QUASIDIAGONALITY AND AMENABILITY

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STARTING POINT: INJECTIVITY $\Rightarrow$ HYPERFINITENESS

CONNES '77: INJECTIVE VNAS ARE HYPERFINITE

Combining this with classification of hyperfinite factors:

- Complete classification of (separably acting) injective factors.

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Classification programme

- Aims to classify separable, nuclear and simple C^*-algebras by "K-theoretic data".

- Nuclear analogous to injective: A nuclear $\iff$ A^{**} injective.
- Simple analogous to factor: weak*-closed ideals in vNa of form $\mathcal{M}p$ for a central projection $p \in \mathcal{M}$

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Connes' 3 ingredients

A (separably acting) injective II_1 factor $\mathcal{M}$

1. is McDuff: $\mathcal{M} \cong \mathcal{M} \overline{\otimes} \mathcal{R}$.
2. has unique morphisms: any two *-hms $\mathcal{M} \to (\mathrm{II}_1$ factor) are approximately unitarily equivalent.
3. has an embedding $\theta : \mathcal{M} \hookrightarrow \mathcal{R}^\omega$

QUASIDIAGONALITY

- $\mathcal{Q}$ universal UHF-algebra. $\mathcal{Q} = \bigotimes_{p\text{ prime}}(\bigotimes_1^\infty M_p)$.
- $\mathcal{Q}_\omega = \ell^\infty(\mathcal{Q})/\{(x_n) \in \ell^\infty(\mathcal{Q}) : \lim_{n\to\omega} \|x_n\| = 0\}$. [$\omega \in \beta\mathbb{N}\setminus\mathbb{N}$ fixed.]
- Has unique trace $\tau_{\mathcal{Q}_\omega}((x_n)_n) = \lim_{n\to\omega} \tau_{\mathcal{Q}}(x_n)$

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- Abelians.
- Subhomogeneous

Closed under

- Subalgebras
- Increasing inductive limits

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Quasidiagonal $\Rightarrow$ stably finite: no infinite projections in $M_n(A)$

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Classification predicts

Stably finite simple nuclear C^*-algebras are approximately subhomogenous, so quasidiagonal.

Traces

Separable nuclear C^*-algebra A is quasidiagonal iff $\exists A \hookrightarrow \mathcal{Q}_\omega$.

Definition

A trace τ_A on A is

1. quasidiagonal if $\exists$ cpc $\phi_i : A \to M_{k_i}$ with
   - $\|\phi_i(ab) - \phi_i(a)\phi_i(b)\| \to 0$ and $\tau_A(a) = \lim \tau_{M_{k_i}}(\phi_i(a))$.
2. amenable if $\exists$ cpc $\phi_i : A \to M_{k_i}$ with
   - $\|\phi_i(ab) - \phi_i(a)\phi_i(b)\|_{2,M_{k_i}} \to 0$ and $\tau_A(a) = \lim \tau_{M_{k_i}}(\phi_i(a))$.

- A sep nuclear: qd traces those factorising $A \xrightarrow{*\text{-hm}} \mathcal{Q}_\omega \xrightarrow{\tau_{\mathcal{Q}_\omega}} \mathbb{C}$.
- τ_A amenable $\Leftrightarrow$ For $A \subset \mathcal{B}(H)$ $\exists$ A-central state extending τ_A.

Second obstruction

Quasidiagonal unital C^*-algebras have amenable traces.

- In particular, as noted by Rosenberg, $\mathrm{C}^*_r(G)$ QD $\Rightarrow$ G amenable.

QUESTIONS

Separable nuclear C^*-algebra A is quasidiagonal iff $\exists A \hookrightarrow \mathcal{Q}_\omega$.

ROSENBERG'S CONJECTURE

$\mathrm{C}^*(G)$ is qd for G discrete amenable.

- Yes for elementary amenable groups (Ozawa, Rørdam, Sato '14) via classification!

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QUESTION

Are all amenable traces quasidiagonal? (Is $\mathcal{R}$ quasidiagonal?)

