Polynomial cohomology and nilpotent groups

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based on joint work with Henrik Densing Petersen

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A PROLOGUE ON NILPOTENT GROUPS

- Two types of groups will play a role in the talk:
- Finitely generated discrete groups like $\mathbb{Z}$, $\mathrm{SL}(3,\mathbb{Z})$, $\mathbb{F}_2$..
- And connected, simply connected (csc) Lie groups like $\mathbb{R}^n$ or $H(3,\mathbb{R})$.
- Often discrete groups are lattices in Lie groups ($\mathbb{Z}^n \leqslant \mathbb{R}^n$).
- A particularly nice class of groups are the nilpotent ones.
- Nilpotency means that the lower central series

$$G \geqslant \underbrace{[G,G]}_{=:G_{[2]}} \geqslant \underbrace{[[G,G],G]}_{=:G_{[3]}} \geqslant \cdots$$

  degenerates to $\mathbf{1}$ after a finite number of steps.
- The prime example of a csc nilpotent Lie group is the Heisenberg group:

$$H(3,\mathbb{R}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$$

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- A lattice Γ in a csc nilpotent Lie group G is automatically cocompact, finitely generated, torsion free and nilpotent.
- Mal'cev proved that the converse is true:

THEOREM (MAL'CEV, 1956)

If Γ is finitely generated, torsion free and nilpotent then there exists a unique csc nilpotent Lie group $G = \Gamma \otimes \mathbb{R}$ in which Γ sits as a cocompact lattice, called the Mal'cev completion of Γ.

- A classical conjecture, originally due to Gromov, claims that if Γ and Λ are quasi-isometric (f.g. torsion free) nilpotent groups then their Mal'cev completions $\Gamma \otimes \mathbb{R}$ and $\Lambda \otimes \mathbb{R}$ are isomorphic.

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QUASI-ISOMETRY

- Recall that metric spaces (X, d_X) and (Y, d_Y) are quasi-isometric if there exists a map $f : X \to Y$ and constants $A, B > 0$ such that for all $x_1, x_2 \in X$

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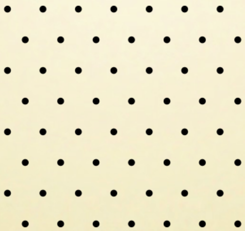

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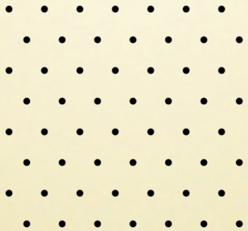

- Γ and Λ are quasi-isometric if they are so with respect to the word metrics arising from some/any finite generating sets.

- Quasi-isometry between amenable groups can be characterized in a more dynamic way:

THEOREM (GROMOV, 1993 (SHALOM, SAUER))

Two amenable discrete groups Γ *and* Λ *are quasi-isometric iff there exists a locally compact space* Ω *with cocompact, free, commuting actions of* Γ *and* Λ *and a Borel measure* η *which is ergodic for the* $\Gamma \times \Lambda$*-action.*

- Such a space (Ω, η) is called a uniform measure equivalence (UME) between Γ and Λ.
- One may define UME for locally compact (unimodular) groups and show that quasi-isometric nilpotent groups have UME Mal'cev completions.
- Thus, Gromov's conjecture boils down to whether or not UME implies isomorphism for csc nilpotent Lie groups.

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For this talk, two results are particularly important. Firstly:

THEOREM (PANSU, '89)

If Γ and Λ are (f.g., torsion free) quasi-isometric, nilpotent groups then the associated graded Lie algebras of their their Mal'cev completions are isomorphic.

- Here the associated graded Lie algebra of a nilpotent Lie algebra $\mathfrak{g}$ is defined as $\oplus_i \mathfrak{g}_{[i]}/\mathfrak{g}_{[i+1]}$.

Secondly there is:

THEOREM (SHALOM '04, SAUER '06)

If Γ and Λ are quasi-isometric nilpotent groups then $\mathrm{H}^n(\Gamma, \mathbb{R}) \simeq \mathrm{H}^n(\Lambda, \mathbb{R})$ for all $n \geqslant 0$, and the isomorphism even respects the ring structure on $\mathrm{H}^*(-, \mathbb{R})$.

