal{A} \to \text{MOD-}\mathbb{Z}A_{\text{fg}}.
\]

Both functors are faithfully flat.

Proof. This follows directly from the Yoneda Lemma \[4.1\] and Lemma \[4.3\] (vi) \( \square \)

The gain of Lemma \[4.10\] is that we have embedded \( \mathcal{A} \) as a full subcategory of \( \text{MOD-}\mathbb{Z}A \Box \) and we can now do certain standard homological constructions in \( \text{MOD-}\mathbb{Z}A \) which a priori make no sense in \( \mathcal{A} \).

The elementary proof of the following lemma based on Lemma \[4.10\] is left to the reader.

Lemma 4.11. An additive category \( \mathcal{A} \) is idempotent complete if and only if every finitely generated projective \( \mathbb{Z}A \)-module is a finitely generated free \( \mathbb{Z}A \)-module.

5. Regularity properties of additive categories

5.1. Definition of regularity properties in terms of the Yoneda embedding. Recall the following standard ring theoretic notions:

Definition 5.1 (Regularity properties of rings). Let \( R \) be a ring and let \( l \) be a natural number.

(i) We call \( R \) Noetherian, if any \( R \)-submodule of a finitely generated \( R \)-module is again finitely generated;
(ii) We call \( R \) regular coherent, if every finitely presented \( R \)-module \( M \) is of type FP;
(iii) We call \( R \) \( l \)-uniformly regular coherent, if every finitely presented \( R \)-module \( M \) admits a \( l \)-dimensional finite projective resolution, i.e., there exist an exact sequence \( 0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0 \) such that each \( P_i \) is finitely generated projective;
(iv) We call \( R \) von Neumann regular, if for any element \( r \in R \) there exists an element \( s \in R \) with \( r = rsr \);
(v) We call \( R \) regular, if it is Noetherian and regular coherent;
(vi) We call \( R \) \( l \)-uniformly regular, if it is Noetherian and \( l \)-uniformly regular coherent;
(vii) We say that \( R \) has global dimension \( \leq l \) if each \( R \)-module \( M \) has projective dimension \( \leq l \).

The notion von Neumann regular should not be confused with the notion regular. It stems from operator theory. A ring is von Neumann regular if and only if it is 0-uniformly regular coherent. For more information about von Neumann regular rings, see for instance [2, Subsection 8.2.2 on pages 325-327].

Let \( A \) be an additive category. Then we define analogously:

**Definition 5.2** (Regularity properties of additive categories). Let \( A \) be an additive category and let \( l \) be a natural number.

(i) We call \( A \) Noetherian if the category \( \text{MOD-}Z \text{A} \) is Noetherian in the sense that any \( Z \text{A} \)-submodule of a finitely generated \( Z \text{A} \)-module is again finitely generated;
(ii) We call \( A \) regular coherent, if every finitely presented \( Z \text{A} \)-module \( M \) is of type FP;
(iii) We call \( A \) \( l \)-uniformly regular coherent, if every finitely presented \( Z \text{A} \)-module \( M \) possesses an \( l \)-dimensional finite projective resolution, i.e., there exist an exact sequence \( 0 \to P_l \to P_{l-1} \to \cdots \to P_0 \to M \to 0 \) such that each \( P_i \) is finitely generated projective;
(iv) We call \( A \) regular, if it is Noetherian and regular coherent;
(v) We call \( A \) \( l \)-uniformly regular, if is Noetherian and \( l \)-uniformly regular coherent;
(vi) We say that \( A \) has global dimension \( \leq l \), if each \( Z \text{A} \)-module \( M \) has projective dimension \( \leq l \).

5.2. The definitions of the regularity properties for rings and additive categories are compatible.

**Lemma 5.3.** Let \( R \) be a ring. The functor

\[
F : R\text{-MOD} \to \text{MOD-}Z R_{\oplus}
\]

sending \( M \) to \( \text{hom}_R(\theta_{B^l}(-), M) \) is an equivalence of additive categories, is faithfully flat, and respects each of the properties finitely generated, free and projective, where the equivalence \( \theta_{B^l} \) has been defined in (1.1).

**Proof.** In the sequel we denote by \([n]\) the \( n \)-fold direct sum in \( R_{\oplus} \) of the unique object in \( R \). Notice that \( \theta([n]) = R^n \). Define

\[
G : \text{MOD-}Z R_{\oplus} \to R\text{-MOD}
\]

by sending \( M \) to \( M(\theta(1)) \). There is a natural equivalence \( G \circ F \to \text{id}_{R\text{-MOD}} \) of functors of additive categories, its value on the \( R \)-module \( M \) is given by evaluating at \( 1 \in R = \theta([1]) \),

\[
G \circ F(M) = \text{hom}_R(\theta([1]), M) \xrightarrow{\cong} M.
\]

Next we construct an equivalence \( S : F \circ G \to \text{id}_{R\text{-MOD}} \) of functors of additive categories. For a \( Z \text{A} \)-module \( N \) and objects \( A_1, \ldots, A_n \) we obtain from Lemma [L3] a natural isomorphism

\[
\bigoplus_{i=1}^n N(pr_i) : \bigoplus_{i=1}^n N(A_i) \xrightarrow{\cong} N\left( \bigoplus_{i=1}^n A_i \right)
\]
where \( \operatorname{pr}_j : \bigoplus_{i=1}^n A_i \to A_j \) is the canonical projection for \( j = 1, 2 \ldots, n \).

Recall that \([n]\) is the \( n \)-fold direct sum of copies of \([1]\), in other words, we have an identification \([n] = \bigoplus_{i=1}^n [1]\). It induces an isomorphism

\[
\bigoplus_{i=1}^n \theta([1]) \xrightarrow{\cong} \theta([n]).
\]

Given an object \([n]\) in \( R_{\geq 0} \) and an \( R \)-module \( M \), we define \( S(M)([n]) \) by the following composite of \( R \)-isomorphisms

\[
F \circ G(M)([n]) = \hom_R(\theta([n]), M(\theta(1))) \xrightarrow{\cong} \hom_R\left( \bigoplus_{k=1}^n \theta([1]), M(\theta(1)) \right)
\]

\[
\xrightarrow{\cong} \bigoplus_{k=1}^n \hom_R(\theta([1]), M(\theta(1))) = \bigoplus_{k=1}^n \hom_R(R, M(\theta(1)))
\]

\[
\xrightarrow{\cong} \bigoplus_{k=1}^n M(\theta(1)) \xrightarrow{\cong} M\left( \bigoplus_{i=1}^n \theta([1]) \right) = M(\theta([n])).
\]

The functor \( F \) is faithfully exact, since for any object \([n]\) in \( R_{\geq 0} \) there is an \( R \)-isomorphism \( \bigoplus_{i=1}^n M \xrightarrow{\cong} F(M)([n]) \), natural in \( M \). Since \( F \) is compatible with direct sums over arbitrary index sets and sends \( R \) to \( \hom_R(\theta(-), R) = \mor_{k_{\geq 0}}(?, [1]) \) it respects the properties finitely generated, free and projective.

The following lemma implies in particular that the inclusion \( i : A \to \operatorname{Idem}(A) \) induces equivalences

\[
\begin{array}{ccc}
\text{MOD-}\mathbb{Z}A & \xrightarrow{i_*} & \text{MOD-}\mathbb{Z}\operatorname{Idem}(A).
\end{array}
\]

**Lemma 5.4.** Let \( i : A \to A' \) be an inclusion of an additive subcategory \( A \) of the additive subcategory \( A' \), which is full and cofinal, for instance \( A \to A' = \operatorname{Idem}(A) \).

Then:

(i) If \( M \) is a \( \mathbb{Z}A \)-module, then the adjoint

\[
\alpha(M) : M \xrightarrow{\cong} i^*i_*M
\]

of \( \id_{i_*M} \) under the adjunction \( i_* \) is an isomorphism of \( \mathbb{Z}A \)-modules, natural in \( M \);

(ii) The restriction functor \( i^* : \text{MOD-}\mathbb{Z}A' \to \text{MOD-}\mathbb{Z}A \) is faithfully flat. It sends a finitely generated \( \mathbb{Z}A' \)-module to a finitely generated \( \mathbb{Z}A \)-module and a projective \( \mathbb{Z}A' \)-module to a projective \( \mathbb{Z}A \)-module;

(iii) The induction functor \( i_* : \text{MOD-}\mathbb{Z}A \to \text{MOD-}\mathbb{Z}A' \) is faithfully flat. It sends a finitely generated \( \mathbb{Z}A \)-module to a finitely generated \( \mathbb{Z}A' \)-module and a projective \( \mathbb{Z}A \)-module to a projective \( \mathbb{Z}A' \)-module;

(iv) If \( M' \) is a \( \mathbb{Z}A' \)-module, then the adjoint

\[
\beta(M') : i_*i^*M' \xrightarrow{\cong} M'
\]

of \( \id_{i_*M'} \) under the adjunction \( i_* \) is an isomorphism of \( \mathbb{Z}A' \)-modules, natural in \( M' \);

(v) \( A \) is Noetherian if and only \( A' \) is Noetherian;

(vi) The category \( A \) is regular coherent or \( l \)-uniformly regular coherent respectively if and only if \( A' \) is regular coherent or \( l \)-uniformly regular coherent;

(vii) The category \( A \) is of global dimension \( \leq l \) if and only if \( A' \) is of global dimension \( \leq l \).
Lemma 4.1. Since \( P \) is a projective \( k \)-module, \( \exists \) an inverse of \( \alpha(M) \) is given by

\[
i^*i_*M = M(?) \otimes_{\mathcal{A}} \text{mor}_{\mathcal{A}'}(i(?), i(?)) = M(?) \otimes_{\mathcal{A}} \text{mor}_{\mathcal{A}'}(?, ?) \xrightarrow{\sim} M(?'),
\]

\[
x \otimes \phi \mapsto x \phi = M(\phi)(x).
\]

(iii) Obviously \( i^* \) is flat.

Consider a sequence of \( \mathcal{A}' \)-modules \( M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_1 \) such that this restriction

with \( i \) yields the exact sequence of \( \mathcal{A} \)-modules \( i^*M_0 \xrightarrow{i^*f_0} i^*M_1 \xrightarrow{i^*f_1} i^*M_1 \). We have to show for any object \( A' \) in \( \mathcal{A}' \) that the sequence of \( R \)-modules \( M_0(A') \xrightarrow{f_0(A')} M_1(A') \) is exact. Since \( \mathcal{A} \) is by assumption cofinal in \( \mathcal{A}' \), we can find an object \( A \) in \( \mathcal{A} \) and and morphisms \( j: A' \to i(A) \) and \( r: i(A) \to A' \) in \( \mathcal{A}' \) satisfying \( r \circ i = \text{id}_{A'} \).

We obtain the following commutative diagram of \( R \)-modules

\[
\begin{array}{ccc}
M_0(A') & \xrightarrow{f_0(A')} & M_1(A') & \xrightarrow{f_1(A')} & M_2(A') \\
\downarrow{M_0(i)} & & \downarrow{M_1(i)} & & \downarrow{M_2(i)} \\
M_0(i(A)) & \xrightarrow{f_0(i(A))} & M_1(i(A)) & \xrightarrow{f_1(i(A))} & M_2(i(A)) \\
M_0(A') & \xrightarrow{f_0(A')} & M_1(A') & \xrightarrow{f_1(A')} & M_2(A')
\end{array}
\]

such that the composite of the two vertical arrows appearing in each of the three columns is the identity. Since the middle horizontal sequence is exact, an easy diagram chase shows that the upper horizontal sequence is exact. This shows that \( i^* \) is faithfully flat.

Consider an object \( A' \) in \( \mathcal{A}' \). Since \( \mathcal{A} \) is by assumption cofinal in \( \mathcal{A} \), we can find an object \( A \) in \( \mathcal{A} \) and and morphism \( j: A' \to i(A) \) and \( q: i(A) \to A' \) in \( \mathcal{A}' \) satisfying \( q \circ j = \text{id}_{A'} \). Composition with \( q \) and \( j \) yield maps of \( \mathcal{A}' \)-modules \( J: \text{mor}_{\mathcal{A}'}(?, A') \to \text{mor}_{\mathcal{A}}(?, i(A)) \) and \( Q: \text{mor}_{\mathcal{A}'}(?, i(A)) \to \text{mor}_{\mathcal{A}}(?, A') \) satisfying \( Q \circ J = \text{id}_{\text{mor}_{\mathcal{A}'}(?, A')} \). If we apply \( i^* \), we obtain homomorphisms of \( \mathcal{A}' \)-modules \( i^*J: i^*\text{mor}_{\mathcal{A}'}(?, A') \to i^*\text{mor}_{\mathcal{A}}(?, i(A)) \) and \( i^*Q: i^*\text{mor}_{\mathcal{A}'}(?, i(A)) \to i^*\text{mor}_{\mathcal{A}}(?, A') \) satisfying \( i^*Q \circ i^*J = \text{id}_{i^*\text{mor}_{\mathcal{A}'}(?, A')} \). Since \( i^*\text{mor}_{\mathcal{A}}(?, i(A)) = \text{mor}_{\mathcal{A}}(i(?), i(A)) = \text{mor}_{\mathcal{A}'}(?, A) \), the \( \mathcal{A} \)-module \( i^*\text{mor}_{\mathcal{A}'}(?, A') \) is a direct summand in \( \text{mor}_{\mathcal{A}'}(?, A) \) and hence a finitely generated projective \( \mathcal{A} \)-module.

(iv) Let \( M' \) be a finitely generated \( \mathcal{A}' \)-module. Fix an epimorphism \( \text{mor}_{\mathcal{A}'}(?, A') \to M' \) for some object \( A' \) in \( \mathcal{A}' \). We conclude that the \( \mathcal{A} \)-module \( i^*M \) is a quotient of \( \text{mor}_{\mathcal{A}}(?, A) \) for some object \( A \) in \( \mathcal{A} \) and hence finitely generated. Hence \( i^* \) respects the property finitely generated.