Questions and some answers

Separable nuclear C^*-algebra A is quasidiagonal iff $\exists A \hookrightarrow \mathcal{Q}_\omega$.

Theorem (Tikuisis, W, Winter)

Every faithful trace on a separable nuclear C^*-algebra in the UCT class is quasidiagonal.

- Having a faithful qd trace ensures quasidiagonality.
- Via Brown: all traces on a quasidiagonal separable nuclear C^*-algebra in the UCT class are qd.

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UCT = UNIVERSAL COEFFICIENT THEOREM OF ROSENBERG AND SCHOCHET

- A has UCT iff it is KK-equivalent (weak homotopy kind of statement) to an abelian C^*-algebra;
- Open whether all separable nuclear C^*-algebras have UCT.

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Rosenberg's conjecture

$\mathrm{C}^*(G)$ is qd for G discrete amenable.

- Yes for elementary amenable groups (Ozawa, Rørdam, Sato '14) via classification!
- Yes in general. $C_r^*(G)$ is in the UCT class by Tu.
- In fact $C_r^*(G)$ is AF-embeddible.

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Blackadar-Kirchberg conjecture

Stably finite nuclear C^*-algebras are quasidiagonal.

- Holds in the simple UCT case.

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- Extended by Gabe to obtain quasidiagonality of faithful amenable traces on separable exact algebras (= subalgebras of nuclears) in the UCT class.

Back to classification (briefly)

Gong-Lin-Niu '15 (the long paper)

Identifies, and classifies (assuming UCT), a class of stably finite algebras which exhausts Elliott's invariant.

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Theorem (Matui-Sato '13)

Let A be simple, separable, unital and nuclear with unique trace. Suppose

1. $A \cong A \otimes \mathcal{Q}$ (for those in the know, $\mathcal{Z}$-stability suffices).
2. A is quasidiagonal.

Then A is in the GLN-class (in fact in a somewhat simpler class).

- In UCT case, 2 is now automatic: thus get classification from a tensorial absoption hypothesis.

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- Nuclear dimension (Winter, Zacharias): natural non-commutative generalising of covering dimension to nuclear C^*-algebras.
- e.g. $C(X) \rtimes G$ has finite nuclear dimension for free minimal action of finitely generated nilpotent on finite dimensional X.

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A stably finite simple unital separable of finite nuclear dimension in UCT class such that all traces are QD. Then A is in the GLN-class.

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A stably finite simple unital separable of finite nuclear dimension in UCT class such that all traces are QD. Then A is in the GLN-class.

Theorem (Many hands)

Simple separable infinite dimensional unital C^*-algebras with finite nuclear dimension and UCT are classified by Elliott's invariant.

Strategy of proof

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Every faithful trace on a separable nuclear C*-algebra in the UCT class is quasidiagonal.

Theorem (Voiculescu)

Quasidiagonality is homotopy invariant.

- Cones $C_0(0,1] \otimes A$ are always quasidiagonal.

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Proposition: Kirchberg-Rørdam, Sato-W-Winter, Gabe, …

All amenable traces on cones $C_0(0,1] \otimes A$ are quasidiagonal.

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STRATEGY OF PROOF

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FIX SUCH FAITHFUL TRACE τ_A

1. $\exists\acute{\Phi} : C_0(0,1]\otimes A \to \mathcal{Q}_\omega$ realising $\mu_{\mathrm{leb}}\otimes\tau_A$.

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A priori, these have nothing in common; but can adjust s.t.

- $\acute{\Phi}$ and $\grave{\Phi}$ agree on $C_0(0,1)\otimes 1_A$.
- $\acute{\Phi}(\mathrm{id}_{(0,1]}\otimes 1_A) + \grave{\Phi}((1-\mathrm{id}_{(0,1]})\otimes 1_A)) = 1_{\mathcal{Q}_\omega}$.

$\therefore$ scalar parts of $\acute{\Phi}, \grave{\Phi}$ restrictions of unital *-hm $\theta : C[0,1]\to\mathcal{Q}_\omega$

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τ_A is qd $\Leftrightarrow$ $\acute{\Phi}$ and $\grave{\Phi}$ are unitarily equivalent on $C_0(0,1) \otimes A$.

