A Brief Overview of Bi-Free Probability

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The Left Regular Representation

Let $\mathbb{F}_2$ denote the free group on two elements a and b. The *left regular representation of* $\mathbb{F}_2$ is the group homomorphism $\lambda : \mathbb{F}_2 \to \mathcal{U}(\ell_2(\mathbb{F}_2))$ defined for all $g, h \in \mathbb{F}_2$ by

$$\lambda(g)\delta_h = \delta_{gh}.$$

If $L(\mathbb{F}_2) = \lambda(\mathbb{F}_2)''$, then the linear map $\tau : L(\mathbb{F}_2) \to \mathbb{C}$ defined by

$$\tau(T) = \langle T\delta_e, \delta_e\rangle$$

is a tracial state on $L(\mathbb{F}_2)$. Indeed if $n_j, m_j \in \mathbb{Z} \setminus \{0\}$, then

$$\tau(\lambda(a)^{n_1}\lambda(b)^{m_1}\lambda(a)^{n_2}\lambda(b)^{m_2}\cdots\lambda(a)^{n_k}\lambda(b)^{m_k}) = 0.$$

Free Independence

Definition

A *non-commutative probability space* is a pair $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear map with $\varphi(I_{\mathcal{A}}) = 1$.

Definition

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. A collection $\{A_k\}_{k \in K}$ of unital subalgebras of $\mathcal{A}$ are said to be *freely independent with respect to* φ if

$$\varphi(Z_1 \cdots Z_n) = 0$$

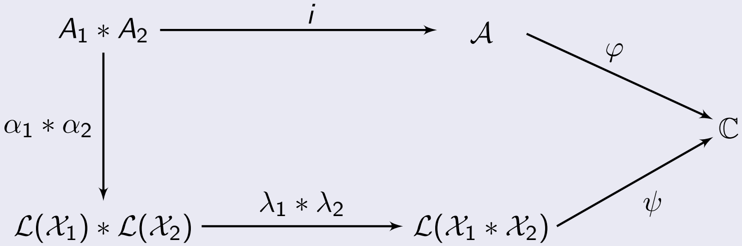
for all $n \geq 1$, for all $k_1, \ldots, k_n \in K$ with $k_m \neq k_{m+1}$ for all m, and for all $Z_m \in A_{k_m}$ with $\varphi(Z_m) = 0$.

For example, $\mathrm{alg}(\lambda(a))$ and $\mathrm{alg}(\lambda(b))$ are freely independent with respect to τ.

Free Independence via Free Products

Theorem (Voiculescu; 1985)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Unital subalgebras A_1 and A_2 of $\mathcal{A}$ are freely independent if and only if there exist vector spaces $\mathcal{X}_k$ and unital homomorphisms $\alpha_k : A_k \to \mathcal{L}(\mathcal{X}_k)$ such that the following diagram commutes:



Central Limit Distributions

Theorem (Central Limit Theorem)

For each $N \in \mathbb{N}$, let $X_1, \ldots, X_N$ be independent, identically distributed real-valued random variables with expectation zero and variance one. Then

$$S_N = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} X_k \quad \xrightarrow[N\to\infty]{distribution} \quad \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

Theorem (Free Central Limit Theorem; Voiculescu)

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$$S_N = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} X_k \quad \xrightarrow[N\to\infty]{distribution} \quad \frac{1}{\sqrt{2\pi}} \sqrt{4 - x^2}\, dx.$$

Bi-Free Independence

What about the right regular representation?

Definition (Voiculescu; 2013)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Pairs of unital algebras $(A_{\ell,1}, A_{r,1})$ and $(A_{\ell,2}, A_{r,2})$ of $\mathcal{A}$ are said to be *bi-freely independent* if there exist vector spaces $\mathcal{X}_k$ and unital homomorphisms unital homomorphisms $\alpha_k : A_{\ell,k} \to \mathcal{L}(\mathcal{X}_k)$ and $\beta_k : A_{r,k} \to \mathcal{L}(\mathcal{X}_k)$ such that the following diagram commutes:

$$
\begin{array}{ccccc}
A_{\ell,1} * A_{r,1} * A_{\ell,2} * A_{r,2} & \xrightarrow{\;\; i \;\;} & \mathcal{A} & \xrightarrow{\;\; \varphi \;\;} & \mathbb{C} \\
\Big\downarrow {\scriptstyle \alpha_1 * \beta_1 * \alpha_2 * \beta_2} & & & & \Big\uparrow {\scriptstyle \psi} \\
\mathcal{L}(\mathcal{X}_1) * \mathcal{L}(\mathcal{X}_1) * \mathcal{L}(\mathcal{X}_2) * \mathcal{L}(\mathcal{X}_2) & & \xrightarrow{\;\; \lambda_1 * \rho_1 * \lambda_2 * \rho_2 \;\;} & & \mathcal{L}(\mathcal{X}_1 * \mathcal{X}_2)
\end{array}
$$

A Simple Characterization of Bi-Freeness?

