II_1 factors with exactly two crossed product decompositions

NCGOA 2016 – Lecture 3

Hausdorff Institute for Mathematics, Bonn, May 17-25, 2016

KU LEUVEN

Stefaan Vaes*

* Supported by ERC Consolidator Grant 614195

Determining all gms decompositions

Setup :

- Take $\Gamma = \mathbb{F}_\infty$, $G = \Gamma \times \Gamma$ and $R = R_0^\Gamma = \overline{\bigotimes}_{g\in\Gamma}(R_0, \tau)$.
- Take $G \curvearrowright^\alpha R$ by left-right translation :

  $\alpha_{(g,h)}(\pi_k(a)) = \pi_{gkh^{-1}}(a)$ for all $g, k, h \in \Gamma$ and $a \in R_0$.
- Recall : $\pi_k : R_0 \to R$ embedding as k'th tensor factor.

Goal of this lecture

Prove that every gms decomposition of $R \rtimes G$ is unitarily conjugate to one of the form $(B_0 \rtimes \Lambda_0)^\Gamma \rtimes G = B_0^\Gamma \rtimes (\Lambda_0^{(\Gamma)} \rtimes G)$,

where $R_0 = B_0 \rtimes \Lambda$ is a gms decomposition of R_0.

Remark : yesterday, we had the extra action $\Gamma \curvearrowright^\beta R_0$.

It is essential, but only adds notational difficulty for today's goal.

Main intermediate step

Write $M = R_0^\Gamma \rtimes (\Gamma \times \Gamma)$.

Assume that $M = B \rtimes \Lambda$ is a gms decomposition.

⤳ Define the dual coaction

$$\Delta : M \to M \overline{\otimes} M : \Delta(bv_s) = bv_s \otimes v_s \text{ for all } b \in B,\ s \in \Lambda.$$

Main intermediate step

Prove that
$$\Delta(L(\Gamma \times e)) \prec L(\Gamma \times e) \overline{\otimes} L(\Gamma \times e)$$
$$\Delta(L(e \times \Gamma)) \prec L(e \times \Gamma) \overline{\otimes} L(e \times \Gamma)$$

Recall : $\prec$ denotes Popa's intertwining-by-bimodules (inside $M \overline{\otimes} M$).

Popa's spectral gap rigidity for Bernoulli actions

A first crucial ingredient : Popa's theorem, 2008

If Γ is an arbitrary group and $P \subset R_0^\Gamma \rtimes \Gamma$ has a nonamenable relative commutant, then $P \prec L(\Gamma)$.

- Sketch of proof at the end of this lecture.
- The above theorem deals with plain Bernoulli actions.
  We need: left-right translation action $R_0^\Gamma \rtimes (\Gamma \times \Gamma)$.
- We even need its doubling: $\big(R_0^\Gamma \rtimes (\Gamma \times \Gamma)\big) \mathbin{\overline{\otimes}} \big(R_0^\Gamma \rtimes (\Gamma \times \Gamma)\big)$.
- Nevertheless: because $\Delta(L(\Gamma \times e))$ has a large relative commutant (at least $\Delta(L(e \times \Gamma))$),

  we can prove that $\Delta(L(\Gamma \times e)) \prec L(\Gamma \times \Gamma) \mathbin{\overline{\otimes}} L(\Gamma \times \Gamma)$.
- Similarly, $\Delta(L(e \times \Gamma)) \prec L(\Gamma \times \Gamma) \mathbin{\overline{\otimes}} L(\Gamma \times \Gamma)$.

Ozawa's solidity

A second crucial ingredient : Ozawa's solidity theorem, 2003

Let $\Gamma = \mathbb{F}_n$ with $2 \leq n \leq \infty$. Then, $M = L(\Gamma)$ is **solid :**

for every diffuse $P \subset M$, we have that $P' \cap M$ is amenable.

