An operator algebra approach to the classification of certain fusion categories III

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May 24, 2016 at Bonn

Classification of near-group categories

Let $\mathcal{C} \subset \mathrm{End}_0(M)$ be a near-group category with a finite group G.

$$[\rho^2] = \sum_{g \in G} [\alpha_g] + m[\rho].$$

Assume G is non-trivial and $m \neq 0$.

If $d = d(\rho) = \frac{m + \sqrt{m^2 + 4|G|}}{2}$ is irrational, then G is abelian and m is a multiple of $|G|$.
Moreover, the categorifications of $R(G, m)$ are completely classified by explicit polynomial equations.

A quadratic form on G appears in the polynomial equations:

$$\langle g, h \rangle = a(g)a(h)\overline{a(g+h)}, \quad a(-g) = a(g),$$

$$c^3 = \frac{1}{\sqrt{|G|}} \sum_{g \in G} \overline{a(g)}.$$

Character formula

$\alpha_g \circ \rho = \rho$.
$S_e \in (\mathrm{id}, \rho^2)$, $S_g = \alpha_g(S_e) \in (\alpha_g, \rho^2)$.
$U(g) \in (\rho, \rho\alpha_g)$, $U(g)S_e = S_e$.

$$(\rho^2, \rho^2) = \bigoplus_{g\in G} \mathbb{C} S_g S_g^* \oplus B(\mathcal{K}),$$

$$U(g) = \sum_{h\in G} \chi_h(g) S_h S_h^* + U_{\mathcal{K}}(g),$$

where $\mathcal{K} = (\rho, \rho^2)$.

$d = \frac{m+\sqrt{m^2+4|G|}}{2} \in \mathbb{R} \setminus \mathbb{Q} \Rightarrow$

$$\bigoplus_{h\in G} \chi_h \cong \lambda, \quad U_{\mathcal{K}} \cong \frac{m}{|G|}\lambda.$$

Irrational case

Lemma

$\langle g, h \rangle := \chi_h(g)$ *is a non-degenerate (in fact, symmetric) bicharacter.*

Proof.

$\alpha_h(U(g)) \in (\rho, \rho\alpha_g) \Rightarrow \alpha_h(U(g)) = {}^{\exists}c(g,h)U(g)$.

$$
\begin{aligned}
c(g,h)S_e &= c(g,h)U(g)S_e = \alpha_h(U(g))S_e \\
&= \alpha_h(U(g)S_{-h}) = \chi_{-h}(g)\alpha_h(S_{-h}) = \chi_{-h}(g)S_e,
\end{aligned}
$$

and $\chi_{-h}(g) = c(g,h)$ is a non-degenerate character. □

$U_{\mathcal{K}}(g) \in B(\mathcal{K})$ is given by $\mathcal{K} \ni T \mapsto U(g)T$.

Definition of two other representations on board.

Irrational case (continued)

Definition

Let $\mathcal{H}(G)$ be the universal C*-algebra generated by three unitary representations v_0, v_1, v_2 of G, and a unitary w of period 3 satisfying

$$v_{i+1}(g)v_i(h) = \langle h, g\rangle v_i(h)v_{i+1}(g),$$

$$w^* v_i(g) w = v_{i+1}(g),$$

where $i \in \mathbb{Z}/3\mathbb{Z}$.

We have a representation of $\mathcal{H}(G)$ in $\mathcal{K} = (\rho, \rho^2)$.

Irrational case (continued)

Lemma

$\exists 3|G|$ irreducible representations of $\mathcal{H}(G)$, realized in $\ell^2(G)$ as

$$\pi_{a,c}(v_0(g))f(h) = \langle g, h\rangle f(h),$$

$$\pi_{a,c}(v_1(g))f(h) = f(h+g),$$

$$\pi_{a,c}(v_2(g))f(h) = a(h)\overline{a(h-g)}f(h-g),$$

$$\pi_{a,c}(w)f(h) = \frac{c}{\sqrt{n}}\sum_k a(h)\overline{\langle h, k\rangle}f(k),$$

where $a : G \to \mathbb{T}$ and $c \in \mathbb{T}$ satisfy

$$a(g+h)\langle g, h\rangle = a(g)a(h),$$

$$c^3 \sum_{g\in G} a(g) = \sqrt{n}.$$

Quadratic categories with (G, τ, m)

Definition

Let G be a finite group, $\tau \in \mathrm{Aut}(G)$ be an involution, and let $m \in \mathbb{N}$.

A quadratic category of type (G, τ, m) is a fusion category $\mathcal{C}$ with $\mathrm{Irr}(\mathcal{C}) = G \sqcup \{g \otimes \rho\}_{g \in G}$, satisfying

$$[g][h] = [gh], \quad g, h \in G,$$

$$[g][\rho] = [\rho][g^\tau],$$

$$[\rho]^2 = [\mathrm{id}] \oplus m \sum_{g \in G} [g][\rho].$$

The even part of the Haagerup subfactor is a quadratic category of type $(\mathbb{Z}_3, -1, 1)$.