Let \( P \) be a projective \( \mathcal{A}' \)-module. Then we can find a collection of objects \( \{ A'_k \mid k \in K \} \) together with an epimorphism \( \bigoplus_{k \in K} \text{mor}_{\mathcal{A}'}(?, A'_k) \to P \) by the Yoneda Lemma \( \ref{yoneda} \). Since \( P \) is projective, \( P \) is a direct summand in \( \bigoplus_{k \in K} \text{mor}_{\mathcal{A}'}(?, A'_k) \). This implies that \( i^*P \) is a direct summand in the direct sum \( \bigoplus_{k \in K} i^*\text{mor}_{\mathcal{A}'}(?, A'_k) \) of projective \( \mathcal{A} \)-modules and hence itself a projective \( \mathcal{A} \)-module. Hence \( i^* \) respects the property projective.

The faithful flatness follows from assertions (i) and (ii). Since \( i_*\text{mor}_{\mathcal{A}}(?, A) = \text{mor}_{\mathcal{A}}(?, i(A)) \) holds for any object \( A \) in \( \mathcal{A} \), the functor \( i_* \) respects the properties finitely generated and projective.

We begin with the case \( M = \text{mor}_{\mathcal{A}}(?, i(A)) = i_*\text{mor}_{\mathcal{A}}(?, A) \) for some object \( A \) in \( \mathcal{A} \). Then the claim follows from assertion (i) applied to the \( \mathcal{A} \)-module
Consider an object $A'$ in $\mathcal{A}'$. Since $\mathcal{A}$ is by assumption cofinal in $\mathcal{A}$, we can find an object $A$ in $\mathcal{A}$ and a morphism $j: A' \rightarrow i(A)$ such that the composite of the two vertical maps in each of the two columns is the identity and the middle arrow is an isomorphism. Hence the upper arrow is an isomorphism.

For any $\mathbb{Z}\mathcal{A}'$-module $M'$ we can find a collection of objects $\{A'_k \mid k \in K\}$ in $\mathcal{A}'$ together with an epimorphism $f_0: F_0 := \bigoplus_{k \in K} \text{mor}_{\mathcal{A}'}(\mathbb{Z}, A'_k) \rightarrow M'$ by the Yoneda Lemma [13]. Repeating this construction for $\text{ker}(f_0)$ instead of $M'$, we obtain another collection $\{A'_l \mid l \in L\}$ of objects in $\mathcal{A}'$ together with a map $f_1: F_1 := \bigoplus_{l \in L} \text{mor}_{\mathcal{A}'}(\mathbb{Z}, A'_l) \rightarrow F_0$ whose image is $\text{ker}(f_1)$. We obtain from assertions [i], [ii], [iii] and [iv] a commutative diagram of $\mathbb{Z}\mathcal{A}'$-modules with exact rows

$$
\begin{array}{c}
\text{id} \\
\text{id} \\
\text{id}
\end{array}
\begin{array}{c}
i_*i^*F_1 \\
i_*i^*F_0 \\
i_*i^*M'
\end{array}
\begin{array}{c}
\beta(F_1) \\
\beta(F_0) \\
\beta(M)
\end{array}
\begin{array}{c}
F_1 \\
F_0 \\
M'
\end{array}
$$

Since $\beta$ is compatible with direct sums over arbitrary index sets, the maps $\beta(F_1)$ and $\beta(F_0)$ are isomorphisms. Hence $\beta(M')$ is an isomorphism.

They follow now directly from assertions [i], [ii], [iii] and [iv]. □

We conclude from Lemma 5.3 and Lemma 5.4 [v], [vi] and [vii].

**Corollary 5.5.** Let $R$ be a ring and let $l$ be a natural number. Then the following assertions are equivalent:

1. The ring $R$ is Noetherian, regular coherent, $l$-uniformly regular coherent, regular, uniformly $l$-regular, or of global dimension $\leq l$ in the sense of Definition 5.3 respectively;
2. The additive category $\text{Idem}(R)$ is Noetherian, regular coherent, $l$-uniformly regular coherent, regular, uniformly $l$-regular, or of global dimension $\leq d$ in the sense of Definition 5.3 respectively;
3. The additive category $\text{Idem}(R)$ is Noetherian, regular coherent, $l$-uniformly regular coherent, regular, uniformly $l$-regular, or of global dimension $\leq l$ in the sense of Definition 5.3 respectively.

**5.3. Intrinsic definitions of the regularity properties.** One can give an intrinsic definition of the regularity properties above without referring to the Yoneda embedding. The situation is quite nice for regular coherent and $l$-uniformly regular coherent for an idempotent complete additive category as above defined below.

**Lemma 5.6 (Intrinsic Reformulation of regular coherent).** Let $\mathcal{A}$ be an idempotent complete additive category.
(i) Let \( l \geq 2 \) be a natural number. Then \( \mathcal{A} \) is \( l \)-uniformly regular coherent if and only if for every morphism \( f_1: A_1 \to A_0 \) we can find a sequence of length \( l \) in \( \mathcal{A} \)

\[
0 \to A_l \xrightarrow{f_l} A_{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0
\]

which is exact at \( A_i \) for \( i = 1, 2, \ldots, n \);

(ii) \( \mathcal{A} \) is \( 1 \)-uniformly regular coherent if and only if for every morphism \( f: A_1 \to A_0 \) we can find a factorization \( A_1 \xrightarrow{f_2} B \xrightarrow{f_0} A_0 \) of \( f \) such that \( f_1 \) is surjective and \( f_0 \) is injective;

(iii) The following assertions are equivalent:

(a) \( \mathcal{A} \) is 0-uniformly regular coherent;

(b) For every morphism \( f_1: A_1 \to A_0 \) there exists a morphism \( f_0: A_0 \to A_{-1} \) such that \( A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} A_{-1} \to 0 \) is exact;

(c) For every morphism \( f: A_1 \to A_0 \) there exists a morphism \( g: A_0 \to A_1 \) satisfying \( f \circ g \circ f = f \);

(iv) \( \mathcal{A} \) is regular coherent if and only if for every morphism \( f_1: A_1 \to A_0 \) we can find a sequence of finite length in \( \mathcal{A} \)

\[
0 \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0
\]

which is exact at \( A_i \) for \( i = 1, 2, \ldots, n \).

**Proof.** It suffices to prove that the following statements are equivalent:

(a) For any morphisms \( f_1: P_1 \to P_0 \) of finitely generated projective \( \mathbb{Z} \mathcal{A} \)-modules we can find finitely generated projective \( \mathbb{Z} \mathcal{A} \)-modules \( P_2, P_3, \ldots, P_l \) and an exact sequence of \( \mathbb{Z} \mathcal{A} \)-modules

\[
0 \to P_l \xrightarrow{f_l} P_{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0;
\]

(b) For any finitely presented \( \mathbb{Z} \mathcal{A} \)-module \( M \) there exists finitely generated projective \( \mathbb{Z} \mathcal{A} \)-modules \( P_0, P_1, \ldots, P_l \) and an exact sequence of \( \mathbb{Z} \mathcal{A} \)-modules

\[
0 \to P_l \xrightarrow{f_l} P_{l-1} \xrightarrow{f_{l-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0.
\]

The implication (b) \( \implies \) (a) is obvious since \( \operatorname{cok}(f_1) \) is a finitely presented \( \mathbb{Z} \mathcal{A} \)-module. It remains to prove the implication (a) \( \implies \) (b). Let \( f_1: P_1 \to P_0 \) be a \( \mathbb{Z} \mathcal{A} \)-homomorphism of finitely generated projective \( \mathbb{Z} \mathcal{A} \)-modules. By assumption we can find an exact sequence of \( \mathbb{Z} \mathcal{A} \)-modules

\[
0 \to Q_l \xrightarrow{c_l} Q_{l-1} \xrightarrow{c_{l-1}} \cdots \xrightarrow{c_2} Q_1 \xrightarrow{c_1} Q_0 \xrightarrow{c_0} \operatorname{cok}(f_1) \to 0.
\]

Let \( P_1 \) be the 1-dimensional \( \mathbb{Z} \mathcal{A} \)-chain complex whose first differential is \( f_1 \). Let \( Q_* \) be the \( l \)-dimensional \( \mathbb{Z} \mathcal{A} \)-chain complex whose \( i \)th chain module is \( Q_i \) for \( 0 \leq i \leq l \) and whose \( i \)th differential is \( c_i: Q_i \to Q_{i-1} \) for \( 1 \leq i \leq l \). One easily constructs a \( \mathbb{Z} \mathcal{A} \)-chain map \( u_*: P_* \to Q_* \) such that \( H_0(u_*) \) is an isomorphism. Let \( \operatorname{cone}(u_*) \) be the mapping cone. We conclude \( H_i(\operatorname{cone}(u_*)) = 0 \) for \( i \neq 2 \) from the long exact homology sequence associated to the exact sequence \( 0 \to P_* \xrightarrow{i_*} \operatorname{cyl}(u_*) \xrightarrow{p_*} \operatorname{cone}(u_*) \to 0 \) and the fact that the canonical projection \( q_*: \operatorname{cyl}(u_*) \to Q_* \) is a \( \mathbb{Z} \mathcal{A} \)-chain homotopy equivalence with \( q_* \circ i_* = u_* \). Let \( D_* \subseteq \operatorname{cone}(u_*) \) be the \( \mathbb{Z} \mathcal{A} \)-subchain complex, whose \( i \)th chain module is \( \operatorname{cone}(u_*) \) for \( i \geq 3 \), the kernel of the second differential of \( \operatorname{cone}(u_*) \) for \( i = 2 \) and \( \{0\} \) for \( i = 0, 1 \). Then \( D_* \) is finitely generated projective for \( i \geq 0 \) and the inclusion \( k_*: D_* \to \operatorname{cone}(u_*) \) induces isomorphisms on homology groups. Define the \( \mathbb{Z} \mathcal{A} \)-chain complex \( C_* \) by
the pullback

\[ \begin{array}{ccc}
C_* & \xrightarrow{p_*} & D_* \\
\downarrow{k_*} & & \downarrow{k_*} \\
\text{cyl}(u_*) & \xrightarrow{p_*} & \text{cone}(u_*)
\end{array} \]

This can be extended to a commutative diagram of \( \mathbb{Z}A \)-chain complexes with exact rows

\[ \begin{array}{ccc}
0 & \xrightarrow{p_*} & C_* \xrightarrow{p_*} D_* \\
\downarrow{id} & & \downarrow{id} \\
0 & \xrightarrow{i_*} & \text{cyl}(u_*) \xrightarrow{p_*} \text{cone}(u_*)
\end{array} \]

Then \( C_* \) is a \( l \)-dimensional \( \mathbb{Z}A \)-chain complex whose \( \mathbb{Z}A \)-chain modules are finitely generated projective. Since \( D_i = 0 \) for \( i = 0, 1 \), we can identify \( P_1 = C_1 \) and \( P_0 = C_0 \) and the first differentials of \( P_* \) and \( C_* \). Since \( k_* \) induces isomorphisms on homology, the same is true for \( E_* \). Hence \( C_* \) yields the desired extension of \( f_1 \) to an exact sequence

\[ 0 \rightarrow C_1 \rightarrow C_{l-1} \rightarrow \cdots \rightarrow C_2 \rightarrow P_1 \xrightarrow{l_1} P_0 \]

This finishes the proof of assertion (i).

(ii) Suppose that \( A \) is 1-uniformly regular coherent. Consider a morphism \( f: A_1 \rightarrow A_0 \). Let \( M \) be the finitely presented \( \mathbb{Z}A \)-module given by the cokernel of the \( \mathbb{Z}A \)-homomorphism \( \iota(f): \iota(A_1) \rightarrow \iota(A_0) \). By assumption we can find an exact sequence \( 0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0 \) of \( \mathbb{Z}A \)-modules, where \( P_1 \) and \( P_0 \) are finitely generated projective. We conclude from Lemma 4.4 (iv) that the image of \( \iota(f) \) is finitely generated projective. Hence we obtain a factorization if \( \iota(f) \) as a composite \( \iota(f): \iota(A_1) \xrightarrow{f_1} \text{im}(\iota(f)) \xrightarrow{\iota(l_1)} \iota(A_0) \) such that \( \text{im}(\iota(f)) \) is a finitely generated projective \( \mathbb{Z}A \)-module, \( f_1 \) is surjective, and \( f_0 \) is injective. We conclude from Lemma 4.10 and Lemma 4.11 that \( \text{im}(f) \) can be identified with \( \iota(B) \) for some object \( B \in A \) and there are morphisms \( f_1: A_1 \rightarrow B \) and \( f_0: B \rightarrow A_1 \) such that \( f_1 \circ f_0 = \iota(f_1) \) and \( f_0 \circ f_1 = \iota(f_0) \). Moreover, \( f_1 \) is surjective, \( f_0 \) is injective and \( f = f_0 \circ f_1 \).

Suppose that for every morphism \( f: A_1 \rightarrow A_0 \) we can find a factorization \( A_1 \xrightarrow{f_1} B \xrightarrow{f_0} A_0 \) of \( f \) such that \( f_1 \) is surjective and \( f_0 \) is injective. Consider any finitely presented \( \mathbb{Z}A \)-module \( M \). We conclude from Lemma 4.10 that there is a morphism \( f: A_1 \rightarrow A_0 \) in \( A \) and a morphism \( p: \iota(A_0) \rightarrow M \) of \( \mathbb{Z}A \)-modules such that the sequence \( \iota(A_1) \xrightarrow{\iota(f)} \iota(A_0) \xrightarrow{\iota(l_1)} \iota(M) \rightarrow 0 \) is exact. Choose a factorization \( f = f_1 \circ f_0 \) such that \( f_1 \) is surjective and \( f_0 \) is injective. Let \( B \) be the domain of \( f_1 \). We conclude from Lemma 4.10 that we obtain a short exact sequence \( 0 \rightarrow \iota(B) \xrightarrow{\iota(l_1)} \iota(A_0) \xrightarrow{\iota(l_0)} \iota(M) \rightarrow 0 \). This is a 1-dimensional finite projection \( \mathbb{Z}A \)-resolution of \( M \). This finishes the proof of assertion (ii).

