Polynomial cohomology and nilpotent groups

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NCGOA Spring Institute 2016
May 17, 2016

based on joint work with Henrik Densing Petersen

Funded by the Villum Foundation grant 7423.
A PROLOGUE ON NILPOTENT GROUPS

- Two types of groups will play a role in the talk:
  - Finitely generated discrete groups like $\mathbb{Z}$, $\text{SL}(3, \mathbb{Z})$, $\mathbb{F}_2$.
  - And connected, simply connected (csc) Lie groups like $\mathbb{R}^n$ or $H(3, \mathbb{R})$.
- Often discrete groups are lattices in Lie groups ($\mathbb{Z}^n \leq \mathbb{R}^n$).
- A particularly nice class of groups are the nilpotent ones.
- Nilpotency means that the lower central series

$$G \geq [G, G] \geq [[G, G], G] \geq \cdots$$

degenerates to 1 after a finite number of steps.
- The prime example of a csc nilpotent Lie group is the Heisenberg group:

$$H(3, \mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\}$$
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A lattice $\Gamma$ in a csc nilpotent Lie group $G$ is automatically cocompact, finitely generated, torsion free and nilpotent.

Mal’cev proved that the converse is true:

**Theorem (Mal’cev, 1956)**

If $\Gamma$ is finitely generated, torsion free and nilpotent then there exists a unique csc nilpotent Lie group $G = \Gamma \otimes \mathbb{R}$ in which $\Gamma$ sits as a cocompact lattice, called the Mal’cev completion of $\Gamma$.

A classical conjecture, originally due to Gromov, claims that if $\Gamma$ and $\Lambda$ are quasi-isometric (f.g. torsion free) nilpotent groups then their Mal’cev completions $\Gamma \otimes \mathbb{R}$ and $\Lambda \otimes \mathbb{R}$ are isomorphic.
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QUASI-ISOMETRY

- Recall that metric spaces \((X, d_X)\) and \((Y, d_Y)\) are quasi-isometric if there exists a map \(f : X \to Y\) and constants \(A, B > 0\) such that for all \(x_1, x_2 \in X\):
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  1. \[
  \frac{1}{A} d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq A d_X(x_1, x_2) + B
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- Recall that metric spaces \((X, d_X)\) and \((Y, d_Y)\) are **quasi-isometric** if there exists a map \(f : X \to Y\) and constants \(C, A, B > 0\) such that for all \(x_1, x_2 \in X\) and \(y \in Y\):
  1. \(\frac{1}{A} d_X(x_1, x_2) - B \leq d_Y(f(x_1), f(x_2)) \leq A d_X(x_1, x_2) + B\)
  2. \(d_Y(y, f(X)) := \inf_{x \in X} d_Y(y, f(x)) \leq C\)
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- \(\Gamma\) and \(\Lambda\) are quasi-isometric if they are so with respect to the word metrics arising from some/any finite generating sets.
\textbf{Theorem (Gromov, 1993 (Shalom, Sauer))}

Two amenable discrete groups $\Gamma$ and $\Lambda$ are quasi-isometric iff there exists a locally compact space $\Omega$ with cocompact, free, commuting actions of $\Gamma$ and $\Lambda$ and a Borel measure $\eta$ which is ergodic for the $\Gamma \times \Lambda$-action.

\begin{itemize}
  \item Such a space $(\Omega, \eta)$ is called a uniform measure equivalence (UME) between $\Gamma$ and $\Lambda$.
  \item One may define UME for locally compact (unimodular) groups and show that quasi-isometric nilpotent groups have UME Mal’cev completions.
  \item Thus, Gromov’s conjecture boils down to whether or not UME implies isomorphism for csc nilpotent Lie groups.
\end{itemize}
Quasi-isometry between amenable groups can be characterized in a more dynamic way:

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For this talk, two results are particularly important. Firstly:

**Theorem (Pansu, ’89)**

If $\Gamma$ and $\Lambda$ are (f.g., torsion free) quasi-isometric, nilpotent groups then the associated graded Lie algebras of their Mal’cev completions are isomorphic.

- Here the associated graded Lie algebra of a nilpotent Lie algebra $\mathfrak{g}$ is defined as $\bigoplus_i \mathfrak{g}[i]/\mathfrak{g}[i+1]$.

Secondly there is:

**Theorem (Shalom ’04, Sauer ’06)**

If $\Gamma$ and $\Lambda$ are quasi-isometric nilpotent groups then $H^n(\Gamma, \mathbb{R}) \simeq H^n(\Lambda, \mathbb{R})$ for all $n \geq 0$, and the isomorphism even respects the ring structure on $H^*(-, \mathbb{R})$.

- We aim to generalize Shalom’s result, and for that we need the notion of a polynomial.
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- We aim to generalize Shalom’s result, and for that we need the notion of a polynomial.
If $G$ is a (lcsc) group, we consider $C(G) = C(G, \mathbb{R})$ with the (left) regular representation and define for each $g \in G$ the difference operator $\partial_g : C(G) \to C(G)$ by $\partial_g (f) := g.f - f$.

