Subfactors and gapped domain walls between topological phases

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Study of topological phases in terms of subfactors

Formulate physical notions of topological phases, gapped domain walls between them and composition of gapped domain walls in condensed matter physics in terms of subfactors.

$\longrightarrow$ Connections to conformal field theory

Outline of the talk:

1. Topological phases
2. Modular tensor categories and subfactors
3. Physical conjecture of Lan-Wang-Wen
4. Composition of gapped domain walls
5. Connections to conformal field theory

Topological phase

A topological phase is a certain 2-dimensional status of matters and a typical example is a thin liquid on a large plane. A point on the plane can have a special status by excitation. An excited point behaves like a particle and is called an anyon.

Suppose we have finitely many particles and exchange them. In dimension 3, the natural group representing such an exchange is the permutation group, and we have Bose-Einstein statistics or Fermi-Dirac statistics. Depending on which we have, our particle is called a boson or a fermion.

In dimension 2, the natural group for exchanges is the braid group and we have braid group statistics. Then a particle is called an anyon. Anyons have been experimentally detected.

Tensor category and braiding

A subfactor $N \subset M$ with finite index produces a set of (isomorphism classes of) irreducible N-N bimodules from decomposition of ${}_N M \otimes_N \cdots \otimes_N M_N$. If we have only finitely many such bimodules, we say that the subfactor is of finite depth and the set of such bimodules gives a tensor category which is closed under the dual operation and irreducible decomposition of a relative tensor product.

For N-N bimodules X and Y in a tensor category, $X \otimes_N Y$ and $Y \otimes_N X$ are different in general, but when they are isomorphic in some compatible way, we say that the tensor category is braided. We say it is a modular tensor category when the braiding is nondegenerate.

Examples and subfactors

The quantum group subfactors (such as the Jones-Wenzl subfactors corresponding to $SU(n)$ and with index values converging to n^2) produce modular tensor categories of bimodules.

Starting with any subfactor with finite index and finite depth, we get a new subfactor through the quantum double construction (Ocneanu, Longo-Rehren, Popa) and it gives a modular tensor category of bimodules. (Drinfeld center)

A rational chiral conformal field theory formulated in an operator algebraic framework also produces a modular tensor category of bimodules through the Doplicher-Haag-Roberts representation theory. (K-Longo-Müger)

Modular invariants

Suppose we have a modular tensor category having n irreducible bimodules. The braiding produces an n-dimensional unitary representation π of $SL(2,\mathbb{Z})$. The matrices $\pi\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\pi\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are represented as S and T, respectively.

A matrix Z is called a modular invariant if it satisfies the following, where the index 0 means the trivial bimodule (the vacuum).

(1) $Z_{\lambda\mu} \in \{0, 1, 2, \dots\}$.

(2) $Z_{00} = 1$.

(3) $ZS = SZ$, $ZT = TZ$.

They are often classified completely.

Subfactors and Q-systems

Suppose we have a subfactor $N \subset M$ with finite index and finite depth. The irreducible decomposition of ${}_N M_N$ gives $\bigoplus_i n_i X_i$ where X_i's are N-N bimodules in the tensor category the subfactor produces. Conversely, we start with a bimodule $\bigoplus_i n_i X_i$ where X_i's are N-N bimodules in some tensor category and ask when it corresponds to a subfactor.

If we start with a subfactor, then we have a multiplication map ${}_N M \otimes_N M_N \to {}_N M_N$. Conversely, an associative map from $(\bigoplus_i n_i X_i) \otimes_N (\bigoplus_i n_i X_i)$ to $\bigoplus_i n_i X_i$ essentially assures that this bimodule arises from a subfactor. The triple of $\bigoplus_i n_i X_i$, the associative map and the embedding of ${}_N N_N$ into $\bigoplus_i n_i X_i$ is called a Q-system in the tensor category. (a Frobenius algebra)

Conformal field theory and Q-systems

A chiral conformal field theory is described with a family of von Neumann algebras $\{A(I)\}$ parameterized by intervals $I \subset S^1$ subject to some axioms such as isotony and locality. We have $A(I_1) \subset A(I_2)$ if $I_1 \subset I_2$, and $[A(I_1), A(I_2)] = 0$ if $I_1 \cap I_2 = \varnothing$, for example. We can construct such a family from a nice vertex operator algebra. (Carpi-K-Longo-Weiner)

The von Neumann algebras $A(I)$ are automatically type III_1 factors and usually injective. If we have some finite index condition, then its representation theory produces a modular tensor category.