- Sketch of proof at the end of this lecture.
- Above, we arrived at $\Delta(L(\Gamma \times \Gamma)) \prec L(\Gamma \times \Gamma) \mathbin{\overline{\otimes}} L(\Gamma \times \Gamma)$.
- We took $\Gamma = \mathbb{F}_\infty$. Therefore, inside $L(\Gamma)$ there is no room for two commuting nonamenable subalgebras.
- Inside $L(\Gamma \times \Gamma)$, the only room for two commuting subalgebras is inside $L(\Gamma \times e)$ and $L(e \times \Gamma)$, respectively.
- With some more struggle,

$$\Delta(L(\Gamma \times e)) \prec L(\Gamma \times e) \mathbin{\overline{\otimes}} L(\Gamma \times e)$$
$$\Delta(L(e \times \Gamma)) \prec L(e \times \Gamma) \mathbin{\overline{\otimes}} L(e \times \Gamma)$$

Let us cheat a bit

We arrived at
$$\Delta(L(\Gamma \times e)) \prec L(\Gamma \times e) \mathbin{\overline{\otimes}} L(\Gamma \times e)$$
$$\Delta(L(e \times \Gamma)) \prec L(e \times \Gamma) \mathbin{\overline{\otimes}} L(e \times \Gamma)$$

Rather assume now that
$$\Delta(L(\Gamma \times e)) \subset L(\Gamma \times e) \mathbin{\overline{\otimes}} L(\Gamma \times e)$$
$$\Delta(L(e \times \Gamma)) \subset L(e \times \Gamma) \mathbin{\overline{\otimes}} L(e \times \Gamma)$$

Subalgebras invariant under the dual coaction

We have $\Delta(bv_s) = bv_s \otimes v_s$ coming from $M = B \rtimes \Lambda$.

If $P \subset M$ and $\Delta(P) \subset P \mathbin{\overline{\otimes}} P$, then $P = B_1 \rtimes \Lambda_1$.

- Thus, $L(\Gamma \times e) = B_1 \rtimes \Lambda_1$.
- By Ozawa-Popa, B_1 is atomic. We assume $B_1 = \mathbb{C}1$.
  Thus, $L(\Gamma \times e) = L(\Lambda_1)$. Similarly, $L(e \times \Gamma) = L(\Lambda_2)$.
- **Attention :** this does not mean yet that $\Gamma \times e = \Lambda_1$!

What if cheating is not allowed ?

⇝ You have to struggle much more to prove that the **mysterious** group Λ contains **two commuting nonamenable subgroups** Λ_1 and Λ_2.

Important ingredient : Ioana's ultrapower techniques of 2011.

Theorem (Chifan – de Santiago – Sinclair, 2015)

If Γ_1, Γ_2 are non elementary hyperbolic groups and $L(\Gamma_1 \times \Gamma_2) \cong L(\Lambda)$, then $\Lambda \cong \Lambda_1 \times \Lambda_2$ with Λ_1 and Λ_2 nonamenable.

⇝ One of the very rare theorems where a **group structure** property is recovered from the group von Neumann algebra !

Identifying $\Gamma \times \Gamma$ with a subgroup of Λ

Back to our business : $M = R_0^\Gamma \times (\Gamma \times \Gamma)$
and $M = B \rtimes \Lambda$ is a gms decomposition giving $\Delta : M \to M \mathbin{\overline{\otimes}} M$.

- We already "proved" that $L(\Gamma \times \Gamma) = L(\Lambda_1 \times \Lambda_2)$.
- By Popa-V, also $B \prec R_0^\Gamma$.
- Cheat a last time and assume that $B \subset R_0^\Gamma$.
- The unitaries $(v_s)_{s \in \Lambda_1 \times \Lambda_2}$ lie in $L(\Gamma \times \Gamma)$ and normalize $B \subset R_0^\Gamma$.
- This forces $v_s \in \mathbb{T}(\Gamma \times \Gamma)$.
- We have "proved" that $\mathbb{T}(\Lambda_1 \times \Lambda_2) = \mathbb{T}(\Gamma \times \Gamma)$.

But, what if cheating is not allowed ?

We need a method to go from $L(\Gamma) = L(\Lambda)$ to $\mathbb{T}\Gamma = \mathbb{T}\Lambda$.

Theorem (Ioana – Popa – V, 2010)

Let Γ be an icc group and assume that $L(\Gamma) = L(\Lambda)$.

Then, the following are equivalent.

- There exists a unitary u such that $u \, \mathbb{T}\Gamma \, u^* = \mathbb{T}\Lambda$.
- There exists a $\delta > 0$ such that $h_\Gamma(v_s) \geq \delta$ for all $s \in \Lambda$.

