Starting point: Injectivity $\Rightarrow$ hyperfiniteness

**Connes ’77: Injective $\mathcal{V}$NAs are hyperfinite**

Combining this with classification of hyperfinite factors:
- Complete classification of (separably acting) injective factors.
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**Connes ’77: Injective vNAs are hyperfinite**

Combining this with classification of hyperfinite factors:
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**Classification programme**

- Aims to classify separable, **nuclear** and **simple** C*-algebras by “$K$-theoretic data”.

- **Nuclear** analogous to injective: $A$ nuclear $\iff A^{**}$ injective.
- **Simple** analogous to factor: weak*-closed ideals in vNa of form $\mathcal{M}p$ for a central projection $p \in \mathcal{M}$. 
STARTING POINT: Injectivity \implies hyperfiniteness

CLASSIFICATION PROGRAMME

- Aims to classify separable, nuclear and simple C*-algebras by “K-theoretic data”.

- Nuclear analogous to injective: A nuclear \iff A^{**} injective.
- Simple analogous to factor: weak*-closed ideals in vNa of form $M\rho$ for a central projection $\rho \in M$

CONNES’ 3 INGREDIENTS

A (separably acting) injective II$_1$ factor $\mathcal{M}$

1. is McDuff: $\mathcal{M} \cong \mathcal{M} \bar{\otimes} \mathcal{R}$.
2. has unique morphisms: any two *-hms $\mathcal{M} \rightarrow (\text{II}_1 \text{ factor})$ are approximately unitarily equivalent.
3. has an embedding $\theta : \mathcal{M} \hookrightarrow \mathcal{R}^\omega$
Quasidiagonality

- $Q$ universal UHF-algebra. $Q = \bigotimes_{p \text{ prime}} (\bigotimes_1^\infty M_p)$.
- $Q_\omega = \ell^\infty(Q)/\{(x_n) \in \ell^\infty(Q) : \lim_{n \to \omega} \|x_n\| = 0\}$. [$\omega \in \beta\mathbb{N}\setminus\mathbb{N}$ fixed.]
- Has unique trace $\tau_{Q_\omega}((x_n)_n) = \lim_{n \to \omega} \tau_Q(x_n)$

Separable nuclear $C^*$-algebra $A$ is quasidiagonal iff $\exists A \hookrightarrow Q_\omega$. 
**Quasidiagonality**

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**Examples**

- Abelian.
- Subhomogeneous

**Closed under**

- Subalgebras
- Increasing inductive limits
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Obstruction one: stable finiteness
Quasidiagonal $\Rightarrow$ stably finite: no infinite projections in $M_n(A)$
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Separable nuclear \( C^\ast \)-algebra \( A \) is quasidiagonal iff \( \exists A \hookrightarrow Q_\omega \).

Examples

- Abeliens.
- Subhomogeneous

Closed under

- Subalgebras
- Increasing inductive limits

Obstruction one: Stable finiteness

Quasidiagonal \( \Rightarrow \) stably finite: no infinite projections in \( M_n(A) \)

Classification predicts

Stably finite simple nuclear \( C^\ast \)-algebras are approximately subhomogenous, so quasidiagonal.
**Traces**

Separable nuclear C*-algebra $A$ is quasidiagonal iff $\exists A \hookrightarrow Q_\omega$.

**Definition**

A trace $\tau_A$ on $A$ is

1. **quasidiagonal** if $\exists$ cpc $\phi_i : A \to M_{k_i}$ with
   \[
   \|\phi_i(ab) - \phi_i(a)\phi_i(b)\| \to 0 \text{ and } \tau_A(a) = \lim \tau_{M_{k_i}}(\phi_i(a)).
   \]

2. **amenable** if $\exists$ cpc $\phi_i : A \to M_{k_i}$ with
   \[
   \|\phi_i(ab) - \phi_i(a)\phi_i(b)\|_{2,M_{k_i}} \to 0 \text{ and } \tau_A(a) = \lim \tau_{M_{k_i}}(\phi_i(a)).
   \]

- $A$ sep nuclear: qd traces those factorising $A \xrightarrow{*-hm} Q_\omega \xrightarrow{Q_\omega} \mathbb{C}$.
- $\tau_A$ amenable $\iff$ For $A \subset B(H)$ $\exists$ $A$-central state extending $\tau_A$.

**Second Obstruction**

Quasidiagonal unital C*-algebras have amenable traces.

- In particular, as noted by Rosenberg, $C^*_r(G)$ QD $\Rightarrow$ $G$ amenable.
Separable nuclear $C^*$-algebra $A$ is quasidiagonal iff $\exists A \hookrightarrow Q_\omega$.

Rosenberg’s Conjecture

$C^*(G)$ is qd for $G$ discrete amenable.