Freely Independent	$\iff$	Alternating Centred Moments Vanish
Bi-Freely Independent	$\iff$	???

Mastnak and Nica attempted to develop a bi-free analogue of non-crossing partitions.

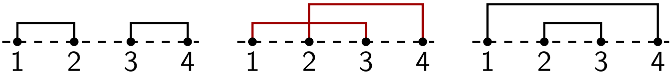
Non-Crossing Partitions

In 1994, Speicher developed a combinatorial approach to free probability via non-crossing partitions.

Definition

An partition π of $\{1, \ldots, n\}$ is said to be *non-crossing* if for all $U, V \in \pi$ with $U \neq V$, for all $a, b \in U$, and for all $c, d \in V$, it is not the case that $a < c < b < d$. The set of non-crossing partitions on $\{1, \ldots, n\}$ is denoted $NC(n)$.

$\{\{1,2\},\{3,4\}\}$ $\{\{1,3\},\{2,4\}\}$ $\{\{1,4\},\{2,3\}\}$



Moment and Cumulant Functions

The free cumulant of $Z_1, \ldots, Z_n$ is a Möbius inversion of the moments.

Theorem (Speicher; 1994)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. A collection $\{A_k\}_{k\in K}$ of unital subalgebras of $\mathcal{A}$ is freely independent if and only if

$$\kappa_{1_n}(Z_1, \ldots, Z_n) = 0$$

for all non-constant $\epsilon : \{1, \ldots, n\} \to K$ and for all $Z_k \in A_{\epsilon(k)}$.

The Permutation

Let $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ designate whether the k^{th} operator is considered a left operator ($\chi(k) = \ell$) or a right operator ($\chi(k) = r$). If

$$\chi^{-1}(\{\ell\}) = \{k_1 < k_2 < \cdots < k_m\}$$
$$\chi^{-1}(\{r\}) = \{k_{m+1} > k_{m+2} > \cdots > k_n\}$$

define the permutation s_χ of $\{1, \ldots, n\}$ via $s_\chi(t) = k_t$.

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Consider $\chi : \{1, \ldots, 7\} \to \{\ell, r\}$ with $\chi^{-1}(\{\ell\}) = \{1, 4, 6, 7\}$.
Suppose $Z_1, \ldots, Z_7$ are operators (either left or right based on χ) for which we want to consider $Z_1 \cdots Z_7 \xi$.

Let $\chi : \{1, \ldots, n\} \to \{\ell, r\}$ designate whether the k^{th} operator is considered a left operator ($\chi(k) = \ell$) or a right operator ($\chi(k) = r$). If

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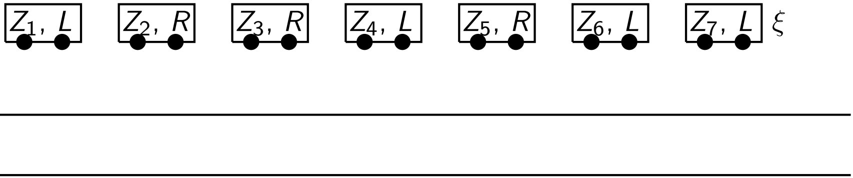

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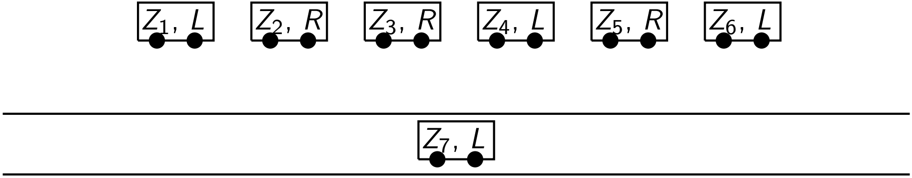

