Report on Research in Groups

Moduli spaces of log del Pezzo pairs and K-stability

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Organizers: Patricio Gallardo, Jesus Martinez-Garcia, Cristiano Spotti.

In this report we start by introducing and motivating the subject of our research. We continue by making precise the goals we intended to achieve and the organisation we followed in order to achieve those goals. In the last section we describe in detail the problem that we tackled and state the theorems that we proved.

Topics

One of the major goals of Algebraic Geometry is to classify all projective varieties into classes, and then to construct parameter spaces of varieties within a class with certain topological properties. The spaces are known as *moduli spaces*. *Fano varieties* (those projective varieties whose anti-canonical line bundle is ample) are an important class of projective varieties, since the Minimal Model Program predicts that Fano varieties are one of the basic *building blocks* for all other varieties. Some well-known Fano varieties include cubic surfaces, or more generally del Pezzo surfaces (Fano varieties of dimension 2 with mild singularities).

Historically, it has been natural to consider families of geometric objects. From the classification of Riemann surfaces, the work of Grothendieck [?] to the current developments by Kollár [?], Alexeev [?] and Kontsevich [?] among others, the study of parameter spaces, those whose points parametrize natural classes of varieties, is a central topic in geometry. In particular, two questions have been a driving force in moduli theory:

Question 1. *How can we construct a meaningful compactification of a given set parametrizing families of smooth varieties?*

Question 2. *Given several compactifications of the same parameter space, how can we understand the relationship between them?*

While much progress has taken place in the construction of moduli spaces for other classes such as varieties of general type, the appropriate notion of moduli space of Fano varieties has only been determined in recent years. Thanks to work of Donaldson, Tian, Spotti, Sun, Yao, Li, Wang, Xu and others [?, ?, ?, ?, ?, ?, ?], there is a characterization of $\mathbb{Q}$-Gorenstein smoothable Fano varieties admitting a Kähler–Einstein metric as those for which the algebro-geometric property of K-stability holds [?, ?, ?]. Using this equivalence Li, Wang, Xu [?] and Odaka [?, ?] showed that a compact coarse moduli space exists for $\mathbb{Q}$-Gorenstein smoothable K-polystable Fano varieties. There is a canonically defined *CM line bundle* Λ^{CM} defined over any flat family of Fano varieties, and in particular over $\overline{M}^K$. Odaka, Spotti and Sun conjectured that Λ^{CM} must be ample over $\overline{M}^K$.

The proof by Chen, Donaldson and Sun of the equivalence of K-stability and the existence of Kähler-Einstein metrics considered a generalization of the problem to log pairs. Namely, one can consider pairs (X, D) where X is a Fano variety and $D = \frac{1}{m}D'$ is an effective $\mathbb{Q}$-divisor such that $D' \in mK_X|$. The pair $(X, (1-\beta)D)$ is a log Fano pair for all $\beta in (0, 1]$. For simplicity, assume that

$m = 1$ and $D = D'$. The proof of Chen-Donaldson-Sun first showed that if $0 < \beta \ll 1$, then the existence of a Kähler–Einstein metric on X with conical singularities along D is equivalent to the pair (X, D) being β-log K-polystable. Then, they showed that this equivalence was preserved as $\beta \to 1^-$. When $\beta = 1$ both notions coincide with the usual definitions of smooth Kähler-Einstein metrics on X and K-polystability of X, respectively. One can naturally expect a similar approach to prove the conjecture of Odaka-Spotti-Sun, namely:

(i) Define a CM line bundle Λ_β^{CM} over families of pairs (X, D).

(ii) Show that there is a compact coarse moduli space of $\mathbb{Q}$-Gorenstein smoothable β-log K-polystable log pairs (X, D).

(iii) Show that Λ_β^{CM} is ample for $0 < \beta \ll 1$.

(iv) Show that the positivity of Λ_β^{CM} is preserved as $\beta \to 1^-$.