Asaeda-Haagerup subfactor can be constructed from a quadratic category of type $(\mathbb{Z}_4, -1, 2)$.

Obstructions

Let $\mathcal{C} \subset \mathrm{End}_0(M)$ be a quadratic category of type (G, τ, m).

$$\alpha_g \circ \alpha_h = \mathrm{Ad}^{\exists}\, U_{g,h} \circ \alpha_{gh},$$
$$\rho \circ \alpha_{g^\tau} = \mathrm{Ad}^{\exists}\, V_g \circ \alpha_g \circ \rho.$$

We seek obstructions to making $U_{g,h} = 1$ and $V_g = 1$.

A quadratic category of type "$(G, \tau, 0)$" is $\mathrm{Vec}^{\omega}_{G \rtimes_\tau \mathbb{Z}_2}$.

Recall that the E_2-term of the spectral sequence computing $H^*(G \rtimes_\tau \mathbb{Z}_2, \mathbb{T})$ is $E_2^{p,q} = H^p(\mathbb{Z}_2, H^q(G, \mathbb{T}))$.

We use this analogy to define invariants of quadratic categories of type (G, τ, m).

$E_2^{0,3}$

$\exists \omega \in Z^3(G, \mathbb{C}^\times)$ satisfying $\alpha_g(U_{h,k})U_{g,hk} = \omega(g,h,k)U_{g,h}U_{gh,k}$.

Lemma

$\exists \xi(g,h) \in \mathbb{T}$ *satisfying*

$$\omega(g,h,k) = \omega(g^\tau, h^\tau, k^\tau)\xi(h,k)\xi(gh,k)^{-1}\xi(g,hk)\xi(g,h)^{-1}.$$

In particular, $[\omega] \in H^3(G,\mathbb{T})^{\mathbb{Z}_2} = H^0(\mathbb{Z}_2, H^3(G,\mathbb{T})) = E_2^{0,3}$.

$$\rho \circ \alpha_{g^\tau} \circ \alpha_{h^\tau} = \rho \circ \mathrm{Ad}\, U_{g^\tau,h^\tau} \circ \alpha_{(gh)^\tau} = \mathrm{Ad}(\rho(U_{g,h})V_{gh}) \circ \alpha_{gh} \circ \rho,$$

$\rho \circ \alpha_{g^\tau} \circ \alpha_{h^\tau} = \mathrm{Ad}(V_g\alpha_g(V_h)) \circ \alpha_g \circ \alpha_h \circ \rho = \mathrm{Ad}(V_g\alpha_g(V_h)U_{g,h}) \circ \alpha_{gh} \circ \rho,$

$\Rightarrow \exists \xi(g,h) \in \mathbb{T}$ satisfying $V_g\alpha_g(V_h)U_{g,h} = \xi(g,h)\rho(U_{g,h})V_{gh}$.

Definition

$\mathfrak{c}^{0,3}(\mathcal{C}) := [\omega] \in H^3(G,\mathbb{T})^\tau$.

$E_2^{1,2}$

Assume $\mathfrak{c}^{0,3}(\mathcal{C}) = 0$.
We may assume α is an action, $\xi \in Z^2(G, \mathbb{T})$ and
$\rho \circ \alpha_{g^\tau} = \mathrm{Ad}\, V_g \circ \alpha_g \circ \rho$,
$V_g \alpha_g(V_h) = \xi(g,h) V_{gh}$.

Lemma

$\exists \eta(g) \in \mathbb{T}$ *satisfying*

$$\xi(g^\tau, h^\tau)\xi(g,h) = \eta(gh)\eta(g)^{-1}\eta(h)^{-1}.$$

In particular, the 2-cocycle $\xi \in Z^2(G, \mathbb{C}^\times)$ gives a class in $H^1(\mathbb{Z}_2, H^2(G, \mathbb{C}^\times)) = E_2^{1,2}$.

Using rigidity, we get $\overline{V_g} \in (\alpha_{g^\tau{}^{-1}} \circ \rho, \rho \circ \alpha_{g^{-1}})$, and $\eta(g) \in \mathbb{T}$ satisfying $\overline{V_{(g^\tau)^{-1}}} = \eta(g) V_g$.

Definition

Define $\mathfrak{c}^{1,2}(\mathcal{C}) \in H^1(\mathbb{Z}_2, H^2(G, \mathbb{T}))$ by the class given by ξ.