- Yes for elementary amenable groups (Ozawa, Rørdam, Sato ’14) via classification!
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**Blackadar-Kirchberg Conjecture**

Stably finite nuclear $C^*$-algebras are quasidiagonal.
Separable nuclear $C^*$-algebra $A$ is **quasidiagonal** iff $\exists A \hookrightarrow Q_\omega$.

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**Blackadar-Kirchberg Conjecture**

Stably finite nuclear $C^*$-algebras are quasidiagonal.

**Question**

Are all amenable traces quasidiagonal? (Is $\mathcal{R}$ quasidiagonal?)
Separable nuclear $C^*$-algebra $A$ is quasidiagonal iff $\exists A \hookrightarrow Q_\omega$.

**Theorem (Tikuisis, W, Winter)**

Every faithful trace on a separable nuclear $C^*$-algebra in the UCT class is quasidiagonal.

- Having a faithful qd trace ensures quasidiagonality.
- Via Brown: all traces on a quasidiagonal separable nuclear $C^*$-algebra in the UCT class are qd.
Questions and Some Answers

Separable nuclear $C^*$-algebra $A$ is quasidiagonal iff $\exists A \hookrightarrow Q_\omega$.

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UCT = Universal Coefficient Theorem of Rosenberg and Schochet

- $A$ has UCT iff it is $KK$-equivalent (weak homotopy kind of statement) to an abelian $C^*$-algebra;
- Open whether all separable nuclear $C^*$-algebras have UCT.
Separable nuclear $C^*$-algebra $A$ is quasidiagonal iff $\exists A \hookrightarrow Q_\omega$.

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**Rosenberg’s conjecture**

$C^*(G)$ is qd for $G$ discrete amenable.

- Yes for elementary amenable groups (Ozawa, Rørdam, Sato ’14) via classification!
- Yes in general. $C^*_r(G)$ is in the UCT class by Tu.
- In fact $C^*_r(G)$ is AF-embeddible.
Separable nuclear $C^*$-algebra $A$ is quasidiagonal iff $\exists A \hookrightarrow Q_\omega$.

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**Blackadar-Kirchberg Conjecture**
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- Holds in the simple UCT case.
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Are all amenable traces quasidiagonal? (Is $\mathcal{R}$ quasidiagonal?)

- Extended by Gabe to obtain quasidiagonality of faithful amenable traces on separable exact algebras (= subalgebras of nuclears) in the UCT class.
Gong-Lin-Niu ’15 (The Long Paper)

Identifies, and classifies (assuming UCT), a class of stably finite algebras which exhausts Elliott’s invariant.
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- Class given by internal conditions (like hyperfinite for vNas).
- Aim: find abstract characterisation of this class.
Back to classification (briefly)

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- Aim: find abstract characterisation of this class.

Theorem (Matui-Sato ’13)

Let $A$ be simple, separable, unital and nuclear with unique trace. Suppose

1. $A \cong A \otimes Q$ (for those in the know, $\mathcal{Z}$-stability suffices).
2. $A$ is quasidiagonal.

Then $A$ is in the GLN-class (in fact in a somewhat simpler class).

- In UCT case, 2 is now automatic: thus get classification from a tensorial absorption hypothesis.
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- Nuclear dimension (Winter, Zacharias): natural non-commutative generalising of covering dimension to nuclear $C^*$-algebras.
- e.g. $C(X) times G$ has finite nuclear dimension for free minimal action of finitely generated nilpotent on finite dimensional $X$.

ELLIOIT-GONG-LIN-NIU + WINTER

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A stably finite simple unital separable of finite nuclear dimension in UCT class such that all traces are QD. Then $A$ is in the GLN-class.

Simple separable infinite dimensional unital C*-algebras with finite nuclear dimension and UCT are classified by Elliott’s invariant.
Strategy of proof

Theorem (Tikuisis, W, Winter)
Every faithful trace on a separable nuclear $C^*$-algebra in the UCT class is quasidiagonal.
**Strategy of Proof**

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**Theorem (Voiculescu)**

Quasidiagonality is homotopy invariant.

- Cones $C_0(0, 1] \otimes A$ are always quasidiagonal.
STRATEGY OF PROOF

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**Proposition: Kirchberg-Rørdam, Sato-W-Winter, Gabe, ...**

All amenable traces on cones $C_0(0, 1] \otimes A$ are quasidiagonal.
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Fix such faithful trace $\tau_\mathcal{A}$
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FIX SUCH FAITHFUL TRACE $\tau_A$

1. $\exists \Phi : C_0(0, 1] \otimes A \rightarrow Q_\omega$ realising $\mu_{\text{leb}} \otimes \tau_A$. 
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Fix such faithful trace $\tau_A$

1. $\exists \hat{\phi} : C_0(0, 1] \otimes A \to Q_\omega$ realising $\mu_{\text{leb}} \otimes \tau_A$.
2. $\exists \hat{\Phi} : C_0[0, 1) \otimes A \to Q_\omega$ realising $\mu_{\text{leb}} \otimes \tau_A$.