While such an endeavour is too ambitious for a 1-month project, we used it to motivate to the work we carried out. See the Results section for details on the problems we carried out and the details of the problems we proved.

Goals

- To construct, using variations of GIT quotients, compact models $\overline{M}_t^{GIT}$ of pairs (X, D) in relevant examples, which serve as birational models of the compactifications $\overline{M}_\beta^K$.
- To define a β-CM line bundle Λ_β^{CM} and study its behaviour in the meaningful examples considered in (i).

Organisation

Although at the time of application the original team consisted of two co-PI's (Patricio Gallardo and Jesus Martinez-Garcia) and three consultants (Yuji Odaka, Julius Ross, Cristiano Spotti), due to other commitments only Gallardo, Martinez-Garcia and Spotti participated in the programme in the end.

During the first week, Gallardo and Martinez-Garcia worked on the problem of classifying all compactifications $\overline{M}_t^{GIT}$ of pairs (X, D) where $X \subset \mathbb{P}^3$ is a cubic surface and $D \in |-K_X|$ is a hyperplane section. Our results form the preprint [?], which has been submitted for publication.

During the second and third weeks, Gallardo and Martinez-Garcia were joined by Spotti and worked on Goal (ii) above. We obtained partial results which we hope to complete by the end of this year [?].

Results

Moduli of cubic surfaces and their anticanonical divisors

The moduli space of (marked) cubic surfaces is a classic space in algebraic geometry. Indeed, its GIT compactification was first described by Hilbert in 1893 [?], and several alternative compactifications have followed it (see [?, ?, ?]). In our article [?], completed during our stay at the Hausdorff Research Institute for Mathematics, we enrich this moduli problem by parametrizing pairs (S, D) where $S \subset \mathbb{P}^3$ is a cubic surface, and $D \in |-K_S|$ is an anticanonical divisor. There are several motivations for our construction. Firstly, it was recently established that the GIT compactification of cubic surfaces corresponds to the moduli space of K-stable del Pezzo surfaces of degree three [?]. The concept of K-stability has a natural generalization to log-K-stability for pairs, and our GIT quotients are the natural candidates for compactifications of log K-stable pairs of cubic surfaces and their anticanonical divisors. Therefore, our description is a first step toward a generalization of [?], which we also addressed during our stay. Secondly, a precise description of the GIT of cubic surfaces is important for describing the complex hyperbolic geometry of the moduli of cubic surfaces, and constructing new examples of ball quotients (see [?]). We expect similar applications for our GIT quotients. Finally, our compactifications explore the setting of variations of GIT quotients for log pairs, which was considered for plane curves by Laza [?]. Our work is, to the best of our knowledge, the first full description of variation of GIT quotients for log pairs in higher dimensions.

The GIT quotients considered depend on a choice of a linearization $\mathcal{L}_t$ of the parameter space $\mathcal{H}$ of cubic forms and linear forms in $\mathbb{P}^3$. It can be shown that the different GIT quotients arising by picking different polarizations of $\mathcal{H}$ are controlled by the parameter $t \in \mathbb{Q}_{>0}$. For each value of t, there is a GIT compactification $\overline{M(t)}$ of the moduli space of pairs (S, D) where S is a cubic surface and $D \in |-K_S|$ is an anticanonical divisor. It follows from the general theory of variations of GIT (see [?, ?], c.f. [?, Theorem 1.1]) that $0 \leqslant t \leqslant 1$ and that there are only finitely many different GIT quotients associated to t. Indeed, there is a *set of chambers* (t_i, t_{i+1}) where the GIT quotients $\overline{M(t)}$ are isomorphic for all $t \in (t_i, t_{i+1})$, and there are finitely many *GIT walls* $t_1, \ldots, t_k$ where the GIT quotient is a birational modification of $\overline{M(t)}$ where $0 < |t - t_i| < \epsilon \ll 1$. Additionally there are initial and end walls $t_0 = 0$ and $t_{k+1} = 1$, respectively.