(iii) We first show (iii)a \( \implies \) (iii)c. Consider a morphism \( f: A_1 \rightarrow A_0 \). Let \( M \) be the finitely presented \( \mathbb{Z}A \)-module given by the cokernel of \( \iota(f): \iota(A_1) \rightarrow \iota(A_0) \). We obtain an exact sequence of \( \mathbb{Z}A \)-modules \( \iota(A_1) \xrightarrow{\iota(f)} \iota(A_0) \xrightarrow{\iota(l_0)} \iota(M) \rightarrow 0 \). By assumption \( M \) is a finitely generated projective \( \mathbb{Z}A \)-module. Let \( \iota(f): \iota(A_1) \xrightarrow{\iota(f)} \text{im}(\iota(f)) \xrightarrow{\iota(l_1)} \iota(A_0) \) be the obvious factorization of \( \iota(f) \). Since \( M \) projective, \( \text{im}(f) \) is a direct summand in \( \iota(A_0) \). We conclude from Lemma 4.10 and Lemma 4.11 that we can identify \( \text{im}(\iota(f)) \) with \( \iota(B) \) for an appropriate object \( B \in A \) and can find morphisms \( r: A_0 \rightarrow B \) and \( s: B \rightarrow A_1 \) in \( A \) such that \( \iota(r) \circ j = \text{id}_{\iota(B)} \)
and \( q \circ \iota(s) = \id_{\langle B \rangle} \). Define \( g : A_0 \to A_1 \) by \( g = s \circ r \). One easily checks that \( \iota(f) \circ \iota(g) \circ s(f) = \iota(f) \). Hence \( f \circ g \circ f = f \).

Next we show \((i) \Rightarrow (ii)\). Let \( f : A_1 \to A_0 \) be a morphism in \( A \). Choose a morphism \( h : A_0 \to A_1 \) with \( f \circ h \circ f = f \). Then \( f \circ h : A_0 \to A_0 \) is an idempotent.

Since \( A \) is idempotent complete, we can find objects \( A_{-1} \) and \( A_{-1} \) and an isomorphism \( u : A_0 \cong A_{-1} \oplus A_{-1} \) in \( A \) such that \( u \circ (\id_{A_0} - f \circ h) \circ u^{-1} = \begin{pmatrix} \id_{A_{-1}} & 0 \\ 0 & 0 \end{pmatrix} \).

Define \( g : A_0 \to A_{-1} \) by the composite \( A_0 \xrightarrow{u} A_{-1} \oplus A_{-1} \xrightarrow{pr_{A_{-1}}} A_{-1} \). One easily checks that the sequence \( A_1 \xrightarrow{f} A_0 \xrightarrow{g} A_{-1} \to 0 \) is exact.

Finally we show \((iii) \Rightarrow (ii)\). Consider a finitely presented \( \mathbb{Z} \)-module \( M \). We conclude from Lemma 4.10 and that we an find a morphism \( f_1 : A_1 \to A_0 \) together with an exact sequence of \( \mathbb{Z} \)-modules \( \iota(A_1) \xrightarrow{\iota(f_1)} \iota(A_0) \xrightarrow{\iota(f_0)} M \to 0 \). Choose a morphism \( f_0 : A_0 \to A_{-1} \) such that the sequence \( A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} A_{-1} \to 0 \) is exact in \( A \). Then we obtain an exact sequence of \( \mathbb{Z} \)-modules \( \iota(A_1) \xrightarrow{\iota(f_1)} \iota(A_0) \xrightarrow{\iota(f_0)} \iota(A_{-1}) \to 0 \) by Lemma 4.10. This implies that \( M \) is \( \mathbb{Z} \)-isomorphic to \( \iota(A_{-1}) \) and hence finitely generated projective. This finishes the proof of assertion \((iii)\).

This follows from assertion \((i)\). This finishes the proof of Lemma 5.8.

Next we deal with the property Noetherian. Consider two morphisms \( f : A \to B \) and \( f' : A' \to B \). We write \( f \subseteq f' \) if there exists a morphism \( g : A \to A' \) with \( f = f' \circ g \). Lemma 4.10 implies

\[(5.7) \quad f \subseteq f' \iff \im(f_*) : \mathrm{mor}_A(?, A) \to \mathrm{mor}_A(?, B) \subseteq \im(f'_*) : \mathrm{mor}_A(?, A') \to \mathrm{mor}_A(?, B).
\]

**Lemma 5.8** (Intrinsic Reformulation of Noetherian). Let \( A \) be an additive category. Then the following assertions are equivalent:

(i) \( A \) is Noetherian.

(ii) Each object \( A \) has the following property: Consider any directed set \( I \) and collections of morphisms \( \{ f_i : A_i \to A \mid i \in I \} \) with \( A \) as target such that \( f_i \subseteq f_j \) holds for \( i \leq j \). Then there exists \( i_0 \in I \) with \( f_i \subseteq f_{i_0} \) for all \( i \in I \).

Proof. Suppose that \((ii)\) is true. Consider a directed set \( I \) and collections of morphisms \( \{ f_i \mid i \in I \} \) such that \( f_i \subseteq f_j \) holds for \( i \leq j \). Define \( M \subseteq \mathrm{mor}_A(?, A) \) by

\[ M = \bigcup_{i \in I} \im((f_i)_*). \]

Since \( I \) is directed, we can find for two elements \( i_0 \) and \( i_1 \) another element \( j \in I \) with \( i_0, i_1 \leq j \). Hence we have \( \im((f_{i_0})_*) \subseteq \im((f_{i_1})_*) \subseteq \im((f_j)_*) \). This implies that \( M \) is \( \mathbb{Z} \)-submodule of \( \mathrm{mor}_A(?, A) \). Since \( A \) is Noetherian, \( M \) is finitely generated. Hence there exists \( i_0 \in I \) with \( \im((f_{i_0})_*) = M \). Hence we get for every \( i \in I \) that \( \im((f_j)_*) \subseteq \im((f_{i_0})_*) \) and hence \( i \leq i_0 \) holds. Hence \( A \) satisfies \((ii)\).

Suppose that property \((ii)\) holds. It remains to show for every finitely generated \( \mathbb{Z} \)-module \( N \) that every \( \mathbb{Z} \)-submodule \( M \subseteq N \) is finitely generated. Choose an object \( A \) and a \( \mathbb{Z} \)-epimorphism \( u : \mathrm{mor}_A(?, A) \to N \). Then \( u^{-1}(M) \) defined by \( u^{-1}(M)(?) = u(?)^{-1}(M(?)) \) is a \( \mathbb{Z} \)-submodule of \( \mathrm{mor}_A(?, A) \) and \( u \) induces an epimorphism \( u^{-1}(M) \to M \). Therefore \( M \) is finitely generated if \( u^{-1}(M) \) is. Hence we can assume without loss of generality \( N = \mathrm{mor}_A(?, A) \).

Let \( \{ M_i \mid i \in I \} \) the collection of finitely generated \( \mathbb{Z} \)-submodules of \( M \) directed by inclusion. For each \( i \in I \) we can choose an object \( A_i \in A \) and a morphism
Given by the morphism \( u : A \rightarrow A \) such that \( \text{im}((f_i)_*) = M_i \) holds. Then \( \{f_i \mid i \in I\} \) satisfies \( f_i \subseteq f_j \) for \( i \leq j \). By assumption there exists \( i_0 \in I \) with \( f_i \subseteq f_{i_0} \) and hence \( \text{im}((f_i)_*) = \text{im}((f_{i_0})_*) \) for all \( i \in I \). This implies \( M = \text{im}((f_{i_0})_*) \). Hence \( M \) is finitely generated. \( \square \)

6. VANISHING OF NIL-TERMS

6.1. Nil-categories. The next definition is taken from [4, Definition 7.1].

**Definition 6.1** (Nilpotent morphisms and Nil-categories). Let \( \mathcal{A} \) be an additive category and \( \Phi \) be an automorphism of \( \mathcal{A} \).

(i) A morphism \( f : \Phi(A) \rightarrow A \) of \( \mathcal{A} \) is called \( \Phi \)-nilpotent if for some \( n \geq 1 \), the \( n \)-fold composite

\[
f^{(n)} := f \circ \Phi(f) \circ \cdots \circ \Phi^{n-1}(f) : \Phi^n(A) \rightarrow A.
\]

is trivial;

(ii) The category \( \text{Nil}(\mathcal{A}, \Phi) \) has as objects pairs \( (A, \phi) \) where \( \phi : \Phi(A) \rightarrow A \) is a \( \Phi \)-nilpotent morphism in \( \mathcal{A} \). A morphism from \( (A, \phi) \) to \((B, \mu)\) is a morphism \( u : A \rightarrow B \) in \( \mathcal{A} \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
\Phi(A) & \xrightarrow{\phi} & A \\
\downarrow{\Phi(u)} & & \downarrow{u} \\
\Phi(B) & \xrightarrow{\mu} & B
\end{array}
\]

The category \( \text{Nil}(\mathcal{A}, \Phi) \) inherits the structure of an exact category from \( \mathcal{A} \), a sequence in \( \text{Nil}(\mathcal{A}, \Phi) \) is declared to be exact if the underlying sequence in \( \mathcal{A} \) is split exact.

Let \( \Phi : \mathcal{A} \xrightarrow{\cong} \mathcal{A} \) be an automorphism of an additive category \( \mathcal{A} \). It induces an automorphism \( \Phi^{-1} : \text{MOD-ZA} \xrightarrow{\cong} \text{MOD-ZA} \) of abelian categories by precomposition with \( \Phi^{-1} : \mathcal{A} \xrightarrow{\cong} \mathcal{A} \). It sends \( \text{MOD-ZA}_{\text{fgf}} \) to itself, since \( \Phi^{-1} \circ \text{mor}_\mathcal{A}(? , \Phi(A)) \) is isomorphic to \( \text{mor}_\mathcal{A}(? , \Phi(A)) \).

Thus we obtain an automorphism of additive categories \( \Phi^{-1} : \text{MOD-ZA}_{\text{fgf}} \xrightarrow{\cong} \text{MOD-ZA}_{\text{fgf}} \).

**Lemma 6.2.** There is an equivalence of exact categories

\( \iota : \text{Nil}(\mathcal{A}, \Phi) \xrightarrow{\cong} \text{Nil}(\text{MOD-ZA}_{\text{fgf}}, \Phi^{-1}) \)

**Proof.** The desired functors \( \iota \) sends an object \((A, f)\) in \( \text{Nil}(\mathcal{A}, \phi) \) given by a morphism \( f : \Phi(A) \rightarrow A \) to the object in \( \text{Nil}(\text{MOD-ZA}_{\text{fgf}}, \Phi^{-1}) \) given by the composite

\[
\Phi^{-1} \circ \text{mor}_\mathcal{A}(?, A) = \text{mor}_\mathcal{A}(\Phi^{-1}(?), A) \xrightarrow{\Phi} \text{mor}_\mathcal{A}(?, \Phi(A)) \xrightarrow{\text{mor}_\mathcal{A}(?, f)} \text{mor}_\mathcal{A}(?, A).
\]

A morphism \( u : (A, f) \rightarrow (A', f') \) in \( \text{Nil}(\mathcal{A}, \Phi) \), which given by a morphism \( u : A \rightarrow A' \) in \( \mathcal{A} \) satisfying \( f' \circ \Phi(u) = u \circ f \), is sent to the morphism in \( \text{Nil}(\text{MOD-ZA}_{\text{fgf}}, \Phi^{-1}) \) given by the morphism \( u : \text{mor}_\mathcal{A}(?, A) \rightarrow \text{mor}_\mathcal{A}(?, A') \). It defines indeed a morphism from \( \iota(A, f) \) to \( \iota(A', f') \) by the commutativity of the following diagram

\[
\begin{array}{ccc}
\text{mor}_\mathcal{A}(\Phi^{-1}(?), A) & \xrightarrow{\Phi} & \text{mor}_\mathcal{A}(?, \Phi(A)) \\
\downarrow{\text{mor}_\mathcal{A}(\Phi^{-1}(?, u))} & & \downarrow{\text{mor}_\mathcal{A}(?, \Phi(u))} \\
\text{mor}_\mathcal{A}(\Phi^{-1}(?), A') & \xrightarrow{\Phi} & \text{mor}_\mathcal{A}(?, \Phi(A')) \\
\downarrow{\text{mor}_\mathcal{A}(?, u)} & & \downarrow{\text{mor}_\mathcal{A}(?, f')} \\
\end{array}
\]

It is an equivalence of additive categories by Lemma 4.10. \( \square \)
6.2. Connective $K$-theory.

**Lemma 6.3.** Let $\mathcal{A}$ be an idempotent complete additive category. Suppose that $\mathcal{A}$ is regular coherent. Let $\Phi: \mathcal{A} \to \mathcal{A}$ be any automorphism of additive categories. Denote by $J: \mathcal{A} \to \text{Nil}(\mathcal{A}, \Phi)$ the inclusion sending an object $A$ to the object $(A, 0)$.

Then the induced map on connective $K$-theory $K(J): K(A) \to K(\text{Nil}(\mathcal{A}, \Phi))$ is a weak homotopy equivalence.

**Proof.** We abbreviate $\Psi = \Phi^{-1}$. We have the following commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{J} & \text{Nil}(\mathcal{A}, \Phi) \\
\downarrow \iota & & \downarrow \iota \\
\text{MOD-}Z\mathcal{A}_{\text{fgf}} & \xrightarrow{J} & \text{Nil}(\text{MOD-}Z\mathcal{A}_{\text{fgf}}, \Psi)
\end{array}
$$

where the vertical arrows are equivalences of exact categories given by Yoneda embeddings, see Lemma 6.10 and Lemma 6.2, and the lower horizontal arrow is the obvious analogue of the upper horizontal arrow. Hence it suffices to show that the map $K(J): K(\text{MOD-}Z\mathcal{A}_{\text{fgf}}) \to K(\text{Nil}(\text{MOD-}Z\mathcal{A}_{\text{fgf}}, \Psi))$ is a weak homotopy equivalence.

Denote by $\text{MOD-}Z\mathcal{A}_{\text{FL}}$ the full subcategory of $\text{MOD-}Z\mathcal{A}$ consisting of $Z\mathcal{A}$-modules which are of type FL, i.e., possesses a finite dimensional resolution by finitely generated free $Z\mathcal{A}$-modules.

Consider the following commutative diagram

$$
\begin{array}{ccc}
K(\text{MOD-}Z\mathcal{A}_{\text{fgf}}) & \xrightarrow{} & K(\text{Nil}(\text{MOD-}Z\mathcal{A}_{\text{fgf}}, \Psi)) \\
\downarrow & & \downarrow \\
K(\text{MOD-}Z\mathcal{A}_{\text{FL}}) & \xrightarrow{} & K(\text{Nil}(\text{MOD-}Z\mathcal{A}_{\text{FL}}, \Psi))
\end{array}
$$

where all arrows are induced by the obvious inclusions of categories.

