Strong solidity and classification of free Araki-Woods factors

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Plan of the talk

1. **Introduction to free AW factors**
   (review of D. Shlyakhtenko's work from the 90s)

2. **Strong solidity of free AW factors**
   (joint with R. Boutonnet and S. Vaes)
   `arXiv:1512.04820`

3. **Classification of non almost periodic free AW factors**
   (joint with D. Shlyakhtenko and S. Vaes)
   `arXiv:1605.06057`

Introduction to free Araki-Woods factors

Voiculescu's free Gaussian functor

Let $H_{\mathbf{R}}$ be any separable real Hilbert space and write

$$H = H_{\mathbf{R}} \otimes_{\mathbf{R}} \mathbf{C} = H_{\mathbf{R}} \oplus \mathrm{i} H_{\mathbf{R}}$$

Define the **full Fock space** of H by

$$\mathcal{F}(H) := \mathbf{C}\Omega \oplus \bigoplus_{k \geq 1} H^{\otimes k}$$

Define the **left creation operators** $\ell(\xi) : \mathcal{F}(H) \to \mathcal{F}(H)$ by

$$\begin{cases} \ell(\xi)\Omega = \xi \\ \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_k) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_k \end{cases}$$

Voiculescu's free Gaussian functor

Theorem (Voiculescu 1985)

The $\mathbf{R}$*-linear mapping*

$$H_{\mathbf{R}} \to \mathbf{B}(\mathcal{F}(H)) : \xi \mapsto W(\xi) := \ell(\xi) + \ell(\xi)^*$$

satisfies the following properties:

1. *For any* $\xi \in H_{\mathbf{R}}$*, the distribution of* $W(\xi)$ *wrt to the state* $\langle \cdot \Omega, \Omega \rangle$ *is the* **semicircle law** *on* $[-2\|\xi\|, 2\|\xi\|]$*.*
2. *For any* $\xi \perp \eta \in H_{\mathbf{R}}$*,* $W(\xi)$ *and* $W(\eta)$ *are* $*$-**free** *wrt to the state* $\langle \cdot \Omega, \Omega \rangle$*.*

Therefore we have

$$\{W(\xi) : \xi \in H_{\mathbf{R}}\}'' \cong \mathrm{L}(\mathbf{F}_{\dim H_{\mathbf{R}}})$$

Changing the isometric embedding $H_{\mathbf{R}} \to H$

Let $U : \mathbf{R} \curvearrowright H_{\mathbf{R}}$ be any orthogonal representation and still denote by $U : \mathbf{R} \curvearrowright H$ the corresponding unitary representation. Write

$$U_t = A^{\mathrm{i}t} \quad \text{where } A : H \to H \text{ is the infinitesimal generator}$$

Then the **R**-linear mapping

$$H_{\mathbf{R}} \to H : \xi \mapsto \left(\frac{2}{A^{-1} + 1}\right)^{1/2} \xi$$

define a new isometric embedding whose image $K_{\mathbf{R}}$ satisfies

$$K_{\mathbf{R}} \cap \mathrm{i}K_{\mathbf{R}} = \{0\} \quad \text{and} \quad K_{\mathbf{R}} + \mathrm{i}K_{\mathbf{R}} \text{ is dense in } H$$

Shlyakhtenko's free Araki-Woods factors

Definition (Shlyakhtenko 1996)

Let $U : \mathbf{R} \curvearrowright H_{\mathbf{R}}$ and write $K_{\mathbf{R}} = (\frac{2}{A^{-1}+1})^{1/2} H_{\mathbf{R}}$ where $U_t = A^{\mathrm{i}t}$.
The **free Araki-Woods von Neumann algebra** is defined by

$$\Gamma(H_{\mathbf{R}}, U)'' := \{W(\eta) : \eta \in K_{\mathbf{R}}\}''$$

where $W(\eta) := \ell(\eta) + \ell(\eta)^*$ for $\eta \in K_{\mathbf{R}}$.

The **free quasi-free** state is faithful and normal on $\Gamma(H_{\mathbf{R}}, U)''$

$$\varphi_U := \langle \cdot \Omega, \Omega \rangle$$

The modular automorphism group σ^{φ_U} of the state φ_U is given by **free Bogoljubov transformations**

$$\sigma_t^{\varphi_U}(W(\eta)) = W(U_t \eta)$$

Functorial property of $\Gamma(H_{\mathbf{R}}, U)''$

Any representation $U : \mathbf{R} \curvearrowright H_{\mathbf{R}}$ decomposes uniquely as

$$U = U^{\mathrm{ap}} \oplus U^{\mathrm{wm}}$$

where

- $U^{\mathrm{ap}} : \mathbf{R} \curvearrowright H_{\mathbf{R}}^{\mathrm{ap}}$ is **almost periodic** (i.e. $\bigoplus$ fin. dim. rep.)
- $U^{\mathrm{wm}} : \mathbf{R} \curvearrowright H_{\mathbf{R}}^{\mathrm{wm}}$ is **weakly mixing** (i.e. $U_t \to 0$ weakly)