Lemma 1 ([?, Lemma 1.1])**.** *The GIT walls are*

$$t_0 = 0,\ t_1 = \frac{1}{5},\ t_2 = \frac{1}{3},\ t_3 = \frac{3}{7},\ t_4 = \frac{5}{9},\ t_5 = \frac{9}{13},\ t_6 = 1.$$

Given $t \in \mathbb{Q}_{>0}$ we say that a pair (S, D) is *t-stable* (respectively *t-semistable*) if it is t-stable (respectively t-semistable) under the natural $\mathrm{SL}(4, \mathbb{C})$-action on $\mathcal{H}$. A pair is *strictly t-semistable* if it is t-semistable but not t-stable. The space $M(t)$ parametrizes t-stable pairs and $\overline{M(t)}$ parametrizes closed strictly t-semistable orbits. In the next two theorems we fully classify all the pairs represented by these points.

The quotient $\overline{M(0)}$ is isomorphic to the GIT of cubic surfaces and the quotient $\overline{M(1)}$ is the GIT of plane cubic curves (see [?, Lemma 4.1]). These spaces are classical and have been thoroughly studied (see [?]). Therefore we only consider the casae $t \in (0, 1)$. A nice feature of $M(t)$ is that for each $t \in (0, 1)$ and each t-stable pair (S, D), the surface S has isolated ADE singularities.

Theorem 2 ([?, Theorem 1.3])**.** *Consider a pair (S, D) formed by a cubic surface S and a hyperplane section $D \in |-K_S|$.*

(i) Let $t \in (0, \frac{1}{5})$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_2$ and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S.

(ii) Let $t = \frac{1}{5}$. The pair (S, D) is t-stable if and only if S has finitely many singularities at worst of type $\boldsymbol{A}_2$, D is reduced, and if $P \in D$ is a surface singularity, then P is at worst an $\boldsymbol{A}_1$ singularity of S.

(iii) *Let* $t \in (\frac{1}{5}, \frac{1}{3})$*. The pair* (S, D) *is t-stable if and only if* S *has finitely many singularities at worst of type* $\boldsymbol{A}_3$*,* D *is reduced and if* $P \in D$ *is a surface singularity, then* P *is at worst an* $\boldsymbol{A}_1$ *singularity of* S*.*

(iv) *Let* $t = \frac{1}{3}$*. The pair* (S, D) *is t-stable if and only if* S *has finitely many singularities at worst of type* $\boldsymbol{A}_3$*,* D *is reduced and if* $P \in D$ *is a surface singularity, then* P *is at worst an* $\boldsymbol{A}_1$ *singularity of* S *and* D *has at worst a cuspidal singularity at* P*.*

(v) *Let* $t \in (\frac{1}{3}, \frac{3}{7})$*. The pair* (S, D) *is t-stable if and only if* S *has finitely many singularities at worst of type* $\boldsymbol{A}_4$*,* D *is reduced and if* $P \in D$ *is a surface singularity, then* P *is at worst an* $\boldsymbol{A}_1$ *singularity of* S *and* D *has at worst a normal crossing singularity at* P*.*

(vi) *Let* $t = \frac{3}{7}$*. The pair* (S, D) *is t-stable if and only if* S *has finitely many singularities at worst of type* $\boldsymbol{A}_4$*,* D *has at worst a tacnodal singularity and if* $P \in D$ *is a surface singularity, then* P *is at worst an* $\boldsymbol{A}_1$ *singularity of* S *and* D *has at worst a normal crossing singularity at* P*.*

(vii) *Let* $t \in (\frac{3}{7}, \frac{5}{9})$*. The pair* (S, D) *is t-stable if and only if* S *has finitely many singularities at worst of type* $\boldsymbol{A}_5$ *or* $\boldsymbol{D}_4$*,* D *has at worst a tacnodal singularity and if* $P \in D$ *is a surface singularity, then* P *is at worst an* $\boldsymbol{A}_1$ *singularity of* S *and* D *has at worst a normal crossing singularity at* P*.*