The left vertical arrow in the diagram is a weak homotopy equivalence by the Resolution Theorem, see [7, Theorem 4.6 on page 41].

Next we show that the lower horizontal arrow in the diagram is a weak homotopy equivalence. Consider an object $(M, f)$ in $\text{Nil}(\text{MOD-}Z\mathcal{A}_{\text{FL}}, \Psi)$.

Recall that nilpotent means that for some natural number $n \geq 0$ the composite

$$f^{(n)}: \Psi^n(M) \xrightarrow{\Psi^{n-1}(f)} \Psi^{n-1}(M) \xrightarrow{\Psi^{n-2}(f)} \cdots \xrightarrow{\Psi(f)} \Psi(M) \xrightarrow{f} M$$

is trivial. We get a filtration of $(M, f)$ by subobjects

$$(M, f) \supseteq (\text{im}(f), f|_{\text{im}(f)}) \supseteq (\text{im}(f^{(2)}), f|_{\text{im}(f^{(2)})}) \supseteq \cdots \supseteq (\text{im}(f^{(n-1)}), f|_{\text{im}(f^{(n-1)})}) \supseteq (\text{im}(f^{(n)}), f|_{\text{im}(f^{(n)})}) = ([0], \text{id}_0),$$

where we consider $\Psi(\text{im}(f^{(i)}))$ as a $Z\mathcal{A}$-submodule of $\Psi(M)$ by the injective map $\Psi(\text{im}(f^{(i)})) \to \Psi(M)$ which is obtained by applying $\Psi$ to the inclusion $\text{im}(f^{(i)}) \to M$. We get exact sequences of $Z\mathcal{A}$-modules

$$0 \to \text{im}(f^{(i)}) \to M \to M/\text{im}(f^{(i)}) \to 0;$$

$$0 \to \text{im}(f^{(i+1)}) \to \text{im}(f^{(i)}) \to \text{im}(f^{(i)}/\text{im}(f^{(i+1)}) \to 0.$$
Since $M$ is finitely presented and $\text{im}(f^i)$ is finitely generated, $M/\text{im}(f^i)$ is finitely presented. Since $\mathcal{A}$ is regular coherent and idempotent complete by assumption, $M$ and $M/\text{im}(f^i)$ for all $i$ are of type $\text{FL}$. We conclude by induction over $i = 0, 1, \ldots$ from Lemma 6.3 that $\text{im}(f^i)$ and $\text{im}(f^i)/\text{im}(f^{i+1})$ belong to $\text{MOD}-\mathcal{Z}\mathcal{A}_{\text{FL}}$ again. The quotient of $(\text{im}(f^i), f|_{\text{im}(f^i)})$ by $(\text{im}(f^{i+1}), f|_{\text{im}(f^{i+1})})$ is given by $(\text{im}(f^i)/\text{im}(f^{i+1}), 0)$, and hence belongs to $\text{MOD}-\mathcal{Z}\mathcal{A}_{\text{FL}}$ for all $i$. Now the lower horizontal arrow in diagram (6.4) is a weak homotopy equivalence by the Devissage Theorem, see [7, Theorem 4.8 on page 42].

Next we show that the right vertical arrow in the diagram (6.4) induces split injections on homotopy groups. For this purpose we consider the following commutative diagram of exact categories

\[
\begin{array}{ccc}
\text{Nil}(\text{MOD}-\mathcal{Z}\mathcal{A}_{\text{gf}} \Psi) & \xrightarrow{I_1} & \text{HNil}(\text{Ch}(\text{MOD}-\mathcal{Z}\mathcal{A}_{\text{gf}} \Psi)) \\
I_2 & & I_4 \\
I_3 & & I_5 \\
\text{HNil}(\text{Ch}_{\text{res}}(\text{MOD}-\mathcal{Z}\mathcal{A}_{\text{gf}} \Psi)) & \xrightarrow{H_0} & \text{Nil}(\text{MOD}-\mathcal{Z}\mathcal{A}_{\text{FL}} \Psi)
\end{array}
\]

The category $\text{HNil}(\text{Ch}(\text{MOD}-\mathcal{Z}\mathcal{A}_{\text{gf}} \Psi))$ is given by finite-dimensional chain complexes $C$ over $\text{MOD}-\mathcal{Z}\mathcal{A}_{\text{gf}}$ (with $C_i = 0$ for $i \leq -1$) together with chain maps $\phi: C_\ast \to C_\ast$, which are homotopy nilpotent, and $\text{HNil}(\text{Ch}_{\text{res}}(\text{MOD}-\mathcal{Z}\mathcal{A}_{\text{gf}} \Psi))$ is the full subcategory of $\text{HNil}(\text{Ch}(\text{MOD}-\mathcal{Z}\mathcal{A}_{\text{gf}} \Psi))$ consisting of those chain complexes for which $H_i(C_\ast) = 0$ for $i \geq 1$. The maps $I_k$ for $k = 1, 2, 3, 4$ are the obvious inclusions, the functor $H_0$ is given by taking the zeroth homology group. The upper horizontal arrow induces a weak homotopy equivalence on connective $K$-theory by [3, page 173]. The functor $H_0$ induces a weak homotopy equivalence on connective $K$-theory by the Approximation Theorem of Waldhausen, see for instance [4, Theorem 4.18]. Hence the map induced by $I_3$ on connective $K$-theory, which is the right vertical arrow in the diagram (6.4), induces split injections on homotopy groups.

We conclude that all arrows appearing in the diagram (6.4) induce weak homotopy equivalences on connective algebraic $K$-theory. This finishes the proof of Lemma 6.3. $\square$

**Theorem 6.5** (The connective $K$-theory of additive categories). Let $\mathcal{A}$ be an additive category which is idempotent complete and regular coherent. Consider any automorphism $\Phi: \mathcal{A} \xrightarrow{\simeq} \mathcal{A}$ of additive categories.

Then we get a map of connective spectra

\[a: \text{T}_K(\Phi^{-1}) \to K(\mathcal{A}_\Phi[t, t^{-1}])\]

such that $\pi_n(a)$ is bijective for $n \geq 1$.

**Proof.** This follows from Theorem 5.4 since Lemma 6.3 implies $\pi_n(\text{E}(R, \Phi)) = 0$ for $n \geq 0$ and hence $\pi_n(\text{NK}(\mathcal{A}_\Phi[t])) = \pi_n(\text{NK}(\mathcal{A}_\Phi[t^{-1}])) = 0$ for all $n \geq 1$. $\square$

We will need later the following consequence of Lemma 5.3 where we can drop the assumption that $\mathcal{A}$ is idempotent complete.

**Lemma 6.6.** Let $\mathcal{A}$ be an additive category. Suppose that $\mathcal{A}$ is regular coherent. Let $\Phi: \mathcal{A} \xrightarrow{\simeq} \mathcal{A}$ be any automorphism of additive categories. Denote by $J: \mathcal{A} \to \text{Nil}(\mathcal{A}, \Phi)$ the inclusion sending an object $A$ to the object $(A, 0)$. 
Then the induced map

$$
\pi_n(K(J)) : \pi_n(K(A)) \to \pi_n(K(\text{Nil}(A, \Phi)))
$$

is bijective for \( n \geq 1 \).

**Proof.** We have the obvious commutative diagram coming from the inclusion \( A \to \text{Idem}(A) \).

\[
\begin{array}{ccc}
\pi_n(K(A)) & \longrightarrow & \pi_n(K(\text{Nil}(A, \Phi))) \\
\downarrow & & \downarrow \\
\pi_n(K(\text{Idem}(A))) & \longrightarrow & \pi_n(K(\text{Nil}(\text{Idem}(A)), \text{Idem}(\Phi)))
\end{array}
\]

The left vertical arrow is bijective for \( n \geq 1 \) by Lemma 2.3 (i). The lower horizontal arrow is bijective for \( n \geq 1 \) by Lemma 6.3 since \( \text{Idem}(A) \) is regular coherent by Lemma 5.4 (vi). Hence we have to show that the right vertical arrow is bijective for \( n \geq 1 \). For this purpose it suffices to show because of Lemma 2.3 (i) that \( \text{Nil}(A, \Phi) \) is a cofinal full subcategory of \( \text{Nil}(\text{Idem}(A), \text{Idem}(\Phi)) \). This follows from the fact that \( A \) is a cofinal full subcategory of \( \text{Idem}(A) \). \( \square \)

### 6.3. Non-connective \( K \)-theory

In the sequel define \( A[\mathbb{Z}^m] \) inductively over \( m \) by \( A[\mathbb{Z}^m] := A[\mathbb{Z}^{m-1}][t, t^{-1}] \), where \( A[\mathbb{Z}^{m-1}][t, t^{-1}] \) is the (untwisted) finite Laurent category associated to \( A[\mathbb{Z}^{m-1}] \) and the automorphism given by the identity, see Subsection 1.4.

**Lemma 6.7.** Let \( A \) be an additive category. Suppose that \( A[\mathbb{Z}^m] \) is regular coherent for every \( m \geq 0 \). Consider any automorphism \( \Phi : A \cong A \) of additive categories. Denote by \( J : A \to \text{Nil}(A, \Phi) \) the inclusion sending an object \( A \) to the object \( (A, 0) \).

Then the induced map on non-connective \( K \)-theory

$$
K^\infty(J) : K^\infty(A) \to K^\infty_{\text{Nil}}(\text{Nil}(A, \Phi))
$$

is a weak homotopy equivalence.

**Proof.** Fix \( n \in \mathbb{Z} \). We have to show that \( \pi_n(K^\infty(J)) \) is bijective. This follows from Lemma 6.6 for \( n \geq 1 \) and is proved in general as follows.

From the definitions and the construction in [5, Section 6] one obtains for every \( n \in \mathbb{Z} \) a commutative diagram

\[
\begin{array}{ccc}
\pi_n(K^\infty(A)) & \longrightarrow & \pi_n(K^\infty_{\text{Nil}}(A, \Phi)) \\
\downarrow i & & \downarrow j \\
\pi_{n+1}(K^\infty(A[Z])) & \longrightarrow & \pi_{n+1}(K^\infty_{\text{Nil}}(A[Z], \Phi[Z])) \\
\downarrow r & & \downarrow s \\
\pi_n(K^\infty(A)) & \longrightarrow & \pi_n(K^\infty_{\text{Nil}}(A, \Phi))
\end{array}
\]

where \( r \circ i = \text{id} \) and \( j \circ s = \text{id} \) and these maps are part of the corresponding (untwisted) Bass-Heller-Swan decompositions. Iterating this, one obtains for every
$m \geq 0$ a commutative diagram

$$
\begin{array}{ccc}
\pi_{-n}(K^\infty(A)) & \xrightarrow{i} & \pi_{-n}(K^\infty_{\Nil}(A, \Phi)) \\
\downarrow & & \downarrow \\
\pi_{n+m}(K^\infty(A[Z^m])) & \xrightarrow{r} & \pi_{n+m}(K^\infty_{\Nil}(A[Z^m], \Phi[Z^m])) \\
\downarrow & & \downarrow \\
\pi_{-n}(K^\infty(A)) & \xrightarrow{s} & \pi_{-n}(K^\infty_{\Nil}(A, \Phi))
\end{array}
$$

where $roi = \text{id}$ and $j \circ s = \text{id}$ holds. Now choose $m$ such that $n + m \geq 1$ holds. Then the middle horizontal arrow can be identified by construction with its connective version

$$
\pi_{n+m}(K(A[Z^m])) \rightarrow \pi_{n+m}(K(\Nil(A[Z^m], \Phi[Z^m])))
$$

Since this map is a bijection by Lemma 6.3 the upper horizontal arrow is a retract of an isomorphism and hence itself an isomorphism. □

**Theorem 6.8** (The non-connective $K$-theory of additive categories). Let $A$ be an additive category. Suppose that $A[Z^m]$ is regular coherent for every $m \geq 0$.

Consider any automorphism $\Phi: A \xrightarrow{\cong} A$ of additive categories.

Then we get a weak homotopy equivalence of non-connective spectra

$$a^\infty: T_{K^\infty(\Phi^{-1})} \xrightarrow{\cong} K^\infty(A[t, t^{-1}]).$$

Proof. This follows from Theorem 5.1 since Lemma 6.7 implies $\pi_n(E^\infty(R, \Phi)) = 0$ and hence $\pi_n(NK^\infty(R[t])) = \pi_n(NK^\infty(A[t^{-1}])) = 0$ for all $n \in \mathbb{Z}$. □

7. NOETHERIAN ADDITIVE CATEGORIES

**Theorem 7.1** (Hilbert Basis Theorem for additive categories).

Consider an additive category $A$ together with an automorphism $\Phi: A \xrightarrow{\cong} A$.

(i) If the additive category $A$ is Noetherian, then the additive categories $A[t]$, $A[t^{-1}]$, and $A[t, t^{-1}]$ are Noetherian;

(ii) If the additive category $A[t]$ is Noetherian, then the additive category $A[t, t^{-1}]$ is Noetherian.

Proof. [11] We only treat $A[t]$, the proof for $A[t^{-1}]$ is analogous. For $A[t, t^{-1}]$ the claim will follow then from [11].

Consider a finitely generated $ZA[t]$-module $N$ and a $ZA[t]$-submodule $M \subseteq N$. We have to show that $M$ is finitely generated. Lemma 5.3 implies that there is an epimorphism $\phi: \text{mor}_{A[t]}(?, A) \rightarrow N$ for some object $A$. If $\phi^{-1}(M)$ is finitely generated, then $M$ is finitely generated since $f$ induces an epimorphism $f^{-1}(M) \rightarrow M$. Hence we can assume without loss of generality $N = \text{mor}_{A[t]}(?, A)$.

Fix an object $Z$ in $A$. Consider a non-trivial element $f: Z \rightarrow A$ in $N(Z)$. We can write it as a finite sum $\sum_{k=0}^{d(f)} f_k \cdot t^k$, where $f_k: \Phi^k(Z) \rightarrow A$ is a morphism in $A$ and $f_{d(f)} \neq 0$. We call the natural number $d(f)$ the degree of $f$ and $R(f) = f_{d(f)}: \Phi^{d(f)}(Z) \rightarrow A$ the leading coefficient of $f$. We put $d(0: Z \rightarrow A) = -\infty$ and $R(0: Z \rightarrow A) = 0$.