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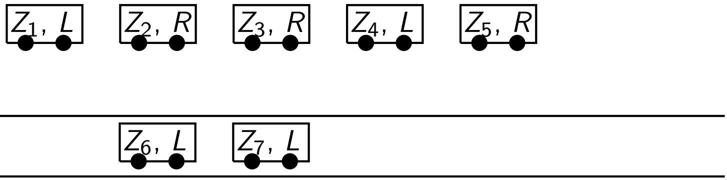

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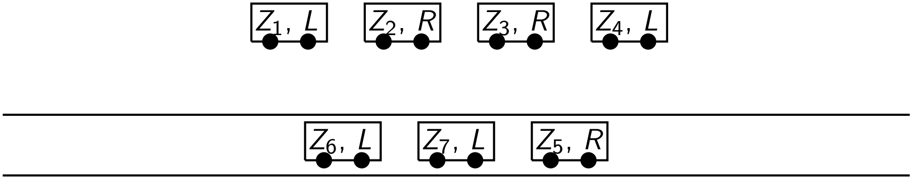

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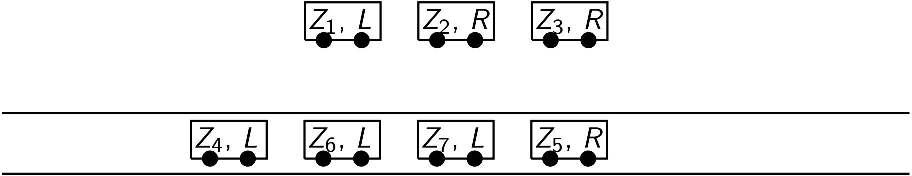

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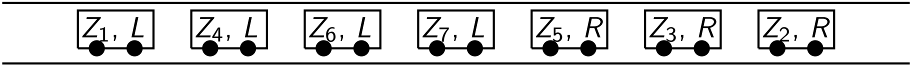

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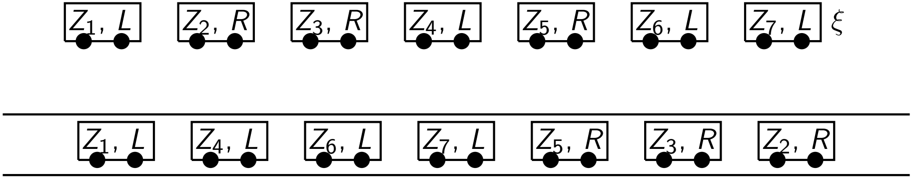

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define the permutation s_χ of $\{1, \ldots, n\}$ via $s_\chi(t) = k_t$.

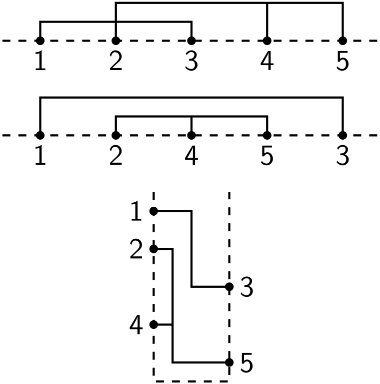
Definition (Mastnak, Nica; 2013)

Given $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, a partition π of $\{1, \ldots, n\}$ is said to be *bi-non-crossing with respect to* χ if the partition $s_\chi^{-1} \cdot \pi$ (the partition obtained by applying s_χ^{-1} to each block of π) is non-crossing.

Bi-Non-Crossing Partitions

Let $\chi^{-1}(\{\ell\}) = \{1, 2, 4\}$, $\chi^{-1}(\{r\}) = \{3, 5\}$, and

$$\pi = \Big\{\{1,3\},\{2,4,5\}\Big\} = s_\chi \cdot \Big\{\{1,5\},\{2,3,4\}\Big\}.$$



$$\mu_{BNC}(\pi, \sigma) = \mu_{NC}(s_\chi^{-1} \cdot \pi, s_\chi^{-1} \cdot \sigma)$$

Bi-Freeness and Mixed Cumulants

Theorem (Charlesworth, Nelson, Skoufranis; 2014)

Let $(\mathcal{A}, \varphi)$ *be a ncps and let* $\{(A_{\ell,k}, A_{r,k})\}_{k\in K}$ *be pairs of unital subalgebras of* $\mathcal{A}$*. Then the following are equivalent:*

