II_1 factors with exactly two crossed product decompositions

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Non uniqueness of Cartan subalgebras

Connes – Jones, 1982

They construct a free, **strongly ergodic**, pmp action $\mathcal{G} \curvearrowright (X, \mu)$ such that $M = L^\infty(X) \rtimes \mathcal{G}$ is **McDuff**.

- This means that $M \cong M \mathbin{\overline{\otimes}} R$.
- The Cartan subalgebras coming from $L^\infty(X) \mathbin{\overline{\otimes}} B \subset M \mathbin{\overline{\otimes}} R$ are not strongly ergodic.
- And thus not even conjugate to $L^\infty(X)$ by an automorphism.

Non uniqueness of Cartan subalgebras

Concrete example (Ozawa – Popa, 2008)

- Consider $\mathbb{Z}^2 \hookrightarrow \mathbb{T}^2$ via $(x, y) \mapsto (z_0^x, z_0^y)$ for an irrational angle $z_0 \in \mathbb{T}$.
- Consider $\mathrm{SL}(2, \mathbb{Z})$ acting on $\mathbb{Z}^2$ and $\mathbb{T}^2$ in a compatible way.
- So we have $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}) \curvearrowright \mathbb{T}^2$.
- In the crossed product $M = L^\infty(\mathbb{T}^2) \rtimes (\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}))$, we have at least two Cartan subalgebras : $L^\infty(\mathbb{T}^2)$ and $L(\mathbb{Z}^2)$.

Open problem : are these the only Cartan of M up to unitary conjugacy ?

Open problem : construct a II_1 factor M with exactly two Cartan up to unitary conjugacy.

Open problem : construct a II_1 factor M where uniqueness fails, but all Cartan can be determined up to unitary conjugacy.

Krogager – V, 2015 : solutions when restricting to **gms** Cartan.

The example systematized

Let H be a countable abelian group, K a compact abelian group and $H \hookrightarrow K$ a dense embedding.

- **Example :** $\mathbb{Z} \hookrightarrow \mathbb{T} : n \mapsto z_0^n$.
- Always, $L^\infty(K) \rtimes H \cong R$, the hyperfinite II_1 factor.
- Assume that $G \curvearrowright^\alpha K$ with $\alpha_g(H) = H$ for all $g \in G$.
- Then, $G \curvearrowright L^\infty(K) \rtimes H \cong L^\infty(\widehat{H}) \rtimes \widehat{K}$.

We get $L^\infty(K) \rtimes (H \rtimes G) \cong L^\infty(\widehat{H}) \rtimes (\widehat{K} \rtimes G)$.

Typically, both $L^\infty(K)$ and $L^\infty(\widehat{H})$ are Cartan subalgebras.

II_1 factors with two gms Cartan

Recall : $\mathcal{G} \curvearrowright (X, \mu)$ is W*-superrigid if every W*-equivalent action must be conjugate.

Theorem (Krogager – V, 2015)

For every integer $n \geq 1$, we construct

- free ergodic pmp actions $\mathcal{G} \curvearrowright (X, \mu)$ that are W*-equivalent with **exactly** n actions up to conjugacy,
- II_1 factors M that have **exactly** n gms Cartan up to conjugacy by an automorphism of M,
- II_1 factor M that have **exactly** 2^n gms Cartan up to unitary conjugacy.

Approach : $M = R \rtimes^{\alpha} G$ where R is the hyperfinite II_1 factor, $G = \mathbb{F}_\infty \times \mathbb{F}_\infty$ and the action α is nice.

We then determine all gms decompositions of M.

The "obvious" group measure space decompositions

Let $M = R \rtimes^{\alpha} G$.

- Whenever $R = B_0 \rtimes \Lambda_0$ such that
- $\alpha_g(B_0) = B_0$ and $\alpha_g(\Lambda_0) = \Lambda_0$ for all $g \in G$,
- we get a crossed product decomposition $M = B_0 \rtimes (\Lambda_0 \times G)$.

Definition

Let $G \curvearrowright^{\alpha} R$. An (α_g)-**invariant gms decomposition** of R is a gms decomposition $R = B_0 \rtimes \Lambda_0$ satisfying $\alpha_g(B_0) = B_0$ and $\alpha_g(\Lambda_0) = \Lambda_0$ for all $g \in G$.

Plan for today

Construct $G \curvearrowright^{\alpha} R$ such that

- **First step :** every gms decomposition of $M = R \rtimes^{\alpha} G$ comes from an (α_g)-invariant gms decomposition of R.
- **Second step :** all (α_g)-invariant gms decompositions of R can be explicitly determined.

Concrete construction

- Take $G = \Gamma \times \Gamma$ with $\Gamma = \mathbb{F}_\infty$.
- Put $R = R_0^\Gamma = \overline{\bigotimes}_{g \in \Gamma}(R_0, \tau)$.
- Fix some action $\Gamma \curvearrowright^\beta R_0$ that will be specified later.
- Define $G \curvearrowright^\alpha R$ given by $\alpha_{(g,h)}(\pi_k(a)) = \pi_{gkh^{-1}}(\beta_h(a))$ for all $g, h, k \in \Gamma$, $a \in R_0$.
- Here : $\pi_k : R_0 \to R_0^\Gamma$ is the embedding as the k'th tensor factor.

