

Hausdorff Institute Research in Groups (Harada) May 2011: Report

Prepared by Megumi Harada with contributions from all participants

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1 Introduction

The following eight mathematicians,

Nigel (Nige) Ray
Matthias Franz
Anthony (Tony) Bahri
Gareth Williams
David (Dave) Anderson
Sam Payne
Tara Holm
Megumi Harada

representing the research areas of algebraic topology, algebraic geometry, and symplectic geometry, gathered at Hausdorff Institute for 3 weeks in May 2011 with the generous support of HIM. Below I have summarized comments from the participants regarding their experiences at HIM. Section 2 addresses generalities while Section 3 itemizes the concrete mathematical progress made during our stay at HIM. Concluding remarks are in Section 4.

2 General comments on the HIM Research in Groups experience

Below I itemize some comments (slightly edited) received from participants.

- From Nigel Ray:

The Research in Groups (Harada) workshop was one of the most enjoyable, exciting and productive mathematical events I have ever attended. The unpredictability of each day was a special pleasure, and breakfasts of croissants, latte, and strawberries in the garden sunshine were unforgettable! I like to think that I interacted in a significant mathematical fashion with every member of the group, and I expect to remain in contact with them all for the foreseeable future.

Megumi should be awarded 3 gold stars for the persistence, enthusiasm and person-management skills she demonstrated in planning the workshop, and ensuring that it ran so smoothly.

- From Tara Holm:

The three weeks in Bonn were fruitful. They will have an important effect on my research program. I am very grateful to the Hausdorff Institute for its hospitality and support. It was an unusual working format. It was a longer period than typical, and fewer participants. This allowed us each to give very detailed lectures, and each to have meaningful mathematical conversations with every other member of the group. I learned a great deal, both in terms of important tools from other areas and in intuition and examples that others explained. I also was able to discuss possible projects, in varying level of detail, with several other members of the group.

In addition to tangible research possibilities, I learned about a variety of topics. It is hard to predict how they will impact my work, but I have no doubt that they will. The main techniques I learned about during our three weeks are toric degeneration, the Atiyah-Hirzebruch spectral sequence, Bredon cohomology, and cobordism.

- From Dave Anderson:

I'd like to emphasize that in addition to the concrete mathematical progress, I felt several flashes of understanding coming out of conversations between the topologists, symplectic geometers, and algebraic geometers. This kind of thing is hard to quantify, but I'm sure knowing more about things like symplectic reduction and equivariant cobordism will be useful in the future.

- From Tony Bahri:

I appreciate very much fine organizational work Megumi did. Her planning threaded a careful path, linking participants' expertise and interests into a focused attack on a selected set of well chosen problems.

Also, from the first week's lectures, I came away with a better understanding of the way in which Megumi and Tara think about toric objects in the realm of symplectic geometry. Megumi's and Dave's talks were fine exposeés of the GKM approach to equivariant cohomology. I thought all the talks were both useful and very well presented.

- From Gareth Williams:

Thank you, Megumi, for organising such a great three weeks in Bonn. I have enjoyed myself very much – and made progress. What more could one ask?

3 Concrete mathematical progress made

Below, I briefly summarize the concrete mathematical progress made during the workshop, as described by the participants. They are roughly organized by theme.

3.1 Computation of $K_T^*(X)$ (topological equivariant K -theory) for a simplicial toric variety X

- Discussion among Gareth, Megumi, Tara, and Nige led to a clear understanding of the relation between the geometric and algebraic approaches to computation of $K_T^*(\mathbb{P}(\chi))$, where χ is a tuple of successively divisible weights. We also believe our techniques pave the way for understanding a broader class of examples of orbifold toric varieties. Specifically, Tara and Megumi first explained to Nige and Gareth the details of an old(er) paper of Harada-Henriques-Holm which extends classical GKM theory to certain G -CW complexes. In ensuing discussions it became clear that Nige, Tony and Matthias' iterated Thom complex construction of certain weighted projective spaces is relevant in order to apply these GKM techniques, which then yields an identification of the K_T^* of these weighted projective spaces with the ring of piecewise Laurent polynomials, or PLPs (the term "piecewise Laurent polynomial" appears to be new and is discussed further below). We are optimistic that this project will bear fruit and we will be able, firstly, to generalize the methods to more general weighted projective spaces $\mathbb{P}(\chi)$ where χ is not necessarily a tuple of successively divisible weights, and secondly, possibly to an even more general class of singular toric varieties. Our proof (sketched during our stay in Bonn) for the special case $\mathbb{P}(\chi)$ where χ is successively divisible, turns out to be quite simple but deserves to be written down. An essential and exciting feature of our collaboration is to approach the problem from two different points of view: the geometric (GKM theory) and algebraic (the equivariant Atiyah-Hirzebruch spectral sequence and PLPs). We anticipate that the algebraic and topological techniques can be played off each other, with the expectation that each will inform the other.