Following Leibman, $f \in C(G)$ is called a polynomial if there exists a $d \in \mathbb{N}_0$ such that for all $g_1, \ldots, g_{d+1} \in G$

$$\partial_{g_1} \circ \cdots \circ \partial_{g_{d+1}} (f) = 0$$

The minimal such $d$ is called its degree.

- $\text{Pol}_d (G)$ denotes set polynomials of degree at most $d$.
- Degree 1: a homomorphism plus a constant.
  - so only interesting for non-compact groups
- When $G = \mathbb{R}$ or $\mathbb{Z}$ (or $\mathbb{R}^n$ or $\mathbb{Z}^n$) we recover the ordinary polynomials (in $n$ variables).
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If $G$ is the Heisenberg group then

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\begin{pmatrix}
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\mapsto \begin{cases}
\quad a & \text{is a degree 1 polynomial} \\
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\end{cases}
\]

Actually $\text{Pol}(G)$ is an algebra and spanned by products of these and 1.

\[\Rightarrow\] A similar description (in terms of a Mal’cev basis) holds for general csd nilpotent Lie groups.

If $\deg(\xi) \leq d$ then $\xi|_{G_{[d+1]}} = \xi(1)$.

This is analogous to the fact that homomorphisms $G \rightarrow \mathbb{R}$ factor through $G/[G, G]$. 
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$\leadsto$ A similar description (in terms of a Mal’cev basis) holds for general csc nilpotent Lie groups.

• If $\deg(\xi) \leq d$ then $\xi \upharpoonright_{G[d+1]} \equiv \xi(1)$.

• This is analogous to the fact that homomorphisms $G \to \mathbb{R}$ factor through $G/[G, G]$. 
If $G$ is the Heisenberg group then

\[
\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{cases} a & \text{is a degree 1 polynomial} \\
 b & \text{is a degree 1 polynomial} \\
 c & \text{is a degree 2 polynomial} \end{cases}
\]

Actually $\text{Pol}(G)$ is an algebra and spanned by products of these and 1.

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**Polynomial Cohomology**

- Note that the map $\partial_g : \xi \mapsto g.\xi - \xi$ makes sense for any $G$-module $E$.
- So we can define the $d'$th order invariants as
  \[ E^{G(d)} := \{ \xi \in E \mid \partial_{g_1} \circ \cdots \circ \partial_{g_d} \xi = 0 \text{ for all } g_1, \ldots, g_d \in G \} \]
- In this language, $\text{Pol}_d(G) = C(G)^{G(d+1)}$
- One may check that $(-)^{G(d)}$ is a left exact endo-functor on the category of (topological) $G$-modules.
- As such it has right derived functors — we denote them $H^n_{(d)}(G, -)$ and call them the polynomial cohomology of $G$.

**Proposition (K-Petersen)**

- $H^n_{(d)}(G, \mathbb{R}) \simeq H^n(G, \text{Pol}_{d-1}(G))$ (*linear description*)
- $H^1_{(d)}(G, \mathbb{R}) \simeq \text{Pol}_d(G)/\text{Pol}_{d-1}(G)$ (*inhomogeneous picture*)
POLYNOMIAL COHOMOLOGY

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PROPOSITION (K-PETERSEN)

- \( H_{(d)}^n(G, \mathbb{R}) \cong H^n(G, \text{Pol}_{d-1}(G)) \) \textit{(linear description)}
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\[\hookrightarrow\] blackboard justification
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Theorem (K-Petersen)

If $G$ and $H$ are UME csc nilpotent Lie groups then $H^n_{(2)}(G, \mathbb{R})$ is isomorphic to $H^n_{(2)}(H, \mathbb{R})$ for all $n \in \mathbb{N}_0$.

- Like Shalom’s theorem but with coefficients in $\text{Pol}_1(-)$; recall that $H^n_{(2)}(G, \mathbb{R}) \simeq H^n(G, \text{Pol}_1(G))$.

Theorem (K-Petersen)

If $G$ and $H$ are 2-step nilpotent groups the isomorphism $H^1_{(2)}(G, \mathbb{R}) \simeq H^1_{(2)}(H, \mathbb{R})$ (pre-)dualizes to an isomorphism $H \simeq G$.

- 2-step analogue of the fact that when both $G$ and $H$ are of the form $\mathbb{R}^k$, any isomorphism $\text{Hom}(G, \mathbb{R}) \simeq \text{Hom}(H, \mathbb{R})$ pre-dualizes to an isomorphism $H \simeq G$.

Corollary (Pansu)

If $\Gamma$ and $\Lambda$ are quasi-isometric, 2-step, nilpotent groups then their Mal’cev completions are isomorphic.
**Theorem (K-Petersen)**

If $G$ and $H$ are UME csc nilpotent Lie groups then $H^{(2)}_{(2)}(G, \mathbb{R})$ is isomorphic to $H^{(2)}_{(2)}(H, \mathbb{R})$ for all $n \in \mathbb{N}_0$.