We define now $I_d$ as the $ZA$-submodule of $\text{mor}_{A}(?, A)$ that is generated by all $R(f)$ with $f \in M(Z)$ and $d(f) = d$ for some object $Z$ from $A$. We have $I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$ and define $I$ to be the $ZA$-submodule $\bigcup_{d \geq 0} I_d$. As $A$ is by assumption Noetherian, $I$ and all the $I_d$ are finitely generated. Therefore we find a finite collection of morphisms $f_i \in M(Z_i) \subseteq \text{mor}_{A[t]}(Z_i, A)$ such that $R(f_i)$ generate $I$. We abbreviate $d_i := d(f_i)$. Since each $f_i$ lies in of the $I_{d_i}$-s, we can find a natural number $d_0$ such that $I = I_{d_0} = I_d$ holds for $d \geq d_0$. Hence we can also...
arrange for the $f_i$ to have the following property: for each $d$ the $R(f_i)$ with $d_i \leq d$ generate $I_d$. We record that $R(f_i) \in I_d(\Phi^d(Z_i)) \subseteq \text{mor}_A(\Phi^d(Z_i), A)$.

We will show that the $f_i$ generate $M$. Let $f \in M(Z)$, $f \neq 0$. We abbreviate $d := d(f)$. We have $R(f) \in I_d(\Phi^d(Z)) \subseteq \text{mor}_A(\Phi^d(Z), A)$. We can write

$$R(f) = \sum_i R(f_i) \circ \varphi_i$$

with $\varphi_i \in \text{mor}_A(\Phi^d(Z), \Phi^d(Z_i))$ and $\varphi_i = 0$ whenever $d(f_i) > d(f)$. Set

$$\tilde{\varphi}_i := \Phi^{-d_i}(\varphi_i) \cdot t^{d-d_i} \in \text{mor}_A[Z, Z_i].$$

Then

$$R\left(\sum_i f_i \circ \tilde{\varphi}_i\right) = \sum_i R(f_i) \circ \Phi^{d_i}(\Phi^{-d_i}(\varphi_i)) = \sum_i R(f_i) \circ \varphi_i.$$

Thus $d(f - \sum_i f_i \circ \tilde{\varphi}) < d$. Now we can repeat the argument for $f' := f - \sum_i f_i \circ \tilde{\varphi}$. By induction on $d(f)$ we now find that $f$ belongs to the submodule of $\text{mor}_A[Z, Z_i]$ generated by the $f_i$. Hence $M$ is a finitely generated $Z\mathcal{A}_0[t]$-module.

It suffices to show for a $\mathcal{A}_0[t, t^{-1}]$-submodule $M$ of $\text{mor}_A[Z, Z_i]$ that $M$ is finitely generated as $\mathcal{A}_0[t, t^{-1}]$-module. For $Z \in \mathcal{A}$ we have $\text{mor}_A[Z, Z] \subseteq \text{mor}_A[Z, Z_i]$. We define the $Z\mathcal{A}_0[t]$-module $M'$ by

$$M'(Z) := M(Z) \cap \text{mor}_A[Z, Z].$$

Since $\mathcal{A}_0[t]$ is Noetherian, we find a finite collection of morphisms $f_i \in M'(Z_i) \subseteq M(Z_i) \subseteq \text{mor}_A[Z, Z_i, A]$ that generate $M'$ as an $Z\mathcal{A}_0[t]$-module. We claim that the $f_i$ also generate $M$ as an $Z\mathcal{A}_0[t, t^{-1}]$-module. Let $f \in M(Z) \subseteq \text{mor}_A[Z, Z_i]$. For $d \geq 0$ we have $\text{id}_Z \cdot t^d \in \text{mor}_A[Z, Z_i](\Phi^{-d}(Z), Z)$. For sufficiently large $d$ we have $f \circ (\text{id}_Z \cdot t^d) \in \text{mor}_A[Z, Z_i](\Phi^{-d}(Z), Z) \cap M(\Phi^{-d}(Z)) = M'(\Phi^{-d}(Z))$. Thus $f \circ (\text{id}_Z \cdot t^d)$ belongs to the $Z\mathcal{A}_0[t, t^{-1}]$-submodule of $M$ generated by the $f_i$. As $(\text{id}_Z \cdot t^d)$ is an isomorphism in $\mathcal{A}_0[t, t^{-1}]$, $f$ also belongs to the $Z\mathcal{A}_0[t, t^{-1}]$-submodule of $M$ generated by the $f_i$.

8. Additive categories with finite global dimension

Let $\Phi: \mathcal{A} \to \mathcal{A}$ be an automorphism of the additive category $\mathcal{A}$. Let $\overline{\Phi}: \mathcal{A}_0[t] \to \mathcal{A}_0[t]$ be the automorphism of additive categories induced by $\Phi$, which sends the morphisms $\sum_{k=0}^{\infty} f_k \cdot t^k: A \to B$ to the morphism $\sum_{k=0}^{\infty} \Phi(f_k) \cdot t^k: \Phi(A) \to \Phi(B)$. Denote by $i: \mathcal{A} \to \mathcal{A}_0[t]$ the inclusion sending $f: A \to B$ to $(f, t^0): A \to B$. Obviously we have $\overline{\Phi} \circ i = i \circ \Phi$.

8.1. The characteristic sequence. Consider a $Z\mathcal{A}_0[t]$-module $M$. Let

$$c: i_* i^* M \to M$$

be the $Z\mathcal{A}_0[t]$-morphism which is the adjoint of the $Z\mathcal{A}$-homomorphism $\text{id}: i^* M \to i^* M$ under the adjunction [4.7]. We get for every object $A$ in $\mathcal{A}$ a morphism $\text{id}_{\Phi(A)}: A \to \Phi(A)$ in $\mathcal{A}_0[t]$. It induces a $Z\mathcal{A}_0[t]$-morphism $M(\text{id}_{\Phi(A)} \cdot t): M(\Phi(A)) \to M(A)$. Since for a morphisms $u: A \to B$ in $\mathcal{A}$ we have

$$(\text{id}_{\Phi(B)} \cdot t) \circ u = (\text{id}_{\Phi(B)} \cdot t) \circ (u \cdot t^0) = \Phi(u) \cdot t$$

$$= (\Phi(u) \cdot t^0) \circ (\text{id}_{\Phi(A)} \cdot t) = i(\Phi(u)) \circ (\text{id}_{\Phi(A)} \cdot t),$$

we obtain a morphism of $Z\mathcal{A}$-modules

$$\alpha': \Phi^* i^* M \xrightarrow{\simeq} i^* M.$$

By applying $i_*$ we obtain a morphism of $Z\mathcal{A}_0[t]$-modules

$$\alpha: i_* \Phi^* i^* M \to i_* i^* M.$$
The morphism \(\text{id}_{\Phi(A)} \cdot t: A \to \Phi(A)\) in \(A_\Phi[\mathbb{t}]\) induces also a \(\mathbb{Z}\)-map \\
\(\beta(A): i_*\Phi^*i^*M(A) = i_*i^*M(\Phi(A)) \to i_*i^*M(A)\).

Since for any morphism \(v = \sum_k^\infty f_k \cdot t^k: A \to B\) in \(A_\Phi[\mathbb{t}]\) we have

\[
(id_{\Phi(B)} \cdot t) \circ v = (id_{\Phi(B)} \cdot t) \circ \left(\sum_k f_k \cdot t^k\right)
\]

\[
= \sum_k (id_{\Phi(B)} \cdot t) \circ (f_k \cdot t^k)
\]

\[
= \sum_k \Phi(f_k) \cdot t^{k+1}
\]

\[
= \sum_k (\Phi(f_k) \cdot t^k) \circ (id_{\Phi(A)} \cdot t)
\]

\[
= \Phi\left(\sum_k f_k \cdot t^k\right) \circ (id_{\Phi(A)} \cdot t)
\]

\[
= \Phi(v) \circ (id_{\Phi(A)} \cdot t)
\]

we get a \(\mathbb{Z}A_\Phi[\mathbb{t}]\)-homomorphism denoted by

\[
\beta: i_*\Phi^*i^*M \to i_*i^*M.
\]

Define the so called characteristic sequence of \(\mathbb{Z}A_\Phi[\mathbb{t}]\)-modules by

\[
0 \to i_*\Phi^*i^*M \xrightarrow{\alpha = \beta} i_*i^*M \xrightarrow{e} M \to 0.
\]

Given an object \(A \in A_\Phi, (\alpha - \beta)(A)\) is explicitly given by

\[
M(\Phi(\تاب)) \otimes_{\mathbb{Z}A} \text{mor}_{A_\Phi[\mathbb{t}]}(A, \تاب) \to M(\تاب) \otimes_{\mathbb{Z}A} \text{mor}_{A_\Phi[\mathbb{t}]}(A, \تاب),
\]

\[
x \otimes (f_k \cdot t^k: A \to \تاب) \mapsto M(id_{\Phi(\تاب)} t^k: \تاب \to (\تاب))(x) \otimes (f_k \cdot t^k: A \to \تاب)
\]

\[
= x \otimes (\Phi(f_k) \cdot t^{k+1}: A \to \Phi(\تاب)).
\]

and \(e(A)\) is explicitly given by

\[
M(\تاب) \otimes_{\mathbb{Z}A} \text{mor}_{A_\Phi[\mathbb{t}]}(A, \تاب) \to M(A),
\]

\[
x \otimes (u: A \to \تاب) \mapsto M(u)(x) = xu.
\]

Lemma 8.3. The characteristic sequence (8.2) is natural in \(M\) and exact.

Proof. It is obviously natural in \(M\). To prove exactness, it suffices to prove the exactness of the sequence of \(\mathbb{Z}A\)-modules

\[
0 \to i_*\Phi^*i^*M \xrightarrow{\alpha = \beta} i_*i^*M \xrightarrow{e} i^*M \to 0.
\]

Let \(N\) be a \(\mathbb{Z}A\)-module. We obtain a \(\mathbb{Z}A\)-isomorphism

\[
S(N): \bigoplus_{k=0}^\infty \Phi^k(N) \xrightarrow{\sim} i^*i_\star N,
\]

which is defined for an object \(A\) in \(A\) by the \(\mathbb{Z}\)-isomorphism

\[
S(N)(A): \bigoplus_{k=0}^\infty N(\Phi^k(A)) \xrightarrow{\sim} i^*i_\star N(A) = N(\تاب) \otimes_{\mathbb{Z}A} \text{mor}_{A_\Phi[\mathbb{t}]}(i(A), \تاب)
\]

sending \((x_k)_{k \geq 0}\) to \(\sum_{k=0}^\infty x \otimes (id_{\Phi(A)} t^k: A \to \Phi^k(A))\). The inverse of \(S(N)(A)\) sends \(y \otimes \left(\sum_{k=0}^\infty f_k \cdot t^k: A \to \تاب\right)\) to \(\sum_{k=0}^\infty N(f_k)(y)\). Applying this to \(N = i^*\Phi^* M =
\]
\[ \Phi^* i^* M \text{ and } N = i^* M, \] we get identifications
\[ i^* s \iota^* \Phi M = \bigoplus_{k=1}^{\infty} (\Phi^k)^* i^* M; \]
\[ i^* s \iota^* M = \bigoplus_{k=0}^{\infty} (\Phi^k)^* i^* M. \]

Consider natural numbers \( m \) and \( n \) with \( m \geq n \). For an object \( A \) let the map \( s_{m,n}(A): (\Phi^m)^* M(A) \rightarrow (\Phi^n)^* M(A) \) be the map obtained by applying \( M \) to the morphism \( \text{id}_{\Phi^m(A)} \cdot t^{m-n}: \Phi^n(A) \rightarrow \Phi^m(A) \) in \( A_\Phi[t] \). This yields a map of \( \mathbb{Z}A \)-modules
\[ s_{m,n}: (\Phi^m)^* i^* M \rightarrow (\Phi^n)^* i^* M. \]

Under these identifications the \( \mathbb{Z}A \)-sequence \( (\Phi) \) becomes the sequence
\[ 0 \rightarrow \bigoplus_{m=1}^{\infty} (\Phi^m)^* i^* M \rightarrow \bigoplus_{n=0}^{\infty} (\Phi^n)^* i^* M \]
\[ \xrightarrow{(\text{id} \ s_{1,0} \ s_{2,0} \cdots)} i^* M \rightarrow 0. \]

Since \( s_{m,n} \circ s_{l,m} = s_{l,n} \) for \( l \geq m \geq n \) and \( s_{m,m} = \text{id} \) hold, this sequence is split exact, with a splitting given by
\[ \bigoplus_{m=1}^{\infty} (\Phi^m)^* M \xleftarrow{(\text{id} \ s_{2,1} \ s_{3,1} \ s_{4,1} \cdots)} \bigoplus_{n=0}^{\infty} (\Phi^n)^* M \xrightarrow{\bigoplus_{n=0}^{\infty} (\Phi^n)^* i^* M} M. \]

\[ \square \]

### 8.2. Localization.

**Definition 8.6 (Local module).** We call a \( \mathbb{Z}A_\Phi[t] \)-module \( M \) local if for any object \( A \) in \( A \) and any natural number \( k \in \mathbb{N} \) the map
\[ M(\text{id}_{\Phi^k(A)} \cdot t^k): M(A) \rightarrow M(\Phi^k(A)) \]
induced by the morphism \( \text{id}_{\Phi^k(A)} \cdot t^k: A \rightarrow \Phi^k(A) \) in \( A_\Phi[t] \) is bijective.

Let \( j: A_\Phi[t] \rightarrow A_\Phi[t, t^{-1}] \) be the inclusion.

**Lemma 8.7.** A \( \mathbb{Z}A_\Phi[t] \)-module \( M \) is local if and only if there is a \( \mathbb{Z}A_\Phi[t, t^{-1}] \)-module \( N \) such that \( M \) and \( j^* N \) are isomorphic as \( \mathbb{Z}A_\Phi[t] \)-modules.

**Proof.** Since the morphism \( \text{id}_{\Phi^k(A)} \cdot t^k: A \rightarrow \Phi^k(A) \) in \( A_\Phi[t] \) becomes invertible when considered in \( A_\Phi[t, t^{-1}] \), a \( \mathbb{Z}A_\Phi[t] \)-module \( M \) is local, if there is a \( \mathbb{Z}A_\Phi[t, t^{-1}] \)-module \( N \) such that \( M \) and \( j^* N \) are isomorphic as \( \mathbb{Z}A_\Phi[t, t^{-1}] \)-modules.