- $\{(A_{\ell,k}, A_{r,k})\}_{k\in K}$ *are bi-freely independent.*
- *For all* $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\epsilon : \{1, \ldots, n\} \to K$, *and* $Z_m \in A_{\chi(m),\epsilon(m)}$,

$$\varphi(Z_1 \cdots Z_m) = \sum_{\pi \in BNC(\chi)} \left[\sum_{\substack{\sigma \in BNC(\chi) \\ \pi \leq \sigma \leq \epsilon}} \mu_{BNC}(\pi, \sigma) \right] \varphi_\pi(Z_1, \ldots, Z_m)$$

- *For all* $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\epsilon : \{1, \ldots, n\} \to K$ *non-constant, and* $Z_m \in A_{\chi(m),\epsilon(m)}$,

$$\kappa_\chi(Z_1, \ldots, Z_n) = 0.$$

Bi-Free Independence from Free Independence

Theorem (Charlesworth, Nelson, Skoufranis; 2014)

Let $(\mathcal{A}, \varphi)$ be a ncps and let $\{(A_{\ell,k}, A_{r,k})\}_{k \in K}$ be pairs of unital subalgebras of $\mathcal{A}$. Suppose

1. *for all $k_1, k_2 \in K$, $\varphi(Z_1 XYZ_2) = \varphi(Z_1 YXZ_2)$ for all $Z_1, Z_2 \in \mathcal{A}$, $X \in A_{\ell,k_1}$, and $Y \in A_{r,k_2}$, and*
2. *for each $k \in K$ and $Y \in A_{r,k}$ there exists a $X \in A_{\ell,k}$ such that $\varphi(ZY) = \varphi(ZX)$ for all $Z \in \mathcal{A}$.*

Then $\{(A_{\ell,k}, A_{r,k})\}_{k \in K}$ are bi-freely independent if and only if $\{A_{\ell,k}\}_{k \in K}$ are freely independent.

Example

If $\mathfrak{M} = \mathfrak{M}_1 *_\tau \mathfrak{M}_2$ and $\lambda, \rho : \mathfrak{M} \to \mathcal{L}(L_2(\mathfrak{M}, \tau))$ are the left and right actions of $\mathfrak{M}$, then $(\lambda(\mathfrak{M}_1), \rho(\mathfrak{M}_1))$ and $(\lambda(\mathfrak{M}_2), \rho(\mathfrak{M}_2))$ are bi-freely independent.

Types of Independence

If $(A_{\ell,1}, A_{r,1})$ and $(A_{\ell,2}, A_{r,2})$ are bi-freely independent, then

- $A_{\ell,1}$ and $A_{\ell,2}$ are freely independent,
- $A_{r,1}$ and $A_{r,2}$ are freely independent,
- $A_{\ell,1}$ and $A_{r,2}$ are classically independent,
- $A_{r,1}$ and $A_{\ell,2}$ are classically independent,
- under additional assumptions on the algebras, $\mathrm{alg}(A_{\ell,1}A_{r,1})$ and $\mathrm{alg}(A_{\ell,2}A_{r,2})$ are Boolean independent, and
- under additional assumptions on the algebras, $\mathrm{alg}(A_{\ell,1}A_{r,1})$ and $A_{\ell,2}$ are monotone independent.

Bi-freeness contains the five universal notions of independent of Speicher and Muraki.

Bi-Free Central Limit Distributions

Definition (Voiculescu; 2013)

A pair $(\{Z_i\}_{i\in I}, \{Z_j\}_{j\in J})$ is said to be a *bi-free central limit distribution* if all cumulants of order at least three vanish.

Theorem (Voiculescu; 2013)

Given non-empty disjoint sets I and J, for each matrix

$$C = [C_{k_1,k_2}]_{k_1,k_2\in I\sqcup J}$$

there exists exactly one centred bi-free central limit distribution with $\kappa_\chi(Z_{k_1}, Z_{k_2}) = C_{k_1,k_2}$. In particular, if $\mathcal{H}$ is a Hilbert space and $h, h^* : I \sqcup J \to \mathcal{H}$ are such that $C_{k_1,k_2} = \langle h(k_2), h^*(k_1)\rangle_{\mathcal{H}}$, then

$$Z_i = l(h(i)) + l^*(h^*(i)) \qquad \text{and} \qquad Z_j = r(h(j)) + r^*(h^*(j))$$

is a realization of said bi-free central limit distribution.