Stable uniqueness

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Thm (Dadarlat-Eilers, c.f. Lin): Stable Uniqueness V1

Let C be unital separable and exact, B unital and $\iota,\phi,\psi : C \to B$ s.t.

1. ϕ,ψ unital nuclear, same class in $KK_{\mathrm{nuc}}(C,B)$.
2. ι unital totally full: $\overline{B\iota(c)B} = B$ for all non-zero $b \in B_+$.

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Then, for all $\varepsilon > 0$ and finite $\mathcal{F} \subset C$, exists $n \in \mathbb{N}$ and unitary in $M_{n+1}(B)$ s.t. $\|(\phi(c) \oplus \iota^{\oplus n}(c)) - u(\psi(c) \oplus \iota^{\oplus n}(c))u^*\| < \varepsilon, \quad c \in \mathcal{F}$.

ϕ and ψ AU equivalent after adding on copies of ι. But

n depends on $\mathcal{F}$ and ε and on B, ϕ, ψ, ι.

STABLE UNIQUENESS

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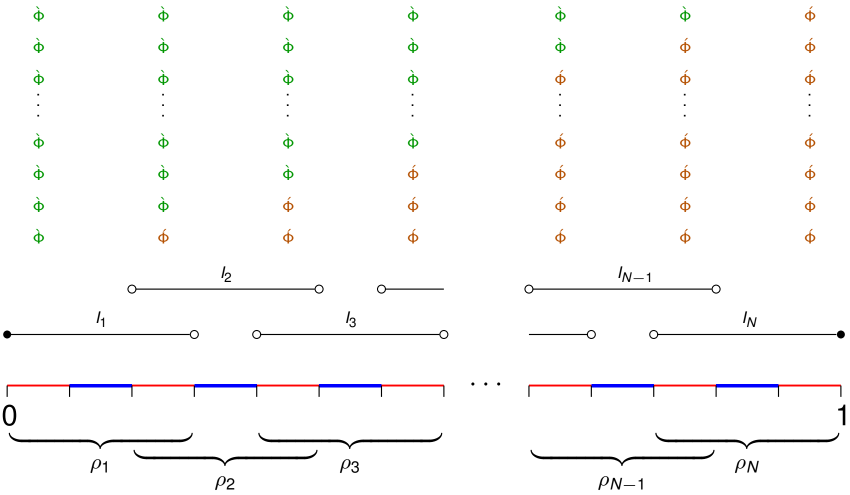
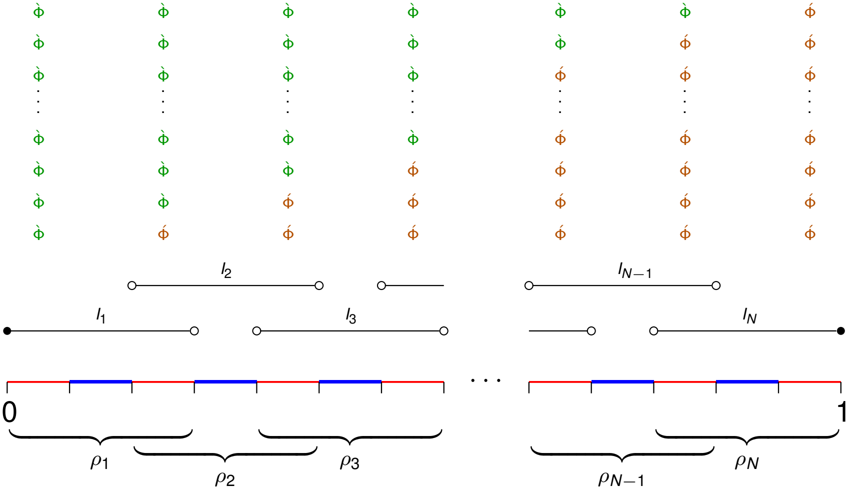
IDEA: DADARLAT-EILERS, LIN

Run sequence of counterexamples: get n to depend only on $\mathcal{F}$, ε.