  Here : $h_\Gamma(x) = \max_{g \in \Gamma} |\tau(x u_g^*)|$ is the largest Fourier coefficient.

⇝ This was an important step in our W*-superrigidity theorem for certain group von Neumann algebras $L(\mathcal{G})$.

End of our proof

Back to our business : $M = R_0^\Gamma \rtimes (\Gamma \times \Gamma)$
and $M = B \rtimes \Lambda$ is a gms decomposition giving $\Delta : M \to M \mathbin{\overline{\otimes}} M$.

- We already "proved" that $\Gamma \times \Gamma \subset \mathbb{T}\Lambda$. Forget about $\mathbb{T}$.
- Thus, $\Delta(u_{(g,h)}) = u_{(g,h)} = u_{(g,h)}$ for all $g, h \in \Gamma$.
- Note that $\pi_e(R_0)$ commutes with $u_{(g,g)}$.
- It follows that $\Delta(\pi_e(R_0)) \subset \pi_e(R_0) \mathbin{\overline{\otimes}} \pi_e(R_0)$.
- But then, $R_0 = B_0 \rtimes \Lambda_0$ and we find that $B = B_0^\Gamma$ and $\Lambda = \Lambda_0^{(\Lambda)}$.

QED

Popa's spectral gap rigidity for Bernoulli actions

Theorem (Popa, 2008)

If Γ is an arbitrary group and $P \subset R_0^\Gamma \rtimes \Gamma$ has a nonamenable relative commutant, then $P \prec L(\Gamma)$.

- Put $R = R_0^\Gamma$ and $\Gamma \curvearrowright^\alpha R$ by Bernoulli shift.
- Popa's malleable deformation : $\theta_t \in \mathrm{Aut}(R \mathbin{\overline{\otimes}} R)$ with $\theta_0 = \mathrm{id}$, with $\theta_1(a \otimes 1) = 1 \otimes a$ and with $\theta_t \circ (\alpha_g \otimes \alpha_g) = (\alpha_g \otimes \alpha_g) \circ \theta_t$.
- Put $M = R \rtimes \Gamma$ and $M_1 = (R \mathbin{\overline{\otimes}} R) \rtimes \Gamma$. Then, $\theta_t \in \mathrm{Aut}(M_1)$.
- Use that ${}_M L^2(M_1 \ominus M)_M$ is weakly contained in the coarse M-bimodule to deduce that $\theta_t \to \mathrm{id}$ uniformly on P.
- Deduce that P and $\theta_1(P)$ are unitarily conjugate.
- This implies $P \prec L(\Gamma)$.

Ozawa's solidity

Theorem (Ozawa, 2003)

Let $\Gamma = \mathbb{F}_n$ with $2 \leq n \leq \infty$. Then, $M = L(\Gamma)$ is **solid :**

for every diffuse $P \subset M$, we have that $P' \cap M$ is amenable.

- The free group belongs to Ozawa's class $\mathcal{S}$: there exists an isometry $V : \ell^2(\Gamma) \to \ell^2(\Gamma) \otimes \ell^2(\Gamma)$ such that $V\lambda_g\rho_h - (\lambda_g \otimes \rho_h)V$ is a compact operator, for all $g, h \in \Gamma$.
- $V(\delta_{\text{word with } n \text{ letters}}) = \frac{1}{\sqrt{n+1}} \sum_{k=0}^{n} \delta_{\text{first } k \text{ letters}} \otimes \delta_{\text{last } n-k \text{ letters}}$
- View $V : L^2(M) \to L^2(M) \otimes L^2(M)$.

  Up to ..., we have $V(x\xi y) = (x \otimes 1)V(\xi)(1 \otimes y)$.
- Use $V(P)$ to produce almost $P' \cap M$-central vectors in $L^2(M) \otimes L^2(M)$. Thus, $P' \cap M$ is amenable.

Conclusion

We have produced concrete II_1 factors of the form $M = R_0^\Gamma \rtimes (\Gamma \times \Gamma)$ with

- in certain cases, exactly n group measure space Cartan subalgebras, up to conjugacy by an automorphism,
- in certain cases, exactly 2^n group measure space Cartan subalgebras, up to unitary conjugacy.

Open problem :

prove that these are the only Cartan subalgebras, up to automorphic, resp. unitary, conjugacy !