Now consider a local \( \mathbb{Z}A_\Phi[t] \)-module \( M \). We have to explain how the \( \mathbb{Z}A_\Phi[t] \)-structure extends to a \( \mathbb{Z}A_\Phi[t, t^{-1}] \)-structure. Consider a morphism \( u: A \rightarrow B \)
in \(A_\Phi[l,t^{-1}]\). Then we can choose a natural number \(m\) such that the composite \(A \xrightarrow{\sim} B \xrightarrow{\Phi_m(B)} \Phi_m(B)\) is a morphism in \(A_\Phi[l]\). Hence we have the \(\mathbb{Z}\)-map \(M((\Phi_m(B) \circ u): M(\Phi_m(B)) \to M(A)\). Since \(M\) is local, the \(\mathbb{Z}\)-map \(M((\Phi_m(B) \circ u): M(\Phi_m(B)) \to M(B)\) is an isomorphism. Now define

\[
M(u) : (M(B) \xrightarrow{M((\phi_m(B) \circ u))} M(A)) \quad \text{by the following calculation}
\]

We leave it to the reader to check that the definition of \(M(u)\) is independent of the choice of \(m\) and that we obtain the desired \(\mathbb{Z}A_\Phi[l,t^{-1}]\)-structure on \(M\) extending the given \(\mathbb{Z}A_\Phi[l]\)-structure.

Let \(M\) be a \(\mathbb{Z}A_\Phi[l]\)-module. We want to assign to it a \(\mathbb{Z}A_\Phi[l,t^{-1}]\)-module \(S^{-1}M\) as follows. Consider an object \(A\) in \(A\). Define the abelian group

\[
S^{-1}M(A) := \{(l, x) \mid l \in \mathbb{Z}, x \in M(\Phi^l(A))\}/\sim
\]

for the equivalence relation \(\sim\), where \((l_0, x_0)\) and \((l_1, x_1)\) are equivalent if and only if there is an integer \(l \in \mathbb{Z}\) with \(l_0, l_1\) such that the elements \(M((\phi_m(B) \circ (\Phi_m(B) \circ u))\) and \(M((\phi_m(B) \circ (\Phi_m(B) \circ u))\) of \(M(\Phi^l(A))\) agree. Given a morphism \(u : A \to B\) in \(A_\Phi[l,t^{-1}]\), we can choose a natural number \(m\) such that the composite \(A \xrightarrow{\sim} B \xrightarrow{\phi_m(B)} \Phi_m(B)\) is a morphism in \(A_\Phi[l]\). Define \(S^{-1}M(u) : S^{-1}M(B) \to S^{-1}M(A)\) by sending \([l, x]\) to the class of \([l - m, M(\Phi^{l-m}(\phi_m(B) \circ u)))(x)\). This is independent of the choice of the representative of \([l, x]\), since we get for the different representative \([l - 1, M(\phi_m(B) \circ u)](x)\)

\[
S^{-1}(M)([l - 1, M(\phi_m(B) \circ u)](x)) = [l - 1 - m, M(\Phi^{l-m}(\phi_m(B) \circ u)) \circ M(\phi_m(B) \circ u)](x)
\]

This is independent of the choice of \(m\) by the following calculation

\[
[l - (m + 1), M(\Phi^{l-(m+1)}(\phi_m(B) \circ u))](x)
\]

We leave it to the reader to check that \(S^{-1}M(v \circ u) = S^{-1}M(u) \circ S^{-1}M(v)\) holds for any two composable morphisms \(u : A \to B\) and \(v : B \to C\) in \(A_\Phi[l,t^{-1}]\) and
$S^{-1}M(\text{id}_A) = \text{id}_{S^{-1}M(A)}$ holds for any object $A$ in $\mathcal{A}$. Notice that the $\mathcal{A}_\wp[t]$-module $j^*S^{-1}M$ is local by Lemma 8.13.

There is a natural map of $\mathcal{A}_\wp[t]$-modules

$$I: M \to j^*S^{-1}M$$

which is given for an object $A$ of $\mathcal{A}$ by the map $I(A): M(A) \to S^{-1}M(A)$ sending $x$ to $(0,x)$. We claim that $I$ is a localization in the sense that for any local $\mathcal{A}_\wp[t]$-module $N$ and any $\mathcal{A}_\wp[t]$-homomorphism $f: M \to N$ there exists precisely one $\mathcal{A}_\wp[t]$-homomorphism $S^{-1}f: S^{-1}M \to N$.

Firstly we explain that there is at most one such map $S^{-1}f$ with these properties. Namely, consider an object $A \in \mathcal{A}$ and an element $[m,x] \in S^{-1}(M)(A)$. If $m \geq 0$, then we compute

$$S^{-1}f(A)([m,x]) = S^{-1}(A)([0,M(\text{id}_{\phi^m(A)} \cdot t^m)(x)])$$

$$= S^{-1}(A) \circ I(A) \circ M(\text{id}_{\phi^m(A)} \cdot t^m)(x)$$

$$= f(A) \circ M(\text{id}_{\phi^m(A)} \cdot t^m)(x).$$

Suppose $m \leq 0$. Since we have $S^{-1}(M)(\text{id}_A \cdot t^{-m})([m,x]) = [0,x]$, we compute for $[m,x] \in S^{-1}M(A)$

$$S^{-1}(N)(\text{id}_A \cdot t^{-m}) \circ S^{-1}f(A)([m,x])$$

$$= S^{-1}f(\phi^m(A)) \circ S^{-1}(M)(\text{id}_A \cdot t^{-m})([m,x])$$

$$= S^{-1}f(\phi^m(A))(0,x)$$

$$= S^{-1}f(\phi^m(A)) \circ I(A)(x)$$

$$= f(\phi^m(A))(x).$$

Since the locality of $N$ implies that $S^{-1}(N)(\text{id}_{\phi^m(A)} \cdot t^m)$ is an isomorphism, we conclude

$$S^{-1}f(A)([m,x]) = S^{-1}(N)(\text{id}_A \cdot t^{-m})^{-1} \circ f(\phi^m(A))(x).$$

Hence $S^{-1}f(A)$ is determined by the equations (8.8) and (8.9). We leave it to the reader to check that it makes sense to define the desired $\mathbb{Z}_\wp[t]$-homomorphism $S^{-1}f(A)$ by the equations (8.8) and (8.9).

The adjoint of $I: M \to j^*S^{-1}M$ under the adjunction (4.7) is denoted by

$$\alpha: j_*M \to S^{-1}M.$$  

The adjoint of $\text{id}_{j_*M}$ under the adjunction (4.7) is the $\mathbb{Z}_\wp[t]$-homomorphism

$$\lambda: M \to j^*j_*M$$

which is explicitly given by $M(\text{id})(? \to \text{mor}_{\mathcal{A}_\wp[t]} (\text{id}, ?) \otimes \mathbb{Z}_\wp[t])$ sending $u \in M(\text{id})$ to $\text{id} \otimes u$. Given an $\mathbb{Z}_\wp[t, t^{-1}]$-module $N$, the adjoint of $\text{id}_{j_*N}$ under the adjunction (4.7) is the $\mathbb{Z}_\wp[t, t^{-1}]$-homomorphism

$$\rho: j_*j^*N \to N$$

which is explicitly given by $N(\text{id}) \otimes \mathbb{Z}_\wp[t] \text{mor}_{\mathcal{A}_\wp[t, t^{-1}]}(\text{id}, ?) \to N(\text{id})$ sending $x \otimes u$ to $N(u)(x) = xu$.

Lemma 8.13.  

(i) The $\mathbb{Z}_\wp[t]$-homomorphism $\lambda: M \to j^*j_*M$ of (8.11) is a localization;

(ii) The $\mathbb{Z}_\wp[t, t^{-1}]$-homomorphism $\alpha: j_*M \to S^{-1}M$ of (8.10) is an isomorphism, which is natural in $M$;

(iii) Let $N$ be a $\mathbb{Z}_\wp[t, t^{-1}]$-module. Then $\mathbb{Z}_\wp[t, t^{-1}]$-map $\rho: j_*j^*N \to N$ of (8.12) is an isomorphism.
Proof. (i) Let \( f : M \to N \) be a \( \mathbb{Z}A_\Phi[t] \)-map with a local \( \mathbb{Z}A_\Phi[t] \)-module as target. Because of Lemma 8.7, there is a \( \mathbb{Z}A_\Phi[t, t^{-1}] \)-module \( N' \) and a \( \mathbb{Z}A_\Phi[t] \)-isomorphism \( u : N \to j^*N' \). Let the \( \mathbb{Z}A_\Phi[t, t^{-1}] \)-map \( v_0 : j_*M \to N \) be the adjoint of \( u \circ f \) under the adjunction (8.7). Because of the naturality of the adjunction (8.7) we get for the composite \( \overline{f} : j^*j_*M \to j^*N' \) there is a natural number such that the composite of \( u \) with \( \text{id}_{\mathbb{Z}A_\Phi[B]} \cdot t^m : \Phi(M) \to \Phi^m(B) \) lies in \( \mathbb{Z}A_\Phi[t] \) that \( \overline{f} \) is uniquely determined by \( \overline{f} \circ \lambda = f \).

(ii) Obviously \( \alpha \) is natural in \( M \). The naturality of the adjunction (8.7) implies
\[
j^*\alpha \circ \lambda = 1.
\]
Since both \( I : M \to j^*S^{-1}M \) and \( \lambda : M \to j^*j_*M \) localizations, \( j^*\alpha \) and hence \( \alpha \) are bijective.

(iii) It suffices to show that \( j^*\rho : j^*j_*s \to j^*N \) is bijective. Assertion (ii) applied to \( j^*N \) and the naturality of the adjunction (8.7) imply that \( j^*\rho : j^*N \to j^*j_*N \) is a localization. Since \( \text{id}_{j^*N} : j^*N \to j^*N \) is a localization, \( j^*\rho \) is an isomorphism. □

Lemma 8.14. The functor \( j_* : \text{MOD-Z}\mathbb{Z}A_\Phi[t] \to \text{MOD-Z}\mathbb{Z}A_\Phi[t, t^{-1}] \) is flat.

Proof. Because of the adjunction (8.7), the functor \( j_* \) is right exact by a general argument, see [10], Theorem 2.6.1, on page 51]. Hence it remains to show that for an injective \( \mathbb{Z}A_\Phi[t] \)-map \( i : M \to N \) the \( \mathbb{Z}A_\Phi[t, t^{-1}] \)-map \( j_*i : j_*M \to j_*N \) is injective. In view of Lemma 8.13 (ii) it suffices to show that \( S^{-1}i : S^{-1}M \to S^{-1}N \) is injective. Consider an object \( A \) in \( \mathcal{A} \) and an element \( [l, x] \) in the kernel of \( S^{-1}i(l)(A) \). Since \( S^{-1}i[l, x] = [l, i(\Phi(A))(x)] \), there is a natural number \( m \leq l \) such that \( N(\text{id}_{\mathbb{Z}A_\Phi[A]} \cdot t^m \cdot i(\Phi(A))(x)) = 0 \). Since \( N(\text{id}_{\mathbb{Z}A_\Phi[A]} \cdot t^m \cdot i(\Phi(A)) \circ i(\Phi(A)) = i(\Phi^{-1}(A)) \circ M(\text{id}_{\mathbb{Z}A_\Phi[A]} \cdot t^m) \) and \( i(\Phi^{-1}(A)) \) is by assumption injective, \( M(\text{id}_{\mathbb{Z}A_\Phi[A]} \cdot t^m) = 0 \). This implies \( [l, x] = 0 \). □

8.3. Global dimension. Recall that an additive category \( \mathcal{A} \) has global dimension \( \leq d \) if the abelian category \( \text{MOD-Z}\mathcal{A} \) has global dimension \( \leq d \), i.e., if every \( \mathbb{Z}A \)-module has a projective \( d \)-dimensional resolution, see Definition 5.2 (vi).

Theorem 8.15 (Global dimension and the passage from \( \mathcal{A} \) to \( \mathbb{Z}A_\Phi[t] \)). Let \( \mathcal{A} \) be an additive category \( \mathcal{A} \) and \( \Phi : \mathcal{A} \to \mathcal{A} \) be an automorphism of additive categories.

(i) Let \( M \) be a \( \mathbb{Z}A_\Phi[t] \)-module. If \( \text{pd}_{\mathcal{A}}(\Phi^{i}(M)) \leq d \), then \( \text{pd}_{\mathcal{A}_\Phi[t]}(M) \leq d+1 \);
(ii) If \( \mathcal{A} \) has global dimension \( \leq d \), then \( \mathbb{Z}A_\Phi[t] \) has global dimension \( \leq (d+1) \).

Proof. (i) Obviously \( \Phi^i : \text{MOD-Z}\mathbb{Z}A_\Phi[t] \to \text{MOD-Z}\mathcal{A} \) is faithfully flat and is compatible with direct sums over arbitrary index sets. Next we show that \( \Phi^i \) sends projective \( \mathbb{Z}A_\Phi[t] \)-modules to projective \( \mathbb{Z}A \)-modules. It suffices to show that \( \text{mor}_{\mathbb{Z}A_\Phi[t]}(\Phi^{i}(M)) \equiv \text{mor}_{\mathcal{A}}(\Phi^{i}(M)) \) is free as a \( \mathbb{Z}A \)-module for any object \( A \). This follows from the \( \mathbb{Z}A \)-isomorphism (8.5), since \( (\Phi^{k})^* \text{mor}_{\mathcal{A}}(\Phi^{i}(M)) \equiv \text{mor}_{\mathcal{A}}(\Phi^{i}(M)) \).