Condition for Cuntz algebra models

Assume further that $\mathfrak{c}^{1,2}(\mathcal{C}) = 0$.
Then we can choose V_g satisfying

$$\rho \circ \alpha_{g^\tau} = \operatorname{Ad} V_g \circ \alpha_g \circ \rho,$$

$$V_g \alpha_g(V_h) = V_{gh}.$$

Thus $\exists W \in \mathcal{U}(M)$ satisfying $V_g = W^{-1}\alpha_g(W)$, and

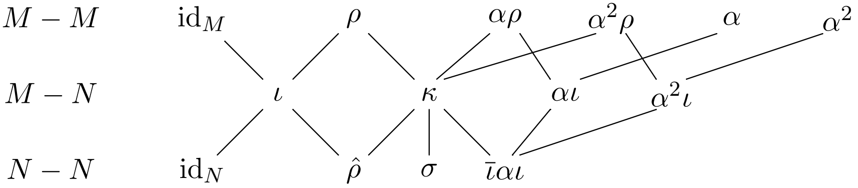
$$\operatorname{Ad} W \circ \rho \circ \alpha_{g^\tau} = \alpha_g \circ \operatorname{Ad} W \circ \rho.$$

Replacing ρ with $\operatorname{Ad} W \circ \rho$, we get

$$\rho \circ \alpha_{g^\tau} = \alpha_g \circ \rho.$$

Summary: To obtain a Cuntz algebra model for $\mathcal{C}$, we need $\mathfrak{c}^{0,3}(\mathcal{C}) = 0$ and $\mathfrak{c}^{1,2}(\mathcal{C}) = 0$.

Vanishing theorem



The Haagerup subfactor is $3^{\mathbb{Z}_3}$.

Theorem (I)

When a quadratic category $\mathcal{C}$ of type $(G,\tau,1)$ comes from a 3^G-subfactor, then $\mathfrak{c}^{0,3}(\mathcal{C}) \in H^3(G,\mathbb{T})^{\mathbb{Z}_2}$ and $\mathfrak{c}^{1,2}(\mathcal{C}) \in H^1(\mathbb{Z}_2, H^2(G,\mathbb{T}))$ vanish.

Proof.

$[\alpha_g][\kappa] = [\kappa] \Rightarrow \mathfrak{c}^{0,3}(\mathcal{C}) = 0.$
$(\kappa,\rho\kappa) \ni T \mapsto V_g\alpha_g(T) \in (\kappa,\rho\kappa)$ gives a projective representation of G with $\dim(\kappa,\rho\kappa) = |G|-1 \Rightarrow \mathfrak{c}^{1,2}(\mathcal{C}) = 0.$ □

Odd G

Theorem (I)

Let $\mathcal{C}$ be a spherical quadratic category with (G, τ, m).
If G is an odd group and m is an odd number, then G is abelian and $g^\tau = g^{-1}$ for any $g \in G$.

Let (π, V_π) be an irreducible representation of $K(\mathcal{C})$.
Then the formal codegree f_π for π is defined by

$$f_\pi = \sum_{X \in \mathrm{Irr}(\mathcal{C})} \mathrm{Tr}(\pi(X))\pi(\overline{X}).$$

Since f_π commutes with $\pi(X)$ for every $X \in \mathrm{Irr}(\mathcal{C})$, it is a scalar.

Theorem (Ostrik 2009)

If $\mathcal{C}$ is spherical, there exists a simple object in the Drinfeld center $\mathcal{Z}(\mathcal{C})$ whose dimension is $\dim \mathcal{C}/f_\pi$.
In particular, $\dim \mathcal{C}/f_\pi$ is a cyclotomic integer.

Odd G

Lemma

If G and m are odd, for any non-trivial irreducible representation π of G, π and π^τ are inequivalent.

Proof.

Suppose that π is a non-trivial irreducible representation of G with $\pi \cong \pi^\tau$. Then π extends to an irreducible representation π' of $K(\mathcal{C})$ whose formal codegree is $f_{\pi'} = 2|G|/\dim \pi$, and

$$\frac{\dim \mathcal{C}}{f_{\pi'}} = \dim \pi + \frac{m|G| \dim \pi \dim \rho}{2}.$$

This is not an algebraic integer. □

Generalized Haagerup categories

Definition

A generalized Haagerup category with a finite abelian group G is a quadratic category $\mathcal{C}$ with $(G, -1, 1)$ satisfying $\mathfrak{c}^{0,3}(\mathcal{C}) = 0$ and $\mathfrak{c}^{1,2}(\mathcal{C}) = 0$.

Caution:
(1) There exist two quadratic categories of type $(\mathbb{Z}_3, -1, 1)$ with $\mathfrak{c}^{0,3}(\mathcal{C}) \neq 0$.
(2) To construct the Asaeda-Haagerup subfactors, we need a quadratic category of type $(\mathbb{Z}_4, -1, 2)$ and $\mathfrak{c}^{0,3}(\mathcal{C}) = 0$.

Classification theorem

Theorem

Generalized Haagerup categories are completely classified by explicit polynomial equations.