The R-Transform

If $(\mathcal{A}, \varphi)$ is a non-commutative probability space and $X \in \mathcal{A}$, the *Cauchy transform* of X is

$$G_X(z) = \varphi((zI_{\mathcal{A}} - X)^{-1}).$$

The *R-Transform* of X is

$$R_X(z) = \sum_{n \geq 0} \kappa_{n+1}(X) z^n.$$

It is possible to show that $G_X\left(R_X(z) + \frac{1}{z}\right) = z$.

Theorem (Voiculescu; 1985)

If X_1 and X_2 are freely independent, then $R_{X_1+X_2} = R_{X_1} + R_{X_2}$.

There exists a combinatorial approach due to Speicher.

The Bi-Free Partial R-Transform

Given $X, Y \in \mathcal{A}$, define the *two-variable Green's function* by

$$G_{X,Y}(z,w) = \varphi((zI_{\mathcal{A}} - X)^{-1}(wI_{\mathcal{A}} - Y)^{-1}) = \frac{1}{zw} + \sum_{\substack{n,m\geq 0 \\ n+m\geq 1}} \frac{\varphi(X^n Y^m)}{z^{n+1}w^{m+1}},$$

The *bi-free partial R-transform* of (X, Y) is

$$R_{X,Y}(z,w) = \sum_{\substack{n,m\geq 0 \\ n+m\geq 1}} \kappa_{n,m}(X,Y) z^n w^m.$$

If (X_1, Y_1) and (X_2, Y_2) are bi-freely independent, then $R_{X_1+X_2, Y_1+Y_2} = R_{X_1,Y_1} + R_{X_2,Y_2}$.

Theorem (Voiculescu; 2013), combinatorial proof (Skoufranis, 2014)

As holomorphic functions near $(0,0)$,

$$R_{X,Y}(z,w) = 1 + zR_X(z) + wR_Y(w) - \frac{zw}{G_{X,Y}\left(R_X(z) + \frac{1}{z}, R_Y(w) + \frac{1}{w}\right)}.$$

Bi-Free with Amalgamation

- Let B be a unital algebra.
- Let $\mathcal{X}$ be a B-B-bimodule that may be decomposed as $\mathcal{X} = B \oplus \mathcal{X}^{\perp}$.
- The projection map $p : \mathcal{X} \to B$ is given by $p(b \oplus \eta) = b$.
- Thus $p(b \cdot \xi \cdot b') = bp(\xi)b'$.
- For $b \in B$, define $L_b, R_b \in \mathcal{L}(\mathcal{X})$ by $L_b(\xi) = b \cdot \xi$ and $R_b(\xi) = \xi \cdot b$.
- Define $E : \mathcal{L}(\mathcal{X}) \to B$ by $E(T) = p(T(1_B \oplus 0))$.
- $E(L_b R_{b'} T) = p(L_b R_{b'}(E(T) \oplus \eta)) = p(bE(T)b' \oplus \eta') = bE(T)b'$.
- $E(TL_b) = p(T(b \oplus 0)) = E(TR_b)$.

B-B-Non-Commutative Probability Space

Definition

A *B-B-non-commutative probability space* is a triple $(\mathcal{A}, E, \varepsilon)$ where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$, $\varepsilon : B \otimes B^{\mathrm{op}} \to \mathcal{A}$ is a unital homomorphism such that $\varepsilon|_{B\otimes I}$ and $\varepsilon|_{I\otimes B^{\mathrm{op}}}$ are injective, and $E : \mathcal{A} \to B$ is a unital linear map such that

$$E(\varepsilon(b_1 \otimes b_2)T) = b_1 E(T) b_2 \quad \text{and} \quad E(T\varepsilon(b \otimes 1_B)) = E(T\varepsilon(1_B \otimes b)).$$

Denote $L_b = \varepsilon(b \otimes 1_B)$ and $R_b = \varepsilon(1_B \otimes b)$.

Every B-B-non-commutative probability space can be embedded into $\mathcal{L}(\mathcal{X})$ for some B-B-bimodule $\mathcal{X}$.