- Gareth learned by talking to Sam and Dave that in the smooth case, every algebraic equivariant K -theory class admits a decomposition as a formal sum/difference of algebraic equivalent line bundles – this inspires confidence in the argument Nigel and Gareth have for the same result in the topological case. For weighted projective spaces, Gareth made dramatic progress with the equivalent Atiyah-Hirzebruch spectral sequence. When he first arrived in Bonn, Gareth could compute the answer only for $\mathbb{P}(1, 1, 2)$, but now he can handle $\mathbb{P}(1, a, b)$ for any positive integers a, b . Moreover, computations for higher dimensional examples are well underway.

3.2 Piecewise Laurent polynomials and bundles over toric varieties

- Nige and Gareth introduced to the group the phrase “piecewise Laurent polynomial”. The term appears to be new, or at least new to algebraic geometers (neither Sam nor Dave had heard it before this meeting). The group quickly adopted the phrase (and PLP as an abbreviation) so it seems that we have some good terminology. Nige and Gareth explained to some members of the group the identification (the proofs, which Nige and Gareth developed before arriving in Bonn, are yet to be written up) of the ring of PLP’s with, firstly, the equivariant K -theory of a smooth toric variety, as well as a K -theoretic face-ring. In further developments, Nige and Gareth worked with Sam and Dave on various algebraic and geometrical viewpoints on PLPs that are closely related, and learned several new techniques for further study. In particular, Sam’s viewpoint for smooth fans seems similar to, but distinct from, Nige and Gareth’s. We expect to pursue these issues further.
- A theorem of Vezzosi and Vistoli says that for any smooth toric variety, the equivariant K -theory is the ring of piecewise Laurent polynomials on the fan. (The use of the term “piecewise Laurent polynomial” might be new; see above comment.) Dave and Sam wondered what interpretation the ring of PLP’s on an arbitrary fan Σ might have — and given the situation with Chow cohomology, it seemed reasonable to conjecture the following:

Conjecture: For an arbitrary toric variety X with fan Σ , the ring of piecewise Laurent polynomials on Σ is isomorphic to the “operational” equivariant algebraic K -theory of X . (NB: Here we only consider K^0 .)

Sam and Dave sketched a proof of this conjecture during their stay in Bonn, and expect to be able to write up a short paper explaining it in the next few weeks. A natural follow-up question is whether and when one can say the operational K -theory agrees with the usual one of (algebraic or topological) vector bundles. This will be true for smooth spaces, but is probably very difficult in general.

- The interpretation of piecewise linear functions on branched covers of fans was also one of the significant breakthroughs of the workshop for Sam. He notes that it opens interesting possibilities for trying to construct bundles of high rank on toric varieties where there are known not to be any nontrivial bundles of small rank.
- Gareth gave a method to construct a PLP, with all coefficients non-negative, on any fan. This answered a question of Sam’s regarding the fan arising from a (distorted) cube (and won Gareth a beer at Sam’s expense).
- Gareth is particularly grateful for the chance to have Sam illustrate Klyachko’s theorem [1]: this is something that Nige and Gareth have wrestled with in the past, but have never fully understood.
- Tara notes that the relationship between topological and algebraic vector bundles is not fully understood. Nige suggested a construction of a topological bundle that may not be algebraic. Tara, Sam, Gareth and Dave discussed this construction, and found explicit K -theory classes that may yield topological bundles that do not have algebraic structures – but there are many details left to check.
- Tony and Gareth discussed piecewise Laurent polynomials in the context of an integrality condition derived from a similar one for piecewise polynomials.