The functor \( \Phi^{i} : \text{MOD-Z}\mathcal{A} \to \text{MOD-Z}\mathbb{Z}A_\Phi[t] \) is compatible with direct sums over arbitrary index sets, is right exact and sends \( \text{mor}_{\mathcal{A}}(\Phi^{i}(M)) \) to \( \text{mor}_{\mathbb{Z}A_\Phi[t]}(\Phi^{i}(M)) \). In particular, \( i_* \) respects the properties finitely generated, free, and projective. Next we want to show that \( i_* \) is flat. For this purpose it suffices to show that \( \Phi^i \circ i_* \) is faithfully flat. This is obvious since \( i_* \circ i_* \) is the functor sending a morphism \( f : M \to N \) to the morphism \( \bigoplus_{k \geq 0} (\Phi^{k})^{*}(f) : \bigoplus_{k \geq 0} (\Phi^{k})^{*}(M) \to \bigoplus_{k \geq 0} (\Phi^{k})^{*}(f) \) under the identification (8.5).

Now consider a \( \mathbb{Z}A_\Phi[t] \)-module \( M \) with \( \text{pd}_{\mathcal{A}}(\Phi^{i}(M)) \leq d \). Since the \( \mathbb{Z}A \)-modules \( \Phi^{i}(M) \) are isomorphic, see (5.1), we get \( \text{pd}_{\mathcal{A}}(\Phi^{i}(M)) \leq d \). Since \( i_* \) is faithfully flat and respects projective modules, we conclude \( \text{pd}_{\mathcal{A}_\Phi[t]}(i_*\Phi^{i}(M)) \leq d \).
and \( \text{pdim}_{A[t]}(i_\ast M) \leq d \). Now Lemma 4.3(iv) and Lemma 8.3 together imply 
\( \text{pdim}_{A[t]}(M) \leq (d + 1) \).

(iii) This follows directly from assertion (i).

\[ \square \]

**Theorem 8.16** (Global dimension and the passage from \( A[t] \) to \( A[t, t^{-1}] \)). Let \( A \) be an additive category \( \Phi: A \to A \) be an automorphism of additive categories.

(i) Let \( M \) be a \( \mathbb{Z}A[t, t^{-1}] \)-module. If we have \( \text{pdim}_{A[t]}(j \ast M) \leq d \), then we get 
\( \text{pdim}_{A[t, t^{-1}]}(M) \leq d \);

(ii) If \( A[t] \) has global dimension \( \leq d \), then \( A[t, t^{-1}] \) has global dimension \( \leq d \).

**Proof.** (i) \( \Phi: A \to A \) be an automorphism of additive categories. Let \( j: \text{MOD-} \mathbb{Z}A[t] \to \text{MOD-} \mathbb{Z}A[t, t^{-1}] \) is flat by Lemma 8.14. Since it respects the property projective, we get \( \text{pdim}_{A[t]}(j \ast M) \leq d \). Lemma 8.13(iii) implies \( \text{pdim}_{A[t, t^{-1}]}(M) \leq d \).

(iii) This follows from assertion (i).

\[ \square \]

9. **Regular additive categories**

Regularity for additive categories \( A \) requires finite resolutions of finitely presented modules, but not for arbitrary modules. In particular, regularity has no consequence for global dimension and we cannot use Theorem 8.3.5 in the following result.

**Theorem 9.1** (Regularity and the passage from \( A \) to \( A[t] \)). Let \( A \) be an additive category \( \Phi: A \to A \) be an automorphism of additive categories. Let \( t \) be a natural number.

(i) Suppose that \( A \) is regular or \( t \)-uniformly regular respectively. Then \( A[t] \) is regular or \( (t + 2) \)-uniformly regular respectively;

(ii) Suppose that \( A[t] \) is regular or \( t \)-uniformly regular respectively. Then \( A[t, t^{-1}] \) is regular or \( t \)-uniformly regular respectively.

**Proof.** (i) We know already that \( A[t] \) is Noetherian because of Theorem 7.4(i). Let \( M \) be a finitely generated \( A[t] \)-module. We have to show that it has a finitely generated projective resolution which is finite-dimensional or \((t+1)\)-dimensional.

Since \( A[t] \) is Noetherian, there exists a finitely generated projective resolution of \( M \) which may be infinite-dimensional. We conclude from Theorem 4.3(iv) that it suffices to show the projective dimension of \( M \) is finite or bounded by \( (t+1) \) respectively. As \( M \) is finitely generated we find a finite collection of elements \( x_j \in M(Z_j) \) with objects \( Z_j \) from \( A \) such that the \( x_j \) generate \( M \) as an \( \mathbb{Z}A[t] \)-module.

For \( d \geq 0 \) consider the morphism \( \text{id}_{Z_j} : \Phi^{-d}(Z_j) \to Z_j \) in \( A[t] \) and set \( x_j[d] := M(\text{id}_{Z_j} \circ t^d)(x_j) \in M(\Phi^{-d}(Z_j)) \). Let \( M_n \) be the \( \mathbb{Z}A \)-submodule of \( i \ast M \) generated by all \( x_j[d] \) with \( d \leq n \). We obtain an increasing subsequence \( M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \) of \( \mathbb{Z}A \)-submodules of \( i \ast M \) with \( i \ast M = \bigcup_{n \geq 0} M_n \). Let \( T_n : i \ast M \to \Phi \ast i \ast M \) be the following \( \mathbb{Z}A \)-morphism. For an object \( Z \) from \( A \) consider \( \text{id}_{\Phi^{-d}(Z)} : t^n \in \text{mor}_{A[t]}(Z, \Phi^{-d}(Z)) \) and define \( T_Z : i \ast M(Z) = M(Z) \to \Phi \ast i \ast M(Z) = M(\Phi(Z)) \) by \( T(Z) : i \ast M \to \Phi \ast i \ast M \). Let \( \text{pr}_n : (\Phi^n)^* M_n \to (\Phi^n)^* (M_{n+1}) \) be the projection. The composition \( f_n \)

\[ M_0 \xrightarrow{T_n} (\Phi^n)^* M_n \xrightarrow{\text{pr}_n} (\Phi^n)^* M_n \] is surjective and we write \( K_n \) for its kernel. We obtain an increasing sequence of \( A \)-submodules \( K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \) of \( M_0 \). Since \( A \) is Noetherian and \( M_0 \) is finitely generated, there exists an integer \( n_0 \geq 1 \) such that \( K_n = K_{n_0} \) holds for \( n \geq n_0 \). Define \( g_n : (\Phi^n)^* M_{n_0} / (\Phi^n)^* M_{n_0-1} \to (\Phi^n)^* M_n / (\Phi^n)^* M_{n-1} \) for \( n \geq n_0 \).
to be the map induced by $\Phi^*(T_{n-n_0})$ for $n \geq n_0$. We obtain for every natural number $n$ with $n \geq n_0$ a commutative diagram of $\mathbb{Z}A$-modules with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_{n_0} & \longrightarrow & M_0 & \xrightarrow{f_0} & (\Phi^{n_0})^* \frac{M_{n_0}}{(\Phi^{n_0})^* M_{n_0-1}} & \longrightarrow & 0 \\
\downarrow{\cong} & & \downarrow{id_{n_0}} & & \downarrow{g_n} & & & & \\
0 & \longrightarrow & K_n & \longrightarrow & M_0 & \xrightarrow{f_n} & (\Phi^n)^* M_n & \longrightarrow & 0
\end{array}
$$

Hence $g_n$ is an isomorphism of $\mathbb{Z}A$-module $n \geq n_0$. As $\Phi^*$ is an isomorphism we have

$$\text{pdim}_{\mathbb{Z}A} M_n/M_{n-1} = \text{pdim}_{\mathbb{Z}A}(\Phi^n)^*(M_n/M_{n-1}) = \text{pdim}_{\mathbb{Z}A}(\Phi^n)^* M_n/\Phi^n M_{n-1}).$$

Thus for $n \geq n_0$ we have $\text{pdim}_{\mathbb{Z}A}(M_n/M_{n-1}) = \text{pdim}_{\mathbb{Z}A}(M_{n_0}/M_{n_0-1})$. We have the short exact sequence $0 \rightarrow M_{n-1} \rightarrow M_n \rightarrow M_n/M_{n-1} \rightarrow 0$ and hence get from Lemma 4.4 (v)

$$\text{pdim}_{\mathbb{Z}A}(M_n) \leq \sup\{\text{pdim}_{\mathbb{Z}A}(M_{n_1}), \text{pdim}_{\mathbb{Z}A}(M_{n_0}/M_{n_0-1})\}.$$

This implies by induction over $n \geq n_0$

$$\text{pdim}_{\mathbb{Z}A}(M_n) \leq \sup\{\text{pdim}_{\mathbb{Z}A}(M_{n_1}), \text{pdim}_{\mathbb{Z}A}(M_{n_0}/M_{n_0-1})\}.$$

Put

$$D := \sup\{\text{pdim}_{\mathbb{Z}A}(M_k) \mid k = 0, 1, \ldots, n_0 - 1\}, \text{pdim}_{\mathbb{Z}A}(M_{n_0}/M_{n_0-1})\}.$$  

Notice that $D < \infty$ if $A$ is regular and $D \leq l$ if $A$ is uniformly $l$-regular. We get

$$\text{pdim}_{\mathbb{Z}A}\left(\bigoplus_{n \geq 0} M_n\right) \leq \sup\{\text{pdim}_{\mathbb{Z}A}(M_n) \mid n \geq 0\} \leq D.$$  

We have the short exact sequence of $A$-modules

$$0 \rightarrow \bigoplus_{n \geq 0} M_n \rightarrow \bigoplus_{n \geq 0} M_n \rightarrow i^* M \rightarrow 0,$$

where the first map is given by $(x_n)_{n \geq 0} \mapsto (x_0, x_1 - x_0, x_2 - x_1, \ldots)$ and the second by $(x_n)_{n \geq 0} \mapsto \sum_{n \geq 0} x_n$. We conclude from Lemma 4.4 (v)

$$\text{pdim}_{\mathbb{Z}A}(i^* M) \leq D + 1.$$  

Now Theorem 8.15 (i) implies

$$\text{pdim}_{\mathbb{Z}A[t]}(M) \leq D + 2.$$  

This finishes the proof if assertion (i)

We know already that $A_0[t, t^{-1}]$ is Noetherian because of Theorem 7.1 (ii). Let $M$ be a finitely generated $A_0[t, t^{-1}]$-module. We can find a finitely generated free $\mathbb{Z}A_0[t]$-module $F_0$ and a free $\mathbb{Z}A_0[t]$-module $F_1$ together with an exact sequence of $\mathbb{Z}A_0[t, t^{-1}]$-modules $j_0 : F_0 \xrightarrow{\phi} j_1 : F_1 \rightarrow M \rightarrow 0$. Here we write $j$ for the inclusion $A_0[t] \rightarrow A_0[t, t^{-1}]$. By composing $\phi$ with an appropriate automorphism of $j_0 F_0$ one can arrange that $f = j_0 g$ for some $\mathbb{Z}A_0[t]$-homomorphism $g : F_0 \rightarrow F_1$. The cokernel of $g$ is a finitely generated $\mathbb{Z}A_0[t]$-module $N$ and there is an obvious exact sequence of $\mathbb{Z}A_0[t, t^{-1}]$-modules $F_1 \xrightarrow{\phi} F_1 \rightarrow N \rightarrow 0$. Since the functor $j_0$ is flat by Lemma 8.13 and respects the property projective, we obtain an $\mathbb{Z}A_0[t, t^{-1}]$-isomorphism $j_0 N \xrightarrow{\cong} M$ and have $\text{dim}_{\mathbb{Z}A_0[t, t^{-1}]}(j_0 N) \leq \text{dim}_{\mathbb{Z}A_0[t]}(N)$. Hence we get $\text{dim}_{\mathbb{Z}A_0[t, t^{-1}]}(M) \leq \text{dim}_{\mathbb{Z}A_0[t]}(N)$. This finishes the proof of Theorem 9.1. \qed

Remark 9.2. We do not know whether Theorem 9.1 remains true if we replace regular by regular coherent. To our knowledge it is an open problem, whether for a regular coherent ring $R$ the rings $R[t]$ or $R[t, t^{-1}]$ are regular coherent again.
10. Directed union and infinite products of additive categories

A functor of additive categories \( F: \mathcal{A} \to \mathcal{B} \) is called flat if for every exact sequence \( A_0 \xrightarrow{i} A_1 \xrightarrow{p} A_2 \) in \( \mathcal{A} \), the sequence \( F(A_0) \xrightarrow{F(i)} F(A_1) \xrightarrow{F(p)} F(A_2) \) in \( \mathcal{B} \) is exact. It is called faithfully flat if it is flat and for every exact sequence \( A_0 \xrightarrow{i} A_1 \xrightarrow{p} A_2 \) in \( \mathcal{A} \), \( F(A_0) \xrightarrow{F(i)} F(A_1) \xrightarrow{F(p)} F(A_2) \) in \( \mathcal{B} \) is exact.

**Lemma 10.1.** Let \( i: \mathcal{A} \to \mathcal{A}' \) and \( j: \mathcal{B} \to \mathcal{B}' \) be inclusions of cofinal full additive subcategories. Suppose that the following diagram of functors of additive categories commutes

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
i & \downarrow & \downarrow j \\
\mathcal{A}' & \xrightarrow{F'} & \mathcal{B}'
\end{array}
\]

Then

(i) The inclusion \( i: \mathcal{A} \to \mathcal{A}' \) is faithfully flat;

(ii) \( F \) is flat or faithfully flat respectively if and only if \( F' \) is flat or faithfully flat respectively.

**Proof.** We first show that \( F' \) is exact or faithfully exact respectively provided that \( F \) is exact or faithfully exact respectively.