More precisely, there exists a one-to-one correspondence between the equivalence classes of generalized Haagerup categories and the $H^2(G,\mathbb{T}) \rtimes \mathrm{Aut}(G)$-orbits of the gauge equivalence classes of the solutions of the polynomial equations.

For a fixed solution, the stabilizer subgroup is isomorphic to the outer automorphism group of the corresponding category.

Polynomial equations for odd G

Variables: $A(g,h) \in \mathbb{C}$ and $\eta \in \mathbb{T}$ with $\eta^3 = 1$.

$$\sum_{h \in G} A(h,0) = -\frac{\overline{\eta}}{d},$$

$$\sum_{h \in G} A(h-g,k)\overline{A(h-g',k)} = \delta_{g,g'} - \frac{\delta_{k,0}}{d},$$

$$A(k,h) = \overline{A(h,k)},$$

$$A(h,k) = A(-k,h-k)\eta = A(k-h,-h)\overline{\eta},$$

$$\sum_{l \in G} A(x+y,l)A(-x,l+p)A(-y,l+q)$$

$$= A(p+x,q+x+y)A(q+y,p+x+y) - \frac{\delta_{x,0}\delta_{y,0}}{d}.$$

Solutions for the polynomial equations

G	# (sols$/H^2(G,\mathbb{T}) \times \mathrm{Aut}(G)$)	With Q-system for $\mathrm{id} \oplus \rho$
$\mathbb{Z}_2$	1	1
$\mathbb{Z}_3$	2	1
$\mathbb{Z}_4$	2	1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	1	1
$\mathbb{Z}_5$	2	1
$\mathbb{Z}_6$	4	2
$\mathbb{Z}_7$	≥ 2	1
$\mathbb{Z}_8$	≥ 1	≥ 1
$\mathbb{Z}_4 \times \mathbb{Z}_2$	≥ 1	≥ 1
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	?	?
$\mathbb{Z}_9$	≥ 2	2
$\mathbb{Z}_3 \times \mathbb{Z}_3$	?	0
$\mathbb{Z}_{10}$	?	?
$\mathbb{Z}_{11}$	≥ 2	2

From generalized Haagerup to near-group

Theorem

Let $\mathcal{C} \subset \mathrm{End}_0(M)$ *be a quadratic category of type* $(G, -1, m)$ *with an odd abelian* G *and* $\mathfrak{c}^{0,3}(\mathcal{C}) = [\omega]$.
Let

$$\iota : M \otimes M^{\mathrm{opp}} \hookrightarrow (M \otimes M^{\mathrm{opp}}) \rtimes_{\alpha \otimes \alpha^{\mathrm{opp}}} G.$$

Then $\iota \circ (\rho \otimes \mathrm{id}) \circ \bar{\iota}$ *generates a near-group category with group* $\mathrm{Irr}(D^{\omega}(G))$ *and multiplicity* $m|G|^2$.

From $G = \mathbb{Z}_3$ and $m = 1$, we get a near group categories for $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9$ with multiplicity 9.

Orbifold (de-equivariantization) of near-group categories I

Theorem

Let $\mathcal{C} \subset \mathrm{End}(M)$ be a near-group category with $m = |G|$.
If H is Lagrangian, i.e. $H = H^{\perp}$ and $a|_H = 1$, then de-equivariantization of $\mathcal{C}$ by H is a quadratic category of type $(G/H, -1, 1)$.

There is a unique near-group category for $\mathbb{Z}_3 \times \mathbb{Z}_3$ with $m = 9$.
It has two Lagragians, giving the Haagerup category and Grossman-Snyder category.

There are two near-group categories for $\mathbb{Z}_9$.
They have Lagrangian $\mathbb{Z}_3$, giving $\mathcal{C}$ with non-trivial $\mathfrak{c}^{0,3}(\mathcal{C})$.

Corollary

There exist exactly 4 quadratic categories type $(\mathbb{Z}_3, -1, 1)$.

Twisted orbifold (de-equivariantization) of near-group categories

Theorem

Let $\mathcal{C} \subset \mathrm{End}(M)$ be a near-group category with $m = |G|$.
Assume $G = K \times H$ and $H = \mathbb{Z}_2^{2l}$.
Assume $\exists \omega \in Z^2(H, \mathbb{T})$ such that $\langle h_1, h_2 \rangle = \omega(h_1, h_2)\overline{\omega(h_2, h_1)}$ is non-degenerate on H.
Then ω-twisted de-equivariantization of $\mathcal{C}$ by H is a near-group category with group K and multiplicity $2^l|K|$.

There are two solutions for $K = \mathbb{Z}_3$, $H = \mathbb{Z}_2 \times \mathbb{Z}_2$ satisfying the above conditions, and they produce 2 near-group categories of $\mathbb{Z}_3$ with multiplicity 6.