B-NCPS via B-B-NCPS

Definition

Let $(\mathcal{A}, E, \varepsilon)$ be a B-B-ncps. The unital subalgebras of $\mathcal{A}$ defined by

$$\mathcal{A}_\ell := \{Z \in \mathcal{A} \mid ZR_b = R_b Z \text{ for all } b \in B\} \text{ and}$$
$$\mathcal{A}_r := \{Z \in \mathcal{A} \mid ZL_b = L_b Z \text{ for all } b \in B\}$$

are called the *left* and *right algebras of* $\mathcal{A}$ respectively. A pair of algebras (A_1, A_2) is said to be a *pair of B-faces* if

$$\{L_b\}_{b \in B} \subseteq A_1 \subseteq \mathcal{A}_\ell \quad \text{and} \quad \{R_b\}_{b \in B^{\mathrm{op}}} \subseteq A_2 \subseteq \mathcal{A}_r.$$

Bi-Free Independence with Amalgamation

Definition

Let $(\mathcal{A}, E_{\mathcal{A}}, \varepsilon)$ be a B-B-ncps. Pairs of B-faces $(A_{\ell,1}, A_{r,1})$ and $(A_{\ell,2}, A_{r,2})$ of $\mathcal{A}$ are said to be *bi-freely independent with amalgamation over B* if there exist B-B-bimodules $\mathcal{X}_k$ and unital B-homomorphisms $\alpha_k : A_{\ell,k} \to \mathcal{L}(\mathcal{X}_k)_\ell$ and $\beta_k : A_{r,k} \to \mathcal{L}(\mathcal{X}_k)_r$ such that the following diagram commutes:

$$\begin{array}{ccccc}
A_{\ell,1} * A_{r,1} * A_{\ell,2} * A_{r,2} & \xrightarrow{\quad i \quad} & \mathcal{A} & \xrightarrow{\quad E_{\mathcal{A}} \quad} & B \\
\Big\downarrow {\scriptstyle \alpha_1 * \beta_1 * \alpha_2 * \beta_2} & & & & \Big\uparrow {\scriptstyle E_{\mathcal{L}(\mathcal{X}_1 * \mathcal{X}_2)}} \\
\mathcal{L}(\mathcal{X}_1)_\ell * \mathcal{L}(\mathcal{X}_1)_r * \mathcal{L}(\mathcal{X}_2)_\ell * \mathcal{L}(\mathcal{X}_2)_r & & \xrightarrow{\quad \lambda_1 * \rho_1 * \lambda_2 * \rho_2 \quad} & & \mathcal{L}(\mathcal{X}_1 * \mathcal{X}_2)
\end{array}$$

Operator-Valued Bi-Freeness and Mixed Cumulants

Theorem (Charlesworth, Nelson, Skoufranis; 2014)

Let $(\mathcal{A}, E, \varepsilon)$ be a B-B-ncps and let $\{(A_{\ell,k}, A_{r,k})\}_{k\in K}$ be pairs of B-faces. Then the following are equivalent:

- $\{(A_{\ell,k}, A_{r,k})\}_{k\in K}$ *are bi-free over B.*
- *For all* $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\epsilon : \{1, \ldots, n\} \to K$, *and* $Z_m \in A_{\chi(m),\epsilon(m)}$,

$$E(Z_1 \cdots Z_m) = \sum_{\pi\in BNC(\chi)} \left[\sum_{\substack{\sigma\in BNC(\chi) \\ \pi\leq\sigma\leq\epsilon}} \mu_{BNC}(\pi, \sigma) \right] \mathcal{E}_\pi(Z_1, \ldots, Z_m)$$

- *For all* $\chi : \{1, \ldots, n\} \to \{\ell, r\}$, $\epsilon : \{1, \ldots, n\} \to K$ *non-constant, and* $Z_m \in A_{\chi(m),\epsilon(m)}$,

$$\kappa_\chi(Z_1, \ldots, Z_n) = 0.$$

Amalgamating Over Matrices

- Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space.
- $\mathcal{M}_N(\mathcal{A})$ is naturally a $\mathcal{M}_N(\mathbb{C})$-ncps where the expectation map $\varphi_N : \mathcal{M}_N(\mathcal{A}) \to \mathcal{M}_N(\mathbb{C})$ is defined via

$$\varphi_N([A_{i,j}]) = [\varphi(A_{i,j})].$$

- If A_1, A_2 are unital subalgebras of $\mathcal{A}$ that are free with respect to φ, then $\mathcal{M}_N(A_1)$ and $\mathcal{M}_N(A_2)$ are free with amalgamation over $\mathcal{M}_N(\mathbb{C})$ with respect to φ_N.
- Is there a bi-free analogue of this result?
- Is $\mathcal{M}_N(\mathcal{A})$ a $\mathcal{M}_N(\mathbb{C})$-$\mathcal{M}_N(\mathbb{C})$-ncps?