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POLYNOMIALS ON GROUPS

- If G is a (lcsc) group, we consider $C(G) = C(G, \mathbb{R})$ with the (left) regular representation and define for each $g \in G$ the difference operator $\partial_g : C(G) \to C(G)$ by $\partial_g(f) := g.f - f$.
- Following Leibman, $f \in C(G)$ is called a polynomial if there exists a $d \in \mathbb{N}_0$ such that for all $g_1, \ldots, g_{d+1} \in G$

$$\partial_{g_1} \circ \cdots \circ \partial_{g_{d+1}}(f) = 0$$

  The minimal such d is called its degree.
- $\mathrm{Pol}_d(G)$ denotes set polynomials of degree at most d.
- Degree 1: a homomorphism plus a constant.

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- When $G = \mathbb{R}$ or $\mathbb{Z}$ (or $\mathbb{R}^n$ or $\mathbb{Z}^n$) we recover the ordinary polynomials (in n variables).

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- Actually $\mathrm{Pol}(G)$ is an algebra and spanned by products of these and 1.

$\rightsquigarrow$ A similar description (in terms of a Mal'cev basis) holds for general csc nilpotent Lie groups.

- If $\deg(\xi) \leqslant d$ then $\xi\restriction_{G_{[d+1]}} \equiv \xi(\mathbf{1})$.
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Polynomial cohomology

- Note that the map $\partial_g \colon \xi \mapsto g.\xi - \xi$ makes sense for *any* G-module E.
- So we can define the d'th order invariants as

$$E^{G(d)} := \{\xi \in E \mid \partial_{g_1} \circ \cdots \circ \partial_{g_d} \xi = 0 \text{ for all } g_1, \ldots, g_d \in G\}$$

- In this language, $\mathrm{Pol}_d(G) = C(G)^{G(d+1)}$
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Proposition (K-Petersen)

- $\mathrm{H}^n_{(d)}(G, \mathbb{R}) \simeq \mathrm{H}^n(G, \mathrm{Pol}_{d-1}(G))$ *(linear description)*
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If G and H are UME csc nilpotent Lie groups then $\mathrm{H}^n_{(2)}(G, \mathbb{R})$ is isomorphic to $\mathrm{H}^n_{(2)}(H, \mathbb{R})$ for all $n \in \mathbb{N}_0$.

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- 2-step analogue of the fact that when both G and H are of the form $\mathbb{R}^k$, any isomorphism $\mathrm{Hom}(G, \mathbb{R}) \simeq \mathrm{Hom}(H, \mathbb{R})$ pre-dualizes to an isomorphism $H \simeq G$.

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If Γ and Λ are quasi-isometric, 2-step, nilpotent groups then their Mal'cev completions are isomorphic.

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Corollary (Pansu)

If Γ and Λ are quasi-isometric, 2-step, nilpotent groups then their Mal'cev completions are isomorphic.

Comments on the proof

- The key to the proof is a reciprocity type theorem à la Monod- Shalom:
- If G and H are UME via (Ω, η) then for all $G \times H$-modules E

$$\mathrm{H}^n(G, L^2_{\mathrm{loc}}(\Omega, E)^H) \simeq \mathrm{H}^n(H, L^2_{\mathrm{loc}}(\Omega, E)^G)$$

- When $E = \mathrm{Pol}_1(G)$ with trivial H-action then:
- The LHS identifies with $\mathrm{H}^n(G, \mathrm{Pol}_1(G))$ — this is not completely immediate and uses Shalom's property H_T and the fact that $\mathrm{H}^n(G, \mathrm{Pol}_1(G))$ is fd.
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- In general, $\mathrm{H}^1_{(2)}(G, \mathbb{R})$ is not a complete invariant of G.
- But the collection $\oplus_d \mathrm{H}^1_{(d)}(G, \mathbb{R}) \simeq \mathrm{Pol}(G)$ is, if one remembers all of its structure (Hopf algebra).
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