A priori, these have nothing in common; but can adjust s.t.

- $\hat{\phi}$ and $\hat{\Phi}$ agree on $C_0(0, 1] \otimes 1_A$.
- $\hat{\phi}(\text{id}_{(0,1]} \otimes 1_A) + \hat{\Phi}((1 - \text{id}_{(0,1]}) \otimes 1_A)) = 1_{Q_\omega}$.

$\therefore$ scalar parts of $\hat{\phi}, \hat{\Phi}$ restrictions of unital *-hm $\theta : C[0, 1] \to Q_\omega$.
**Strategy of proof**

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Every faithful trace on a separable nuclear C*-algebra in the UCT class is quasidiagonal.

**Fix such faithful trace** $\tau_A$

1. $\exists \Phi : C_0(0, 1) \otimes A \to Q_\omega$ realising $\mu_{\text{leb}} \otimes \tau_A$.
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A priori, these have nothing in common; but can adjust s.t.

- $\Phi$ and $\dot{\Phi}$ agree on $C_0(0, 1) \otimes 1_A$.
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$\therefore$ scalar parts of $\Phi$, $\dot{\Phi}$ restrictions of unital *-hm $\theta : C[0, 1] \to Q_\omega$.

**In fact**

$\tau_A$ is qd $\iff$ $\Phi$ and $\dot{\Phi}$ are unitarily equivalent on $C_0(0, 1) \otimes A$. 
**Stable uniqueness**

**In fact**

\( \tau_A \) is qd \( \iff \) \( \hat{\phi} \) and \( \hat{\Phi} \) are unitarily equivalent on \( C_0(0, 1) \otimes A \).

**Thm (Dadarlat-Eilers, c.f. Lin): Stable Uniqueness V1**

Let \( C \) be unital separable and exact, \( B \) unital and \( \iota, \phi, \psi : C \to B \) s.t.

1. \( \phi, \psi \) unital nuclear, same class in \( KK_{nuc}(C, B) \).
2. \( \iota \) unital totally full: \( B\iota(c)B = B \) for all non-zero \( b \in B_+ \).
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Then, for all $\varepsilon > 0$ and finite $\mathcal{F} \subset C$, exists $n \in \mathbb{N}$ and unitary in $M_{n+1}(B)$ s.t. $\| (\phi(c) \oplus \iota^{\oplus n}(c)) - u(\psi(c) \oplus \iota^{\oplus n}(c))u^* \| < \varepsilon$, $c \in \mathcal{F}$.

$\phi$ and $\psi$ AU EQUIVALENT AFTER ADDING ON COPIES OF $\iota$. BUT

$n$ depends on $\mathcal{F}$ and $\varepsilon$ and on $B$, $\phi$, $\psi$, $\iota$. 

**Stuart White (Glasgow) Quasidiagonality & Amenability 8 / 9**
**Stable uniqueness**

**In fact**

\( \tau_A \) is qd \( \Leftrightarrow \) \( \hat{\phi} \) and \( \tilde{\phi} \) are unitarily equivalent on \( C_0(0, 1) \otimes A \).

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\( \phi \) and \( \psi \) AU equivalent after adding on copies of \( \iota \). But

\[ n \text{ depends on } \mathcal{F} \text{ and } \varepsilon \text{ and on } B, \phi, \psi, \iota. \]

**Idea: Dadarlat-Eilers, Lin**

Run sequence of counterexamples: get \( n \) to depend only on \( \mathcal{F}, \varepsilon \).
Stable Uniqueness along the Interval
Stable Uniqueness along the interval

\[ \Phi \oplus \Phi^{N/2} \cong_{au, \mathcal{F}, \varepsilon} \Phi \oplus \Phi^{N/2} \] and \[ \Phi \oplus \Phi^{N/2} \cong_{au, \mathcal{F}, \varepsilon} \Phi \oplus \Phi^{N/2} \]

On blue intervals stable uniqueness gives
**Stable Uniqueness along the interval**

\[
\begin{align*}
\phi & \quad \phi & \quad \phi & \quad \phi & \quad \phi & \quad \phi & \quad \phi \\
\rho_1 & \quad \rho_2 & \quad \rho_3 \\
0 & \quad \rho_1 & \quad \rho_2 & \quad \rho_3 & \quad \rho_{N-1} & \quad \rho_N & \quad 1 \\
\end{align*}
\]

**Patching on the intervals** \(l_i\) gives

approx \*-*hms \(\rho_i : C_0(l_i, A) \to M_{2N}(\mathbb{Q_\omega})\) as specified on red intervals.
**Stable Uniqueness along the Interval**

Glue $\rho_i$ together using partition of unity for $[0, 1]$ gives

$$\text{approx } *\text{-hm } C_0([0, 1], A) \to M_{2N}(\mathcal{Q}_\omega) \text{ realising } \frac{1}{2} \tau_A.$$