A Q-system in this modular tensor category with an extra condition called locality corresponds to an extension $B(I) \supset A(I)$ of a chiral conformal field theory bijectively. (Longo-Rehren)

Gapped domain walls

A topological phase is described with a modular tensor category where each anyon corresponds to an irreducible bimodule. The probability of a certain move of anyons is described with a colored link invariant.

Suppose we have two topological phases described with two modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$. The exterior tensor product $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{opp}}$, where "opp" means reversing the braiding, gives a new modular tensor category.

We have a physical notion of a gapped domain wall between the two topological phases and it is mathematically defined to be an irreducible local Lagrangian Q-system with the bimodule part $\bigoplus Z_{\lambda\mu}\lambda \boxtimes \bar{\mu}$ in $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{opp}}$, where $Z_{\lambda\mu} = 0, 1, 2, \dots$.

Modular invariance and Lagrangian Q-systems

A Q-system with the bimodule part $\bigoplus Z_{\lambda\mu}\lambda \boxtimes \bar\mu$ is called Lagrangian if the square of the Jones index of the corresponding subfactor is equal to the sum of the Jones indices of the irreducible bimodules in the tensor category.

An irreducible local Lagrangian Q-system with the bimodule part $\bigoplus Z_{\lambda\mu}\lambda \boxtimes \bar\mu$ is our main mathematical object now.

Let S_1, S_2 be the S-matrices of the modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, and T_1, T_2 be their T-matrices. We then have the following modular invariance property: $S_1 Z = Z S_2$, $T_1 Z = Z T_2$. This is because Davydov-Nikshych-Ostrik determined a specific form of a Q-system and then Böckenhauer-Evans applies.

Examples from α-induction construction

For a modular tensor category $\mathcal{C}$ and a (not necessarily local) Q-system in it, we have a machinery of α-induction, similar to the induction procedure in the classical representation theory. This process depends on a choice of a braiding, and we use symbols α_λ^+ and α_λ^- for the induced bimodules arising from λ.

Then this α-induction machinery produces a modular invariant through $Z_{\lambda\mu} = \langle \alpha_\lambda^+, \alpha_\mu^- \rangle$ (Böckenhauer-Evans-K) and further an irreducible local Lagrangian Q-system in $\mathcal{C} \boxtimes \mathcal{C}^{\mathrm{opp}}$ with matrix Z (Rehren). Bischoff-K-Longo showed that this construction coincides with the categorical construction of a full center of Fröhlich-Fuchs-Runkel-Schweigert.

Conjecture of Lan-Wang-Wen

In a recent paper (Phys. Rev. Lett. 2015), Lan, Wang and Wen conjectured that if we have a modular invariant matrix Z with $Z_{00} = 1$ for modular tensor categories $\mathcal{C}_1$ and $\mathcal{C}_2$, and the entries of matrix Z satisfy some inequalities about multiplicities, then there would exist a corresponding irreducible local Lagrangian Q-system.

However, subfactor theory easily disproves this conjecture. Actually, the charge conjugation matrix $(\delta_{\lambda\bar{\mu}})$ gives a counterexample for some modular tensor category $\mathcal{C}_1 = \mathcal{C}_2$ with a recent work of Davydov. (K 2015)

This is just another example of well-known phenomena in subfactor theory that compatibility of fusion rules does not imply existence of a subfactor.

Composition of gapped domain walls

In physics literature, we have a notion of composition of two gapped domain walls and its irreducible decomposition. We would like to formulate this notion mathematically.