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If $\Gamma$ and $\Lambda$ are quasi-isometric, 2-step, nilpotent groups then their Mal’cev completions are isomorphic.
**THEOREM (K-PETERSEN)**

*If G and H are UME csc nilpotent Lie groups then $H_{(2)}^{n}(G, \mathbb{R})$ is isomorphic to $H_{(2)}^{n}(H, \mathbb{R})$ for all $n \in \mathbb{N}_0$.***

- Like Shalom’s theorem but with coefficients in $\text{Pol}_1(-)$; recall that $H_{(2)}^{n}(G, \mathbb{R}) \simeq H^{n}(G, \text{Pol}_1(G))$.

**THEOREM (K-PETERSEN)**

*If G and H are 2-step nilpotent groups the isomorphism $H_{(2)}^{1}(G, \mathbb{R}) \simeq H_{(2)}^{1}(H, \mathbb{R})$ (pre-)dualizes to an isomorphism $H \simeq G$.***

- 2-step analogue of the fact that when both G and H are of the form $\mathbb{R}^k$, any isomorphism $\text{Hom}(G, \mathbb{R}) \simeq \text{Hom}(H, \mathbb{R})$ pre-dualizes to an isomorphism $H \simeq G$.

**COROLLARY (PANSU)**

*If Γ and Λ are quasi-isometric, 2-step, nilpotent groups then their Mal’cev completions are isomorphic.*
Theorem (K-Petersen)
If $G$ and $H$ are UME csc nilpotent Lie groups then $H_{(2)}^n(G, \mathbb{R})$ is isomorphic to $H_{(2)}^n(H, \mathbb{R})$ for all $n \in \mathbb{N}_0$.

- Like Shalom’s theorem but with coefficients in $\text{Pol}_1(-)$; recall that $H_{(2)}^n(G, \mathbb{R}) \simeq H^n(G, \text{Pol}_1(G))$.

Theorem (K-Petersen)
If $G$ and $H$ are 2-step nilpotent groups the isomorphism $H_{(2)}^l(G, \mathbb{R}) \simeq H_{(2)}^l(H, \mathbb{R})$ (pre-)dualizes to an isomorphism $H \simeq G$.

- 2-step analogue of the fact that when both $G$ and $H$ are of the form $\mathbb{R}^k$, any isomorphism $\text{Hom}(G, \mathbb{R}) \simeq \text{Hom}(H, \mathbb{R})$ pre-dualizes to an isomorphism $H \simeq G$.

Corollary (Pansu)
If $\Gamma$ and $\Lambda$ are quasi-isometric, 2-step, nilpotent groups then their Mal’cev completions are isomorphic.
THEOREM (K-PETERSEN)
If $G$ and $H$ are UME csc nilpotent Lie groups then $H^\big(2\big)_n(G, \mathbb{R})$ is isomorphic to $H^\big(2\big)_n(H, \mathbb{R})$ for all $n \in \mathbb{N}_0$.

- Like Shalom’s theorem but with coefficients in $\text{Pol}_1(-)$; recall that $H^\big(2\big)_n(G, \mathbb{R}) \simeq H^n(G, \text{Pol}_1(G))$.

THEOREM (K-PETERSEN)
If $G$ and $H$ are 2-step nilpotent groups the isomorphism $H^\big(2\big)_1(G, \mathbb{R}) \simeq H^\big(2\big)_1(H, \mathbb{R})$ (pre-)dualizes to an isomorphism $H \simeq G$.

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COROLLARY (PANSU)
If $\Gamma$ and $\Lambda$ are quasi-isometric, 2-step, nilpotent groups then their Mal’cev completions are isomorphic.
The key to the proof is a reciprocity type theorem à la Monod- Shalom:

- If $G$ and $H$ are UME via $(\Omega, \eta)$ then for all $G \times H$-modules $E$

  \[ H^n(G, L^2_{\text{loc}}(\Omega, E)^H) \cong H^n(H, L^2_{\text{loc}}(\Omega, E)^G) \]

- When $E = \text{Pol}_1(G)$ with trivial $H$-action then:
  - The LHS identifies with $H^n(G, \text{Pol}_1(G))$ — this is not completely immediate and uses Shalom’s property $H_T$ and the fact that $H^n(G, \text{Pol}_1(G))$ is fd.
  - Difficult part: to identify the RHS with $H^n(H, \text{Pol}_1(H))$. 
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The key to the proof is a reciprocity type theorem à la Monod- Shalom:

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In general, $H^1_{(2)}(G, \mathbb{R})$ is not a complete invariant of $G$.

But the collection $\bigoplus_d H^1_{(d)}(G, \mathbb{R}) \simeq \text{Pol}(G)$ is, if one remembers all of its structure (Hopf algebra).

Actually, in the sense of algebraic geometry, $\text{Pol}(G)$ is the Hopf algebra of regular functions on $G$, which is well-known to be a Hopf algebra which completely remembers $G$.

We are working on showing the general result that for UME, csc, nilpotent Lie groups $G$ and $H$ we have

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and that, for $n = 1$, these isomorphisms can be ‘glued together’ to a Hopf algebra isomorphism $\text{Pol}(G) \simeq \text{Pol}(H)$. 


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