Theorem (Krogager – V, 2015)

If $\mathrm{Ker}\,\beta \neq \{e\}$, then every gms decomposition of $R \rtimes^\alpha G$ is unitarily conjugate to a gms decomposition of the form

- $R \rtimes^\alpha G = R_0^\Gamma \rtimes G = (B_0 \rtimes \Lambda_0)^\Gamma \rtimes G = B_0^\Gamma \rtimes (\Lambda_0^{(\Gamma)} \rtimes G)$,
- where $R_0 = B_0 \rtimes \Lambda_0$ is a $(\beta_g)_{g \in \Gamma}$-invariant gms decomposition of R_0.

Conclusions at this point

- The theorem (sketch of proof tomorrow) solves the **first step** : every gms decomposition of $R \rtimes^{\alpha} G$ comes from an invariant gms decomposition of R.
- The theorem partially solves the **second step** : these invariant gms decompositions of R are of the form $(B_0 \rtimes \Lambda_0)^{\Gamma}$.
- To finish the **second step** : determine all $(\beta_g)_{g \in \Gamma}$-invariant gms decomposition of R_0.
- **Big advantage :** we can choose any action $\mathbb{F}_\infty \curvearrowright^{\beta} R_0$ that we want.

$\rightsquigarrow$ We may choose any action of any group to make this task as easy as possible.

Determining all invariant gms decompositions

- Take a countable abelian group H, a compact abelian group K and a dense embedding $H \hookrightarrow K$.
- Assume that H and $\widehat{K}$ are torsion free.
- Realize $R_0 = L^\infty(K^3) \rtimes H^3$ with the natural action $\mathrm{SL}(3, \mathbb{Z}) \curvearrowright^\beta R_0$.

Theorem (Krogager – V, 2015)

The (β_g)-invariant gms decompositions of R_0 are precisely given by

- a decomposition $H_1 \oplus H_2 = H \hookrightarrow K = K_1 \oplus K_2$ that is compatible,
- so that $R_0 = L^\infty(K_1^3 \times \widehat{H_2^3}) \rtimes (H_1^3 \times \widehat{K_2^3})$.

Before proving this theorem, we look at some examples.

Examples

Given a dense embedding $H \hookrightarrow K$, a decomposition $H = H_1 \oplus H_2$ is compatible iff the closures $K_i = \overline{H_i}$ still satisfy $K_1 \cap K_2 = \{0\}$.

- For $\mathbb{Z} \hookrightarrow \mathbb{T} : n \mapsto z_0^n$, we have ... two compatible decompositions, namely $\mathbb{Z} \oplus 0$ and $0 \oplus \mathbb{Z}$.

  Feeding in the above constructions :

  We get a II_1 factor with two gms decompositions up to unitary conjugacy,

  but they are conjugate by an automorphism :

  $\mathbb{Z} \hookrightarrow \mathbb{T}$ is isomorphic with $\widehat{\mathbb{T}} \hookrightarrow \widehat{\mathbb{Z}}$.

- For $\mathbb{Z} \hookrightarrow \mathbb{T}^2 : n \mapsto (z_0^n, z_1^n)$, we get two gms decompositions up to unitary/automorphic conjugacy.

Examples

Given a dense embedding $H \hookrightarrow K$, a decomposition $H = H_1 \oplus H_2$ is compatible iff the closures $K_i = \overline{H_i}$ still satisfy $K_1 \cap K_2 = \{0\}$.

- For $\mathbb{Z}^k \hookrightarrow \mathbb{T}^{2k} : (n_1, \ldots, n_k) \mapsto (z_0^{n_1}, z_1^{n_1}, \ldots, z_0^{n_k}, z_1^{n_k})$

  all decompositions $\mathbb{Z}^k = H_1 \oplus H_2$ are compatible.

  Infinitely many gms decompositions up to unitary conjugacy.

  Exactly $k + 1$ up to conjugacy by an automorphism.

- For $\mathbb{Z}^k \hookrightarrow \mathbb{T}^{2k} : (n_1, \ldots, n_k) \mapsto (y_1^{n_1}, z_1^{n_1}, \ldots, y_k^{n_k}, z_k^{n_k})$

  only the decompositions $\mathbb{Z}^k = \mathbb{Z}^A \oplus \mathbb{Z}^B$
  with $\{1, \ldots, k\} = A \sqcup B$ are compatible.

  Exactly 2^k gms decompositions up to unitary/automorphic conjugacy.

Determining all invariant gms decompositions

- Take a countable abelian group H, a compact abelian group K and a dense embedding $H \hookrightarrow K$.
- Assume that H and $\widehat{K}$ are torsion free.
- Realize $R_0 = L^\infty(K^3) \rtimes H^3$ with the natural action $\mathrm{SL}(3,\mathbb{Z}) \curvearrowright^{\beta} R_0$.

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Sketch of the proof.