3.3 (Toric) topology of toric orbifolds

- Nige, Tony and Matthias made substantial progress in finalizing two papers on the topology of weighted projective spaces: the first on the classification of weighted projective spaces and the second, a study of weighted projective spaces as iterated Thom complexes.
- Tony learned from Tara an example of a moment-angle complex on which there is a locally free action of a torus, specified by a simplicial fan and whose Borel quotient has torsion in odd cohomological degree. This example will impact significantly Tony's thinking about certain problems in the toric topology of orbifolds.
- Tony had the opportunity also to describe recent new topological constructions of toric manifolds/orbifolds for which he hopes that corresponding constructions will be identified in symplectic geometry. Tony's discussions with Dave on this topic seem to have piqued his interest as well, and they plan to remain in contact.
- Tara notes that Tony's new techniques for building topological spaces endowed with a torus action look similar to constructions in symplectic geometry. Tara hopes to think through this in more detail, and discuss it further with him. This was one item that, although Tara did not have time to consider carefully during the three weeks in Bonn, should have some clear answers.

3.4 Equivariant K -theory and the Atiyah(-Bredon) sequence

- Matthias advertised to the group the Franz-Puppe ideas for studying the K -theoretic Atiyah-Bredon sequence(s). These ideas seemed to appeal to everyone, so we may all expect to hear a lot more about them in future. The workshop certainly helped bring them to the fore.
- Matthias notes that discussions with the other members of the group led him to a much better understanding of equivariant K -theory in general and of K_T^* of orbifolds in particular. Based on these discussions he was able to identify important open questions, e.g. the characterization of equivariant formality in K -theory. Guided by the computation of examples, Matthias also developed concrete ideas of what an answer should look like.
- Dave spoke to the group about the idea of treating equivariant cohomology and K -theory as rings of functions on schemes, via the $Spec$ functor, and relating the two by taking complete local rings. This led to some interesting discussions with Matthias about injectivity of the completion map in K -theory, and with Volker Puppe (who visited the group during the first week of our workshop) about the role of evaluating the "t" parameters.

3.5 Cohomology of foliations and orbifold cohomology

- Matthias invited two young mathematicians, Goertsches and Toeben, to visit us during our stay in Bonn and give a talk on their work on equivariant cohomology and foliations. During their visit, the group learned that there are close connections between their work and that of Holm-Matsumura, as well as to the theory of "non-rational toric varieties" (Battaglia-Prato, Barthel-Brasselet-Fieseler-Kaup, Karu etc.). More specifically, Tara notes that the Goertsches-Toeben work is potentially useful for computations involving Battaglia and Pratos computations of cohomological invariants of quasifolds. They have also applied their techniques to K-contact toric manifolds. This is related to recent work of Tara's student, Shisen Luo; it is natural to ask whether one can unify these two approaches.

4 Concluding Remarks

In summary, our 3-week Focused Research Group led to concrete, new mathematics, the most immediate and significant being:

- The new collaboration on equivariant topological K -theory of orbifold toric varieties, involving Nigel Ray, Gareth Williams, Tara Holm, and Megumi Harada. The team already has concrete preliminary results from their stay in Bonn, in addition to ideas for its generalizations.
- During their stay in Bonn, Sam Payne and Dave Anderson sketched a proof of a concrete new result on operational algebraic K -theory of arbitrary fans and its relation to the ring of piecewise Laurent polynomials, and they intend to write a paper on the topic within a few weeks/months.
- While in Bonn, Matthias Franz, Nigel Ray, and Tony Bahri finished two papers on the toric topology of weighted projective spaces.

In addition, as the lists in Section 3 suggest, many participants had important insights into new and open problems, as well as the time to discuss these ideas with other members of the group, and to make specific preliminary progress on them (e.g. computing specific examples or learning of important special cases).

Our stay in Bonn was therefore extremely fruitful, especially considering the relatively short (3-week) stay. It appears evident that the group will continue to actively interact in the foreseeable future, and we expect that a similar follow-up workshop will be desirable within a 1- or 2-year time frame. I quote some of the participants:

- From Nigel Ray: *“We need another similar event (rather than a more traditional workshop – and maybe in about 18 month’s time?) to deal with all the issues raised, and their progeny. An additional thought: it would be excellent if two or three of our graduate students and postdocs could also be involved.”*
- From Gareth Williams: *“We [should] make clear that some kind of follow-up activity is necessary.”*

References

- [1] A. A. Klyachko. Equivariant bundles on toral varieties. *Math. USSR-Izv.* 35(2):337-375, 1990.