II_1 factors with exactly two crossed product decompositions

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KU LEUVEN

Stefaan Vaes*

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Von Neumann algebras

Recall : two big families of von Neumann algebras.

- Group von Neumann algebras $L(G)$.
- Crossed product von Neumann algebras $P \rtimes G$.

  In particular with $P = L^\infty(X)$.

Questions :

- **Flexibility :** unexpected isomorphisms, e.g. for G amenable.
- **Rigidity :**

  recover information on G or on $G \curvearrowright P$ out of $L(G)$ or $P \rtimes G$.

Cartan subalgebras

Some reminders :

- If (P,τ) is tracial and $G \curvearrowright (P,\tau)$ is trace preserving, then $P \rtimes G$ is tracial.
- $L^\infty(X) \subset L^\infty(X) \rtimes G$ is a MASA iff $G \curvearrowright (X,\mu)$ is (essentially) free : for all $g \neq e$, the set $\{x \in X \mid g \cdot x = x\}$ has measure zero.
- Thus : $L^\infty(X) \rtimes G$ is a II_1 factor when $G \curvearrowright (X,\mu)$ is a free ergodic pmp action.
- In that case : $L^\infty(X) \subset L^\infty(X) \rtimes G$ is a Cartan subalgebra.

Definition

A **Cartan subalgebra** A in a II_1 factor M is a MASA such that

$$\mathcal{N}_M(A) = \{u \in \mathcal{U}(M) \mid uAu^* = A\}$$

generates M.

W*-superrigidity

W*-equivalence

We say that free ergodic pmp actions $\mathcal{G} \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ are

- **W*-equivalent** if $L^\infty(X) \rtimes \mathcal{G} \cong L^\infty(Y) \rtimes \Lambda$,
- **conjugate** if there exist $\Delta : (X, \mu) \to (Y, \eta)$ and $\delta : \mathcal{G} \to \Lambda$ such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$.

W*-superrigidity

We say that a free ergodic pmp action $\mathcal{G} \curvearrowright (X, \mu)$ is **W*-superrigid** if every W*-equivalent action is actually conjugate.

⇝ **This means :**

$\mathcal{G}$ and $\mathcal{G} \curvearrowright (X, \mu)$ can be recovered from $L^\infty(X) \rtimes \mathcal{G}$.

W^*-superrigid actions

- Peterson (2009) : existence of (virtually) W^*-superrigid actions.
- Popa – V (2009) : concrete W^*-superrigid actions.
- Ioana (2010) : if G has property (T), then the Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$ is W^*-superrigid.
- Ioana – Popa – V (2010) : if $G = \Gamma_1 \times \Gamma_2$ with Γ_1 infinite and Γ_2 nonamenable, then again the Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$ is W^*-superrigid.

$$\vdots$$

- Gaboriau – Ioana – Tucker-Drob (2016) : the action $\mathrm{PSL}(2, \mathbb{Z}) \times \mathrm{PSL}(2, \mathbb{Z}) \curvearrowright \mathrm{PSL}(2, \mathbb{Z}_p)$ by left and right translation is virtually W^*-superrigid.

How to prove W*-superrigidity ...

... for a given $\mathcal{G} \curvearrowright (X, \mu)$.

Put $A = L^\infty(X)$ and $M = A \rtimes \mathcal{G}$.

Assume that $M = B \rtimes \Lambda$ for some other $\Lambda \curvearrowright (Y, \eta)$.

- **First step :** find $u \in \mathcal{U}(M)$ such that $uBu^* = A$.
- By Popa's theorem, it suffices that $B \prec A$.
- **Conclusion** of the first step (Singer) : the actions are **orbit equivalent**.

  Recall : this means $\exists \Delta : X \to Y$ with $\Delta(\mathcal{G} \cdot x) = \Lambda \cdot \Delta(x)$.
- **Second step :** prove OE-superrigidity. This means : every action that is OE to $\mathcal{G} \curvearrowright (X, \mu)$ is actually conjugate.

How to prove OE-superrigidity ...

... for a given $\mathcal{G} \curvearrowright (X, \mu)$.

Assume that $\Delta : X \to Y$ is an orbit equivalence with $\Lambda \curvearrowright (Y, \eta)$.

- Define Zimmer's rearrangement 1-cocycle $\omega : \mathcal{G} \times X \to \Lambda$ by $\Delta(g \cdot x) = \omega(g, x) \cdot \Delta(x)$.
- Prove that ω is cohomologous to a homomorphism $\delta : \mathcal{G} \to \Lambda$.

  This means : there exists $\varphi : X \to \Lambda$ such that $\omega(g, x) = \varphi(g \cdot x)^{-1} \, \delta(g) \, \varphi(x)$.
- Then, the map $\widetilde{\Delta} : X \to Y : \widetilde{\Delta}(x) = \varphi(x) \cdot \Delta(x)$ satisfies $\widetilde{\Delta}(g \cdot x) = \delta(g) \cdot \widetilde{\Delta}(x)$.

Popa's cocycle superrigidity theorem

Popa's cocycle superrigidity theorem (2005-2006)

When G has property (T), or when G is a product group, or when ..., every 1-cocycle for the Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$ with values in a countable group Λ is cohomologous to a group homomorphism.

Target group Λ may be any closed subgroup of $\mathcal{U}(M)$ with M a $\mathbf{II}_1$ factor.

Conjecture : cocycle superrigidity holds if and only if $\beta_1^{(2)}(G) = 0$.

Focus on the first step

Definition

Let M be a II_1 factor. We call $B \subset M$ a **group measure space (gms) Cartan subalgebra** if $B \subset M$ is a Cartan subalgebra and there exists Λ such that $M = B \rtimes \Lambda$.