STABLE UNIQUENESS ALONG THE INTERVAL



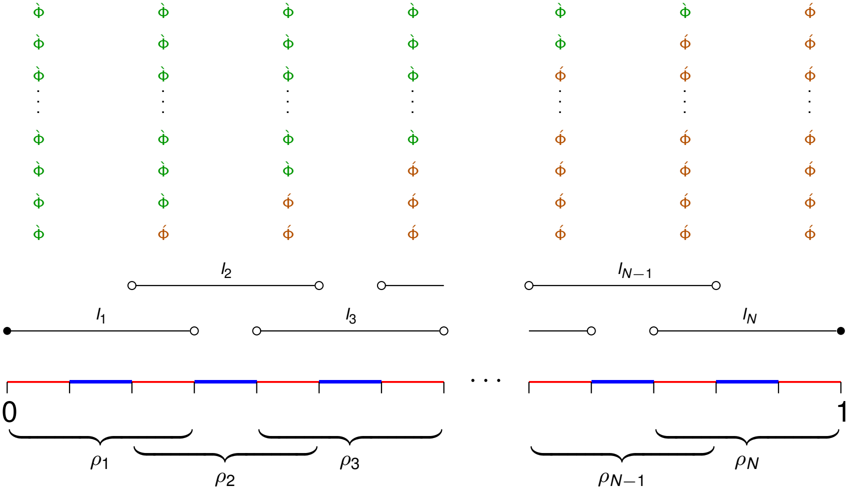
Stable Uniqueness along the interval



On blue intervals stable uniqueness gives

$$\grave{\phi} \oplus \grave{\phi}^{\oplus N/2} \approx_{au,\mathcal{F},\varepsilon} \acute{\phi} \oplus \grave{\phi}^{\oplus N/2} \text{ and } \grave{\phi} \oplus \acute{\phi}^{\oplus N/2} \approx_{au,\mathcal{F},\varepsilon} \acute{\phi} \oplus \acute{\phi}^{\oplus N/2}$$

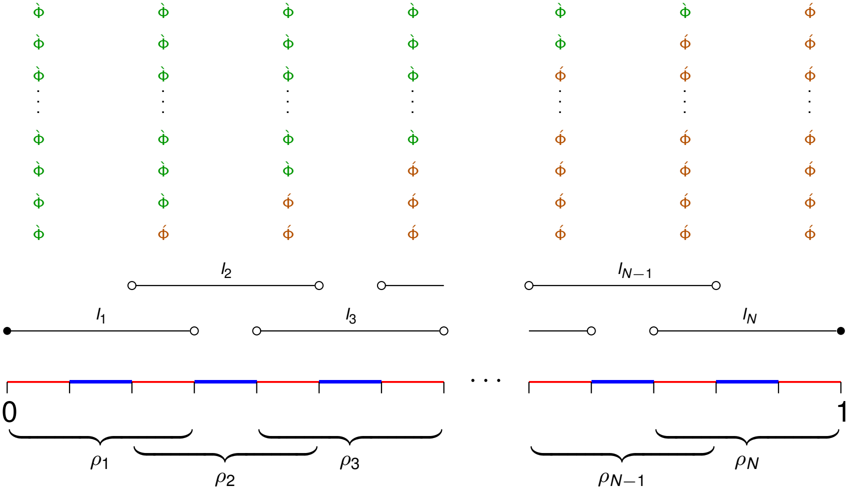
Stable Uniqueness along the interval



Patching on the intervals I_i gives

approx *-hms $\rho_i : C_0(I_i, A) \to M_{2N}(\mathcal{Q}_\omega)$ as specifed on red intervals.

Stable Uniqueness along the interval



Glue ρ_i together using partition of unity for $[0,1]$ gives

approx *-hm $C_0([0,1],A) \to M_{2N}(\mathcal{Q}_\omega)$ realising $\frac{1}{2}\tau_A$.