Consider morphisms \( f': A'_0 \to A'_1 \) and \( g': A'_1 \to A'_2 \) in \( \mathcal{A} \). Choose objects \( A_k \) in \( \mathcal{A} \) and morphisms \( i_k: A'_k \to A_k \) and \( r_k: A_k \to A'_k \) in \( \mathcal{A}' \) satisfying \( r_k \circ i_k = \text{id}_{A_k} \) for \( k = 0, 1, 2 \). Define \( f: A_0 \to A_1 \) and \( g: A_1 \to A_2 \) by \( f = i_1 \circ f' \circ r_0 \) and \( g = i_2 \circ g' \circ r_1 \). Then the following diagram of morphisms in \( \mathcal{A}' \) commutes

\[
\begin{array}{cccc}
A'_0 & \xrightarrow{f'} & A'_1 & \xrightarrow{g'} & A'_2 \\
\downarrow i_0 & & \downarrow i_1 & & \downarrow i_2 \\
A_0 & \xrightarrow{f} & A_1 & \xrightarrow{g \oplus (\text{id}_{A_1} - i_1 \circ r_1)} & A_2 \oplus A_1 \\
\downarrow r_0 & & \downarrow r_1 & & \downarrow r_2 \oplus 0 \\
A'_0 & \xrightarrow{f'} & A'_1 & \xrightarrow{g'} & A'_2
\end{array}
\]

Next we check that the middle row is exact in \( \mathcal{A} \) if and only if the upper row is exact in \( \mathcal{A}' \). Suppose that the middle row is exact in \( \mathcal{A} \). Consider a morphism \( v': B' \to A'_1 \) in \( \mathcal{A}' \) such that \( g' \circ v' = 0 \). Choose an object \( B \) in \( \mathcal{A} \) and maps \( j: B' \to B \) and \( s: B \to B' \) with \( s \circ j = \text{id}_B \). Then we have the morphism \( i_1 \circ v' \circ s: B \to A_1 \) whose composite with \( g \oplus (\text{id}_{A_1} - i_1 \circ r_1) \): \( A_1 \to A_2 \oplus A_1 \) is zero. Hence we can find a morphism \( u_0: B \to A_0 \) with \( f \circ u_0 = i_1 \circ v' \circ s \). Define \( u': B' \to A'_0 \) by the composite \( r_0 \circ u_0 \circ j \). One easily checks that \( f' \circ u' = v' \). Hence the upper row is exact in \( \mathcal{A}' \).

Suppose that the upper row is exact in \( \mathcal{A}' \). Consider a morphism \( v: B \to A_1 \) in \( \mathcal{A} \) such that \( g \oplus (\text{id}_{A_1} - i_1 \circ r_1) \circ v = 0 \). Then \( g \circ v = 0 \) and \( v = i_1 \circ r_1 \circ v \). We conclude

\[
g' \circ (r_1 \circ v) = r_2 \circ i_2 \circ g' \circ r_1 \circ v = r_2 \circ g \circ i_1 \circ r_1 \circ v = r_2 \circ g \circ v = r_2 \circ 0 = 0.
\]

Since the upper row is exact, we can find \( u': B \to A'_0 \) satisfying \( f' \circ u'_0 = r_1 \circ v \). Define \( u: B \to A_0 \) by \( i_0 \circ u \). Then

\[
f \circ u = f \circ i_0 \circ u' = i_1 \circ f' \circ u' = i_1 \circ r_1 \circ v = v.
\]

Hence the middle row is exact.
If we apply $F'$ and put $i'_k = F'(i_k)$ and $r'_k = F'(r_k)$, we get $r_k' \circ i'_k = \text{id} \circ F'(A_k)$ and the commutative diagram

$$
\begin{array}{cccc}
F'(A_0) & F'(f') & F'(A_1) & F'(A_2) \\
\downarrow{i'_0} & \downarrow{i'_1} & \downarrow{i'_2} & \\
F(A_0) & F(f) & F(A_1) & F(A_2) \\
\downarrow{r'_0} & \downarrow{r'_1} & \downarrow{r'_2} & \\
F'(A_0) & F'(f') & F'(A_1) & F'(A_2)
\end{array}
$$

and, by the same argument as above, the middle row is exact in $B$ if and only if the upper row is exact in $B'$. We conclude that the functor $F'$ is exact or faithfully exact respectively, provided that $F$ is exact or faithfully exact respectively.

Since both $\text{id}_A$ and $\text{id}_B$ are faithfully flat, this special case shows that both $i: A \to A'$ and $j: B \to B'$ are faithfully flat.

Suppose that $F'$ is flat or faithfully flat respectively. Then $j \circ F = F' \circ i$ is flat or faithfully flat respectively. This implies that $F$ is flat or faithfully flat respectively. This finishes the proof of Lemma 10.1.

**Lemma 10.2.** Let $A = \bigcup_{i \in I} A_i$ be the directed union of additive subcategories $A_i$ for an arbitrary directed set $I$.

(i) The idempotent completion $\text{Idem}(A)$ is the directed union of the idempotent completions $\text{Idem}(A_i)$;

(ii) Consider $l \geq 1$.

Suppose that $A_i$ is regular coherent or $l$-uniformly regular coherent respectively for every $i \in I$ and for every $i, j \in I$ with $i \leq j$ the inclusion $A_i \to A_j$ is flat. Then the inclusion $\text{Idem}(A_i) \to \text{Idem}(A_j)$ is flat for every $i, j \in I$ with $i \leq j$ and $A$ is regular coherent or $l$-uniformly regular coherent respectively;

(iii) Suppose that $A_i$ is $0$-uniformly regular coherent respectively for every $i \in I$. Then $A$ is $0$-uniformly regular coherent respectively.

**Proof.** (i) This is obvious.

(ii) If the inclusion $A_i \to A_j$ is flat, then also the inclusion $\text{Idem}(A_i) \to \text{Idem}(A_j)$ is flat by Lemma 10.1. In view of Lemma 5.4 (vii) and assertion (i), we can assume without loss of generality that each $A_i$ and $A$ are idempotent complete. Hence we can use the criterion for regular coherent given in Lemma 5.6 in the sequel. We treat only the case $l \geq 2$, the case $l = 1$ is proved analogously.

Consider a morphism $f_1: A_1 \to A_0$ in $A$. Choose an index $i$ such that $f_1$ belongs to $A_i$. Then we can find a sequence of morphisms

$$
0 \to A_n \xrightarrow{f_n} A_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} A_1 \xrightarrow{f_1} A_0
$$

which is in $A_i$ exact at $A_k$ for $k = 1, 2, \ldots, n$. It remains to show that this sequence is exact at $A$ at $A_k$ for $k = 1, 2, \ldots, n$. Fix $k \in \{1, 2, \ldots, n\}$. It remains to show for any object $A \in A$ and morphism $g: A \to A_k$ with $f_k \circ g = 0$ that there exists a morphism $\overline{g}: A \to A_{k+1}$ with $f_{k+1} \circ \overline{g} = g$. We can choose $j \in I$ with $i \leq j$ such that $g$ belongs to $A_j$. Since $A_{k+1} \xrightarrow{f_{k+1}} A_k \xrightarrow{f_k} A_{k-1}$ is exact in $A_i$, we conclude from the assumptions that it is also exact in $A_j$ and hence we can construct the desired lift $\overline{g}$ already in $A_j$.

(iii) In view of Lemma 5.4 (vii) and assertion (i), we can assume without loss of generality that each $A_i$ and $A$ are idempotent complete. Now the claim follows from the equivalence (iii) $\iff$ (iii) appearing in Lemma 5.6 (iii).
Lemma 10.3. Let \( l \) be a natural number. Let \( A = \{ A_i \mid i \in I \} \) be a collection of \( l \)-uniformly regular coherent additive categories \( A_i \) for an arbitrary index set \( I \).

Then \( \bigoplus_{i \in I} A_i \) and \( \prod_{i \in I} A_i \) are \( l \)-uniformly regular coherent additive categories.

Proof. Obviously \( \prod_{i \in I} A_i \) inherits the structure of an additive category. Recall that \( \bigoplus_{i \in I} A_i \) is the full additive subcategory of \( \prod_{i \in I} A_i \) consisting of those objects \( A_i \mid i \in I \) for which only finitely many of the objects \( A_i \) are different from zero. Obviously

\[
\text{Idem}(\bigoplus_{i \in I} A_i) \cong \bigoplus_{i \in I} \text{Idem}(A_i);
\]

\[
\text{Idem}(\prod_{i \in I} A_i) \cong \prod_{i \in I} \text{Idem}(A_i).
\]

Lemma 5.6 implies that \( \bigoplus_{i \in I} \text{Idem}(A_i) \) and \( \prod_{i \in I} \text{Idem}(A_i) \) are \( l \)-uniformly regular coherent if each \( \text{Idem}(A_i) \) is \( l \)-uniformly regular coherent. Now the claim follows from Lemma 5.4 [vi].

\[\square\]

Remark 10.4 (Advantage of the notion \( l \)-uniformly regular coherent). The decisive advantage of the notion \( l \)-uniformly regular coherent is that it satisfies both Lemma 10.2 and Lemma 10.3. None of these lemmas hold for the properties Noetherian, regular, or \( l \)-uniformly regular. Lemma 10.3 is not true if one replaces \( l \)-uniformly regular coherent by regular coherent unless \( I \) is finite.

11. Vanishing of negative K-groups

Theorem 11.1 (Vanishing of negative \( K \)-groups). Let \( A \) be an additive category, such that \( A[t_1, t_2, \ldots, t_m] \) is regular coherent for every \( m \geq 0 \).

Then \( K_{-n}(A) = 0 \) holds for all \( n \leq -1 \).

Proof. Suppose \( A[t] \) is regular coherent. Next next show that \( A[t, t^{-1}] \) is regular coherent and \( K_{-1}(A) = 0 \).

For an additive category \( B \) define \( G_0(B) \) to be the abelian group with isomorphism classes \( [M] \) of finitely presented \( B \)-modules \( M \) as generators such that for each exact sequence of finitely presented \( B \)-modules \( 0 \rightarrow M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow 0 \) we have the relation \( [M_0] - [M_1] + [M_2] = 0 \). Define \( K_0(B) \) analogously but with finitely presented replaced by finitely generated projective. A functor of additive categories \( F: B \rightarrow B' \) induces a homomorphism \( F_*: K_0(B) \rightarrow K_0(B') \) by sending \( [M] \) to \( [F_*M] \). It induces a homomorphism \( F_*: G_0(B) \rightarrow G_0(B') \) by sending \( [M] \) to \( [F_*M] \), if \( F_*: \text{MOD-}\mathbf{A}_B \rightarrow \text{MOD-}\mathbf{A}_{B'} \) is flat. There is the forgetful functor \( U: K_0(B) \rightarrow G_0(B) \). If \( B \) is regular coherent, then \( U \) is a bijection by the Resolution Theorem, see [4] Theorem 4.6 on page 411. The Yoneda embedding induces an isomorphism \( K_0(B) \xrightarrow{\cong} K_0(B) \), natural in \( B \). The functor \( j_*: \text{MOD-}\mathbf{Z}\mathbf{A}[t] \rightarrow \text{MOD-}\mathbf{Z}\mathbf{A}[t, t^{-1}] \) is flat by Lemma 5.4

Let \( M_* \) be a finitely presented \( \mathbf{Z}\mathbf{A}[t, t^{-1}] \)-module. Then we can find a morphism \( f: A \rightarrow A' \) in \( \mathbf{A}[t, t^{-1}] \) together with an exact sequence of \( \mathbf{Z}\mathbf{A}[t, t^{-1}] \)-module

\[
\text{mor}_{\mathbf{A}[t, t^{-1}]}(?, A) \xrightarrow{f_*} \text{mor}_{\mathbf{A}[t, t^{-1}]}(?, A') \rightarrow M \rightarrow 0.
\]

Choose a natural number \( s \) and a morphism \( g: A \rightarrow A' \) in \( \mathbf{A}[t] \) such that \((\text{id}_A \cdot t^s) \circ g = f \) holds in \( \mathbf{A}[t, t^{-1}] \). Since \( \text{id}_A \cdot t^s: A' \cong A' \) is an isomorphism in \( \mathbf{A}[t, t^{-1}] \), we obtain an exact sequence of \( \mathbf{Z}\mathbf{A}[t, t^{-1}] \)-modules

\[
j_* (\text{mor}_{\mathbf{A}[t]}(?), A)) \xrightarrow{j_* (g_*)} j_* (\text{mor}_{\mathbf{A}[t]}(?), A')) \rightarrow M \rightarrow 0.
\]
Let $N$ be the finitely presented $\mathbb{Z}[A[t]]$-module which is the cokernel of the $\mathbb{Z}[A[t]]$-homomorphism $g_*: \text{mor}_A(t)(?, A) \rightarrow \text{mor}_A(t)(?, A')$. Since $j_*$ is flat and in particular right exact, we obtain an isomorphism of finitely presented $\mathbb{Z}[A[t, t^{-1}]]$-modules $j_*N \cong M$. This implies that the homomorphism $j_*: G'_0(\mathbb{Z}[A[t]]) \rightarrow G'_0(\mathbb{Z}[A[t, t^{-1}]]$ is surjective and that $\mathbb{Z}[A[t, t^{-1}]]$ is regular coherent since $\mathbb{Z}[A[t]]$ is regular coherent by assumption.

Hence we obtain a commutative diagram

$$
\begin{array}{ccc}
K_0(\mathbb{Z}[A[t]]) & \longrightarrow & K_0(\mathbb{Z}[A[t, t^{-1}]]) \\
\cong & & \cong \\
K_0(\mathbb{Z}[A[t]]) & \longrightarrow & K_0(\mathbb{Z}[A[t, t^{-1}]]) \\
\cong & & \cong \\
G'_0(\mathbb{Z}[A[t]]) & \longrightarrow & G'_0(\mathbb{Z}[A[t, t^{-1}]])
\end{array}
$$

whose vertical arrows are bijections and whose lowermost horizontal arrow is surjective. Hence the uppermost horizontal arrow is surjective. We conclude from Theorem 3.2 that $K_{-1}(A)$ vanishes.

Next we show by induction for $n = 1, 2, \ldots$ that $K_{-m}(A)$ vanishes for $m = 1, 2, \ldots, n$. The induction beginning $n = 1$ has been taken care of above. The induction step from $n \geq 1$ to $n + 1$ is done as follows. One shows using the claim above by induction for $i = 1, 2, \ldots, n$ that $A[\mathbb{Z}^n][t_{i+1}, t_{i+2}, \ldots, t_{n+1}]$ is regular coherent. In particular $A[\mathbb{Z}^n][t_{n+1}]$ is regular coherent.

We conclude from the $n$-times iterated Bass-Heller-Swan isomorphism, see Theorem 3.2 that $K_{-n-1}(A)$ is a direct summand in $K_{-1}(A[\mathbb{Z}^n])$. Hence it suffices to show that $K_{-1}(A[\mathbb{Z}^n])$ is trivial. This follows from the induction beginning applied to $A[\mathbb{Z}^n]$.

We conclude from Theorem 9.1 and Theorem 11.1.

Corollary 11.2 (Vanishing of negative $K$-groups of regular additive categories).

Let $A$ be an additive category which is regular. Then $K_n(A) = 0$ holds for all $n \leq -1$.

References


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