(viii) *Let* $t = \frac{5}{9}$*. The pair* (S, D) *is t-stable if and only if* S *has finitely many singularities at worst of type* $\boldsymbol{A}_5$ *or* $\boldsymbol{D}_4$*,* D *has at worst an* $\boldsymbol{A}_2$ *singularity and if* $P \in D$ *is a surface singularity, then* P *is at worst an* $\boldsymbol{A}_1$ *singularity of* S *and* D *has at worst a normal crossing singularity at* P*.*

(ix) *Let* $t \in (\frac{5}{9}, \frac{9}{13})$*. The pair* (S, D) *is t-stable if and only if* S *has finitely many singularities at worst of type* $\boldsymbol{A}_5$ *or* $\boldsymbol{D}_5$*,* D *has at worst a cuspidal singularity and if* $P \in D$ *is a surface singularity, then* P *is at worst an* $\boldsymbol{A}_1$ *singularity of* S *and* D *has at worst a normal crossing singularity at* P*.*

(x) *Let* $t = \frac{9}{13}$*. The pair* (S, D) *is t-stable if and only if* S *has finitely many singularities at worst of type* $\boldsymbol{A}_5$ *or* $\boldsymbol{D}_5$*,* D *has at worst normal crossing singularities and if* $P \in D$ *is a surface singularity, then* P *is at worst an* $\boldsymbol{A}_1$ *singularity of* S*.*

(xi) *Let* $t \in (\frac{9}{13}, 1)$*. The pair* (S, D) *is t-stable if and only if* S *has finitely many ADE singularities,* D *has at worst normal crossing singularities and if* $P \in D$ *is a surface singularity, then* P *is at worst an* $\boldsymbol{A}_1$ *singularity of* S*.*

Theorem 3 ([**?**, Theorem 1.4])**.** *Let* $t \in (0, 1)$*. If* $t \neq t_i$*, then* $\overline{M}(t)$ *is the compactification of the stable loci* $M(t)$ *by the closed* $\mathrm{SL}(4, \mathbb{C})$*-orbit in* $\overline{M}(t) \setminus M(t)$ *represented by the pair* (S_0, D_0)*, where* S_0 *is the unique* $\mathbb{C}^*$*-invariant cubic surface with three* $\boldsymbol{A}_2$ *singularities and* D_0 *is the union of the unique three lines in* S_0*, each of them passing through two of those singularities.*

If $t = t_i$*,* $i = 1, 2, 4, 5$*, then* $\overline{M}(t_i)$ *is the compactification of the stable loci* $M(t_i)$ *by the two closed* $\mathrm{SL}(4, \mathbb{C})$*-orbits in* $\overline{M}(t_i) \setminus M(t_i)$ *represented by the uniquely defined pair* (S_0, D_0) *described above and the* $\mathbb{C}^*$*-invariant pair* (S_i, D_i) *uniquely defined as follows:*

(i) *the cubic surface* S_1 *with an* $\boldsymbol{A}_3$ *singularity and two* $\boldsymbol{A}_1$ *singularities and the divisor* $D_1 = 2L + L' \in |-K_S|$ *where* L *and* L' *are lines such that* L *is the line containing both* $\boldsymbol{A}_1$ *singularities and* L' *is the only line in* S *not containing any singularities;*

(ii) the cubic surface S_2 with an $\boldsymbol{A}_4$ singularity and an $\boldsymbol{A}_1$ singularity and the divisor $D_2 \in |-K_S|$ which is a tacnodal curve singular at the $\boldsymbol{A}_1$ singularity of S;

(iii) the cubic surface S_4 with a $\boldsymbol{D}_5$ singularity and the divisor $D_4 \in |-K_S|$ which is a tacnodal curve whose support does not contain the surface singularity;

(iv) the cubic surface S_5 with an $\boldsymbol{E}_6$ singularity and the cuspidal rational curve $D_5 \in |-K_S|$ whose support does not contain the surface singularity.