B-B-NCPS Associated to $\mathcal{A}$

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let B be a unital algebra. Then $\mathcal{A} \otimes B$ is a B-B-bi-module where

$$L_b(a \otimes b') = a \otimes bb', \qquad \text{and} \qquad R_b(a \otimes b') = a \otimes b'b.$$

If $p : \mathcal{A} \otimes B \to B$ is defined by

$$p(a \otimes b) = \varphi(a)b,$$

then $\mathcal{L}(\mathcal{A} \otimes B)$ is a B-B-ncps with

$$E(Z) = p(Z(1_{\mathcal{A}} \otimes 1_B)).$$

If $X, Y \in \mathcal{A}$, defined $L(X \otimes b) \in \mathcal{L}(\mathcal{A} \otimes B)_\ell$ and $R(Y \otimes b) \in \mathcal{L}(\mathcal{A} \otimes B)_r$ via

$$L(X \otimes b)(a \otimes b') = Xa \otimes bb' \qquad \text{and} \qquad R(Y \otimes b)(a \otimes b') = Ya \otimes b'b.$$

Bi-Freeness Preserved Under Tensoring

Theorem (Skoufranis; 2015)

Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space and let $\{(A_{\ell,k}, A_{r,k})\}_{k \in K}$ be bi-free pairs of faces with respect to φ. If B is a unital algebra, then $\{(L(A_{\ell,k} \otimes B), R(A_{r,k} \otimes B))\}_{k \in K}$ are bi-free over B with respect to some E.

Bi-Matrix Models - q-Deformed Fock Space

Let $q \in [-1, 1]$, let $\mathcal{H}$ be a Hilbert space, let $\mathcal{F}_q(\mathcal{H})$ be the q-deformed Fock space, and, for $h \in H$, let $l_q(h), r_q(h), l_q^*(h)$, and $r_q^*(h)$ denote the left/right q-deformed creation and annihilation operators on $\mathcal{F}_q(\mathcal{H})$.

Given an index set K, an $N \in \mathbb{N}$, and an orthonormal set of vectors $\{h_{i,j}^k \mid i, j \in \{1, \ldots, N\}, k \in K\} \subseteq \mathcal{H}$, let

$$L_k(N) := \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} L(l_q(h_{i,j}^k) \otimes E_{i,j}), \quad L_k^*(N) := \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} L(l_q^*(h_{j,i}^k) \otimes E_{i,j})$$

$$R_k(N) := \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} R(r_q(h_{i,j}^k) \otimes E_{i,j}), \quad R_k^*(N) := \frac{1}{\sqrt{N}} \sum_{i,j=1}^{N} R(r_q^*(h_{j,i}^k) \otimes E_{i,j}).$$

Bi-Matrix Models - q-Deformed Fock Space

Theorem (Skoufranis; 2015)

If $E : \mathcal{L}(\mathcal{L}(\mathcal{F}_q(\mathcal{H})) \otimes \mathcal{M}_N(\mathbb{C})) \to \mathcal{M}_N(\mathbb{C})$ *is the expectation, the joint distribution of*

$$\{L_k(N), L_k^*(N), R_k(N), R_k^*(N)\}_{k\in K}$$

with respect to $\frac{1}{N}\mathrm{Tr} \circ E$ *is asymptotically equal the joint distribution of*

$$\{l_0(h^k), l_0^*(h^k), r_0(h^k), r_0^*(h^k)\}_{k\in K}$$

with respect to φ *where* $\{h^k\}_{k\in K} \subseteq \mathcal{H}$ *is an orthonormal set.*

Bi-R-Cyclic Families

Definition

Let I and J be disjoint index sets and let

$$\{[Z_{k;i,j}]\}_{k\in I} \cup \{[Z_{k;i,j}]\}_{k\in J} \subseteq \mathcal{M}_N(\mathcal{A}).$$