- For the Bourbakists : existence of a subgroup $\Lambda < \mathcal{N}_M(B)$ such that $(B \cup \Lambda)'' = M$ and $E_B(v) = 0$ for every $v \in \Lambda \setminus \{1\}$.
- There may exist Cartan subalgebras that are not gms.

The **first step** in the approach to W*-superrigidity then becomes :

Does M have a unique gms Cartan subalgebra, up to unitary conjugacy.

More natural question : does M have a unique Cartan subalgebra, up to unitary conjugacy.

Uniqueness of Cartan subalgebras

Theorem (Ozawa – Popa, 2007)

Let $\mathcal{G} = \mathbb{F}_n$ with $2 \leq n \leq \infty$ and let $\mathcal{G} \curvearrowright (X, \mu)$ be a free ergodic **profinite** action.

This means : $\mathcal{G} \curvearrowright \varprojlim \mathcal{G}/\mathcal{G}_n$.

Then $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \mathcal{G}$ up to unitary conjugacy.

Chifan-Sinclair (2011) : the same for non elementary hyperbolic $\mathcal{G}$.

Theorem (Popa – V, 2011-2012)

Let $\mathcal{G} = \mathbb{F}_n$ or any non elementary hyperbolic group. For **arbitrary** free ergodic pmp actions $\mathcal{G} \curvearrowright (X, \mu)$, we have that $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \mathcal{G}$ up to unitary conjugacy.

A general dichotomy theorem

Theorem (Popa – V, 2011-2012)

Let $\mathcal{G} = \mathbb{F}_n$ or any non elementary hyperbolic group.

Let $M = P \rtimes \mathcal{G}$ be an **arbitrary** tracial crossed product.

If $Q \subset M$ is **amenable relative to** P, then

- either $Q \prec P$,
- or $\mathcal{N}_M(Q)''$ stays amenable relative to P.

Intuition : read $Q \prec P$ as saying that "Q is finite relative to P".

Then guess what relative amenability may be.

For groups : when $\Gamma, \Lambda < G$, we say that Γ is amenable relative to Λ if G/Λ admits a Γ-invariant mean.

Precise definition : ...

Consequences of the general dichotomy theorem

Definition

We say that $\mathcal{G}$ is Cartan-rigid if for every free ergodic pmp action $\mathcal{G} \curvearrowright (X, \mu)$, we have that $L^\infty(X)$ is the unique Cartan subalgebra of $L^\infty(X) \rtimes \mathcal{G}$ up to unitary conjugacy.

Theorem (Ioana, 2012)

All free products $\mathcal{G} = \Gamma_1 * \Gamma_2$ with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$ are Cartan-rigid.

Method : a family of embeddings $\theta_t : L^\infty(X) \rtimes (\Gamma_1 * \Gamma_2) \to P \rtimes \mathbb{F}_2$.

A stability result

Let $\mathcal{C}$ be the smallest class of groups

- containing all "nontrivial" free products and all non elementary hyperbolic groups,
- that is stable under **extensions** (in particular, direct products).

Then all groups in this class are Cartan-rigid.

How to prove uniqueness of Cartan subalgebras ?

Ozawa – Popa (2007) : $M = L^\infty(X) \rtimes \mathbb{F}_n$ with $\mathbb{F}_n \curvearrowright (X, \mu)$ free ergodic profinite.

Then, M has the complete metric approximation property (CMAP) :

- There exist $\varphi_n : M \to M$, such that
- each φ_n is completely bounded and $\limsup_n \|\varphi_n\|_{\mathrm{cb}} = 1$,
- each φ_n has finite rank,
- and for every $x \in M$, we have $\|\varphi_n(x) - x\|_2 \to 0$.

Start of the proof : given a Cartan subalgebra $B \subset M$ (or any amenable von Neumann subalgebra), we get a sequence of normal functionals μ_n on $B \mathbin{\overline{\otimes}} B^{\mathrm{op}}$ given by $\mu_n(a \otimes b^{\mathrm{op}}) = \tau(\varphi_n(a)b)$.

This sequence has nice asymptotic invariance properties, etc, etc, ...

Remarks

- To prove the dichotomy theorem for $P \rtimes \mathbb{F}_n$:

  work "relative" to P and build the "good" von Neumann algebra on which we have the functionals μ_n.

- **Every known uniqueness theorem for Cartan subalgebras ultimately relies on this "miracle" to construct special functionals μ_n using CMAP (or weak amenability).**

- Conceptually, weak amenability has very little to do with (non)uniqueness of Cartan subalgebras.

  Indeed: weak amenability of a group G has nothing to do with G having or not having abelian normal subgroups!

- **Conclusion : uniqueness of Cartan subalgebras is largely non understood !**

Back to uniqueness of gms Cartan subalgebras

Recall : enough to prove W*-superrigidity (in combination with OE-superrigidity).

More conceptual approach (Popa – V, 2009 and Ioana, 2010) :

- any crossed product decomposition $M = B \rtimes \Lambda$ gives rise to
- the **dual coaction** $\Delta : B \rtimes \Lambda \to (B \rtimes \Lambda) \mathbin{\overline{\otimes}} L(\Lambda)$ given by
  $\Delta(bv_s) = bv_s \otimes v_s$ for all $b \in B$, $s \in \Lambda$.
- We view $\Delta : M \to M \mathbin{\overline{\otimes}} M$.

Strategy : if $M = A \rtimes \mathcal{G}$ for a specific group $\mathcal{G}$ and a specific action $\mathcal{G} \curvearrowright A$ (for instance, Bernoulli), we might be able to determine "all" embeddings $\Delta : M \to M \mathbin{\overline{\otimes}} M$.

And thus, determine all possible group measure space decompositions $M = B \rtimes \Lambda$. **That's our program for the coming days.**