The space $\overline{M}(t_3)$ is the compactification of the stable loci $M(t_3)$ by the three closed $\mathrm{SL}(4,\mathbb{C})$*-orbits in $\overline{M}(t_3) \setminus M(t_3)$ represented by the $\mathbb{C}^*$-invariant pairs uniquely defined as follows:*

(i) the pair (S_0, D_0) described above;

(ii) the pair (S_3, D_3) where S_3 is the cubic surface with a $\boldsymbol{D}_4$ singularity and and Eckardt point and D_3 consists of the unique three coplanar lines intersecting at the Eckardt point;

(iii) the pair (S_3', D_3') where S_3' is the cubic surface with an $\boldsymbol{A}_5$ and an $\boldsymbol{A}_1$ singularity and the divisor D_3' which is an irreducible curve with a cuspidal point at the $\boldsymbol{A}_1$ singularity of S_3'.

Figure 1: Pairs in $\overline{M(t)} \setminus M(t)$ for each $t \in (0,1)$. The dotted lines represent the divisor D. The bold points are singularities of the surface.

Moduli space of log K-polystable hypersurfaces

Given a projective manifold X and an ample line bundle L, the existence of a Kähler metric with constant scalar curvature in $c_1(L)$ is conjectured (by Yau-Tian-Donaldson [?]) to be equivalent to the algebro-geometric notion of K-polystability of (X, L). The conjecture is known to hold when $L = -K_X$ (and hence X is Fano) by Chen-Donaldson-Sun [?, ?, ?] and in some other special cases. These notions have been extended to (possibly singular) pairs (X, D), where D is a divisor in D. In particular, if $D \in |-K_X|$ and $L = -K_X$, the existence of a Kähler-Einstein metric with conical singularities of angle $2\pi\beta$ along D is equivalent to the β-K-polystability of (X, D), where $\beta \in (0, 1]$. These notions coincide with smooth Kähler-Einstein metrics and K-polystability, respectively, when $\beta = 1$.

Given a flat fibration $p\colon \mathcal{X} \to B$ of projective varieties and a p-ample line bundle, Paul and Tian [?, ?] introduced a line bundle Λ^{CM} over B, naming it *CM line bundle* (after Chow and Mumford), whose weight controlled the K-stability of the members in the family. In the second part of our visit we extended the notion of CM line bundle to families $p\colon (\mathcal{X}, \mathcal{D}) \to \mathcal{B}$ of log pairs (X, D) (formed by a divisor D in the projective variety X) equipped with a p-ample line bundle $\mathcal{L}$ on $\mathcal{X}$. We showed that the log CM line bundle Λ^{CM}_β we consider has analogue properties to that considered by Paul and Tian. In particular, $\Lambda^{CM}_1 \cong \lambda^{CM}$, as expected. Moreover, we also recovered the original motivation from Paul and Tian:

Theorem 4 ([?]). *Suppose the CM line bundle Λ^{CM}_β is G-linearised where $G = \mathrm{SL}(N+1, \mathbb{C})$ and $i\colon X \hookrightarrow \mathbb{P}^N$ such that $L = i^*(\mathcal{O}_{\mathbb{P}^N}(1))$. If $(X, (1-\beta)D, L)$ is K-(semi/poly)stable, then (X, D) is GIT-(semi/poly)stable with respect to the Λ^{CM}_β-polarization of the Hilbert scheme of pairs. If Λ^{CM}_β is ample and $(X, (1-\beta)D, L)$ is K-(semi/poly)stable, then (X, D) is GIT-(semi/poly)stable with respect to the Λ^{CM}_β-polarization of the Hilbert scheme of pairs.*

In addition, we computed the degree of the log CM line bundle:

Theorem 5 ([**?**]). *Let (X, D, L) be the restriction of $(\mathcal{X}, \mathcal{D}, \mathcal{L})$ to a general $b \in \mathcal{B}$ and let $\mu(L) = \frac{c_1(X)\cdot c_1(L)}{c_1(L)^n}$ and $\mu(L, D) = \frac{D\cdot c_1(L)}{c_1(L)^n}$. Then*

$$\deg(\Lambda^{CM}_{,}\beta) = \pi_* \left(n\mu\left(L\right) c_1\left(\mathcal{L}\right)^{n+1} + \left(n+1\right) c_1\left(\mathcal{L}\right)^n c_1\left(K_{\mathcal{X}/\mathcal{B}}\right) + \\ \left(1-\beta\right)\left(\left(n+1\right) c_1\left(\mathcal{L}\right)^n \cdot \mathcal{D} - n\mu(L,D) c_1\left(\mathcal{L}\right)^{n+1}\right)\right).$$

Moreover, if $\mathcal{L} = -K_{\mathcal{X}/\mathcal{B}}$ and $\mathcal{D}|_{\mathcal{X}_b} \in |-K_{\mathcal{X}_b}|$ for all $b \in \mathcal{B}$, then

$$\deg(\Lambda^{CM}_{,}\beta) = \pi_* \left(c_1\left(-K_{\mathcal{X}/\mathcal{B}}\right)^n \cdot \left(-c_1\left(-K_{\mathcal{X}/\mathcal{B}}\right) + (1-\beta)\left(\left(n+1\right)\mathcal{D} - nc_1\left(-K_{\mathcal{X}/\mathcal{B}}\right)\right)\right)\right).$$

Finally, we considered a slight generalisation of the setting for cubic surfaces and anti-canonical divisors. Namely, we considered pairs (X, H) formed by a hypersurface of degree d and a hyperplane in $\mathbb{P}^{n+1}$. The GIT setting for these pairs was studied by Gallardo and Martinez-Garcia in [**?**], giving rise to a parametrisation space $\mathcal{H} \cong \mathbb{P}^N \times P^{n+1}$ (where N is determined by n and d) for these pairs and a natural group action of the reductive group $G = \mathrm{SL}(n+2)$ on H. The GIT quotients obtained are a generalization of the case of cubic surfaces and anti-canonical divisors described above. In the case where $d \leqslant n+1$ (e.g. the case of cubic surfaces, or more generally the case of Calabi-Yau and Fano hypersurfaces), we can identify H with the divisor $D = X \cap H$. However, finding a precise description of the stable locus as in Theorem 2 in term of the singularities of the pairs for arbitrary d and n is not deemed to be possible, since the classification of singularities is not known in general.

Nevertheless, given n and d, there is a finite number of GIT quotients M_t parametrised by $t = \frac{b}{a} \in \mathbb{Q}_{>0}$, where $\mathcal{O}(a, b)$ is a G-linearised line bundle on $\mathcal{H}$. We can find a Zariski open set $\mathcal{U} \subset \mathcal{H}$ with codimension larger or equal than 2. Hence, any line bundle defined on $\mathcal{U}$ extends uniquely to a line bundle on $\mathcal{H}$. If $\mathcal{X} \to \mathcal{U}$ is the universal family of pairs (X, D), then we show:

Theorem 6 ([**?**]). *Let $\mathcal{L} = -K_{\mathcal{X}/U}$ be relatively very ample, then $\Lambda^{CM}_{\beta}(-K_{\mathcal{X}/U}) \cong \mathcal{O}(a, b)$ for $a > 0$ and $b > 0$ such that $t = \frac{a}{b}$ and*

$$t(\beta) := \frac{d(n+1)(1-\beta))}{\left(1-\beta\right)\left(n+1\right)\left(1-nd\right) - (1+n(1-\beta))(n+2-d)^n(n+2)(1-d)}.$$

In particular, for $n+1 = d$, it holds

$$t(\beta) = \frac{d^2(1-\beta)}{d^2-\beta}.$$

In particular, the above theorem allows us to identify the log CM line bundle (a natural bundle for the compactification of log K-polystable pairs) to the line bundle determining the different GIT compactifications. It is our hope that in our upcoming work [**?**] we can show that these compactifications are isomorphic, at least for some β, and explore how the compact moduli of log K-polystable pairs changes as β varies.