The pair

$$(\{[Z_{k;i,j}]\}_{k\in I}, \{[Z_{k;i,j}]\}_{k\in J})$$

is said to be *R-cyclic* if for every $n \geq 1$, $\omega : \{1, \ldots, n\} \to I \sqcup J$, and $1 \leq i_1, \ldots, i_n, j_1, \ldots, j_n \leq d$,

$$\kappa^{\mathbb{C}}_{\chi_\omega}(Z_{\omega(1);i_1,j_1}, Z_{\omega(2);i_2,j_2}, \ldots, Z_{\omega(n);i_n,j_n}) = 0$$

whenever at least one of

$$j_{s_\chi(1)} = i_{s_\chi(2)}, j_{s_\chi(2)} = i_{s_\chi(3)}, \ldots, j_{s_\chi(n-1)} = i_{s_{\chi(n)}}, j_{s_\chi(n)} = i_{s_\chi(1)}$$

fail.

Bi-R-Cyclic Families and Bi-Free over the Diagonal

For example, if K is an index set and $\{h^k_{i,j} \mid i,j \in \{1,\ldots,N\}, k \in K\} \subseteq \mathcal{H}$ is an orthonormal set of vectors, then

$$(\{[l(h^k_{i,j})],[l^*(h^k_{j,i})]\},\{[r(h^k_{i,j})],[r^*(h^k_{j,i})]\})_{k\in K}$$

is an R-cyclic family.

Theorem (Skoufranis; 2015)

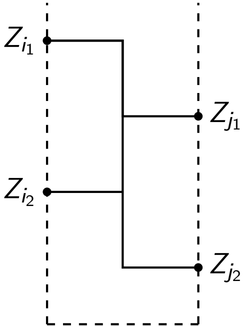
Let $(\mathcal{A},\varphi)$ be a non-commutative probability space and let

$$\{[Z_{k;i,j}]\}_{k\in I} \cup \{[Z_{k;i,j}]\}_{k\in J} \subseteq \mathcal{M}_N(\mathcal{A}).$$

Then the following are equivalent:

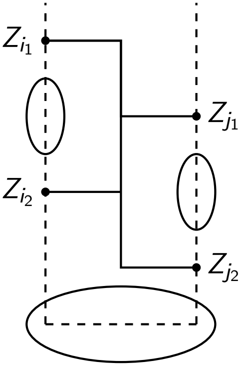
- $(\{[Z_{k;i,j}]\}_{k\in I},\{[Z_{k;i,j}]\}_{k\in J})$ *is R-cyclic.*
- $(\{L([Z_{k;i,j}])\}_{k\in I},\{R([Z_{k;i,j}])\}_{k\in J})$ *is bi-free from* $(L(\mathcal{M}_N(\mathbb{C})), R(\mathcal{M}_N(\mathbb{C})^{\mathrm{op}}))$ *with amalgamation over* $\mathcal{D}_N$ *with respect to* $F \circ E_N$ *where* $F : \mathcal{M}_N(\mathbb{C}) \to \mathcal{D}_N$ *is the expectation onto the diagonal.*

Suppose $\{Z_i\}_{i\in I} \subseteq \mathcal{A}_\ell$ and $\{Z_j\}_{j\in J} \subseteq \mathcal{A}_r$. Suppose we wanted to describe all B-valued moments involving Z_{i_1}, Z_{j_1}, Z_{i_2}, and Z_{j_2} each occurring once in that order.



Operator-Valued Bi-Free Distributions

Suppose $\{Z_i\}_{i\in I} \subseteq \mathcal{A}_\ell$ and $\{Z_j\}_{j\in J} \subseteq \mathcal{A}_r$. Suppose we wanted to describe all B-valued moments involving Z_{i_1}, Z_{j_1}, Z_{i_2}, and Z_{j_2} each occurring once in that order.



Operator-Valued Bi-Free S-Transform Formula

Theorem (Skoufranis; 2015)

If (X_1, Y_1) *and* (X_2, Y_2) *are bi-free over a unital algebra* B*, then*

$$S_{X_1 X_2, Y_1 Y_2}(b, c, d)$$

equals

$$Z_\ell S_{X_1, Y_1}\left(Z_\ell^{-1} b Z_\ell,\ Z_\ell^{-1} S_{X_2, Y_2}(b, c, d) Z_r^{-1},\ Z_r d Z_r^{-1}\right) Z_r$$

where $Z_\ell = S^\ell_{X_2}(b)$ *and* $Z_r = S^r_{Y_2}(d)$.

Thanks for Listening!