Suppose we have three topological phases described with three modular tensor categories $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively. We further assume to have two irreducible local Lagrangian Q-systems with the bimodule parts $\bigoplus Z^1_{\lambda\mu}\lambda \boxtimes \bar{\mu}$ in $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{opp}}$ and $\bigoplus Z^2_{\mu\nu}\mu \boxtimes \bar{\nu}$ in $\mathcal{C}_2 \boxtimes \mathcal{C}_3^{\mathrm{opp}}$.

We would like to have a new Q-system with the bimodule part $\bigoplus(\sum_\mu Z^1_{\lambda\mu} Z^2_{\mu\nu})\lambda \boxtimes \bar{\nu}$. That is, the matrix part is just given by a matrix multiplication.

Locality and modular invariance of composition

We can construct a Q-system with the bimodule part $\bigoplus(\sum_\mu Z^1_{\lambda\mu}Z^2_{\mu\nu})\lambda\boxtimes\bar\nu$ by considering a tensor product functor and taking an intermediate Q-system, but this is reducible in general. We have a notion of irreducible decomposition of a Q-system corresponding to irreducible decomposition of a subfactor.

We show that after irreducible decomposition, each Q-system is local and Lagrangian. (K 2016) Being Lagrangian is shown to be equivalent to modular invariance property, which was originally conjectured by Rehren. (Müger, K-Longo)

On the matrix level, we thus have a decomposition $\sum_\mu Z^1_{\lambda\mu}Z^2_{\mu\nu}=\sum_i Z^{3,i}_{\lambda\nu}$.

Fusion rules of modular invariants

If we have $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3$, the matrices Z^1 and Z^2 are usual modular invariant matrices for $\mathcal{C}_1$. In this case, the decomposition $\sum_\mu Z^1_{\lambda\mu} Z^2_{\mu\nu} = \sum_i Z^{3,i}_{\lambda\nu}$ produces a fusion rule of modular invariants.

Such fusion rules have been already studied by Evans-Pinto and Fuchs-Runkel-Schweigert. In their setting, each irreducible local Lagrangian Q-system is known to arise from the α-induction construction of Rehren. (Bischoff-K-Longo, Kong-Runkel)

Then a so-called braided product of two (not necessarily local) Q-systems through Rehren's α-induction construction gives a fusion rule of modular invariant.

Examples

In the case of the modular tensor category $SU(2)_k$, the modular invariant matrices are labeled with the A-D-E Dynkin diagrams where the Coxeter number of each is equal to $k+2$.

Some of the fusion rules are given as follows.

$$D_{10} \otimes D_{10} = A_{17},$$
$$D_{10} \otimes E_7 = E_7 \otimes D_{10} = 2D_{10},$$
$$E_7 \otimes E_7 = D_{10} \oplus E_7.$$

In the case of Evans-Pinto and Fuchs-Runkel-Schweigert, the fusion product on the left hand side is given through the above realization of Rehren's α-induction Q-system.

Full conformal field theory and Q-systems

A $\mathbf{2}$-dimensional full conformal field theory can be also formulated with a family of von Neumann algebras. Through the restriction procedure, we have two chiral conformal field theories $\{A_L(I)\}$ and $\{A_R(I)\}$, and an extension $B(I \times J) \supset A_L(I) \otimes A_R(J)$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the modular tensor categories corresponding to $\{A_L(I)\}$ and $\{A_R(I)\}$, respectively. Then we have an irreducible local Lagrangian Q-system in $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\mathrm{opp}}$ if the extension $\{B(I \times J)\}$ has a trivial representation theory. This is a formulation of a heterotic full conformal field theory.

Our composition of such Q-systems should have an interpretation in heterotic full conformal field theory, but the meaning is not clear. (cf. Boundary conformal field theory of Bischoff-K-Longo-Rehren)

Defects and fusion

Bartels-Douglas-Henriques study chiral conformal theory within an operator algebraic framework which is a little bit different from the above and is called coordinate-free.

When we have two chiral conformal field theories in their setting, they have a notion of a defect between them, and further study the fusion product of two defects when we have three chiral conformal field theories.

Their mathematical structure is similar to ours, and we expect some direct relations between them, but it is not clear. A typical example of their defect arises from an extension of a chiral conformal field theory, so its understanding in the setting of abstract categories would help understanding of the possible relations.