Shlyakhtenko's Γ functor gives the **free product splitting**

$$\Gamma(H_{\mathbf{R}}, U)'' = \Gamma(H_{\mathbf{R}}^{\mathrm{ap}}, U^{\mathrm{ap}})'' * \Gamma(H_{\mathbf{R}}^{\mathrm{wm}}, U^{\mathrm{wm}})''$$
$$\varphi_U = \varphi_{U^{\mathrm{ap}}} * \varphi_{U^{\mathrm{wm}}}$$

Type classification and computation of Connes invariants

Theorem (Shlyakhtenko 1996-1998)

Let $U : \mathbf{R} \curvearrowright H_{\mathbf{R}}$ be an orthogonal representation with $\dim H_{\mathbf{R}} \geq 2$. Then $M := \Gamma(H_{\mathbf{R}}, U)''$ is a **full factor** *such that:*

1. *M is of type II_1 iff $U = \mathbf{1}_{H_{\mathbf{R}}}$: $M \cong \mathrm{L}(\mathbf{F}_{\dim H_{\mathbf{R}}})$.*
2. *M is of type III_λ, $0 < \lambda < 1$, iff U is $\frac{2\pi}{|\log(\lambda)|}$-periodic.*
3. *M is of type III_1 otherwise.*
4. *U is almost periodic iff M is almost periodic: $\mathrm{Sd}(M) \subset \mathbf{R}^*_+$ is the subgroup generated by the eigenvalues of U.*
5. *$\tau(M)$ is the weakest topology on $\mathbf{R}$ that makes the map $U : \mathbf{R} \to \mathcal{O}(H_{\mathbf{R}})$ $*$-strongly continuous.*

- If $\dim H_{\mathbf{R}}^{\mathrm{ap}} = 0$, $M^{\varphi_U} = \mathbf{C}1$
- If $\dim H_{\mathbf{R}}^{\mathrm{ap}} = 1$, $M^{\varphi_U} = \mathrm{L}(\mathbf{Z})$
- If $\dim H_{\mathbf{R}}^{\mathrm{ap}} \geq 2$, M^{φ_U} is a free group factor (Dykema 1996).

Complete classification of almost periodic free AW factors

Theorem (Shlyakhtenko 1996)

Assume that $U : \mathbf{R} \curvearrowright H_{\mathbf{R}}$ *is not trivial and* **almost periodic**

$$U_t = \mathbf{1}_{\ell^2_{\mathbf{R}}(k)} \oplus \bigoplus_n \begin{pmatrix} \cos(\log(\lambda_n)t) & -\sin(\log(\lambda_n)t) \\ \sin(\log(\lambda_n)t) & \cos(\log(\lambda_n)t) \end{pmatrix}$$

Then $\Gamma(H_{\mathbf{R}}, U)''$ *is exactly classified, up to isomorphism, by the countable subgroup* Λ *of* $(\mathbf{R}, +)$ *generated by* $(\log \lambda_n)_n$.

Moreover, there is a state preserving isomorphism

$$(\Gamma(H_{\mathbf{R}}, U)'', \varphi_U) \cong (\mathbf{B}(\ell^2), \psi_{\exp(\Lambda)}) * (\mathrm{L}(\mathbf{Z}), \tau_{\mathbf{Z}})$$

The classification problem of free Araki-Woods factors beyond the almost periodic case has been wide open since then.

To be continued...

Strong solidity of free Araki-Woods factors

(joint with R. Boutonnet and S. Vaes)

arXiv:1512.04820

A family theorem

Ozawa-Popa (2007) showed that the free group factors $\mathrm{L}(\mathbf{F}_n)$ are strongly solid in the following sense.

Definition

A von Neumann algebra M is **strongly solid** if for any diffuse amenable subalgebra $Q \subset M$ with expectation, the normalizer $\mathcal{N}_M(Q)''$ remains amenable.

Our main result is:

Theorem (BHV 2015)

All free Araki-Woods factors are strongly solid.

Our result provides the first class of strongly solid type III factors.

Corollary (H-Ricard 2010)

Free Araki-Woods factors have no Cartan subalgebra.

(Stable) normalizers

An inclusion of von Neumann algebras $Q \subset M$ is **with expectation** if there is a faithful normal conditional expectation $\mathrm{E}_Q : M \to Q$.

Definition (Stable normalizers)

Denote by $\mathcal{N}_M(Q)''$ the **normalizer** of Q inside M where

$$\mathcal{N}_M(Q) = \{u \in \mathcal{U}(M) : uQu^* = Q\}$$

Denote by $s\mathcal{N}_M(Q)''$ the **stable normalizer** of Q inside M where

$$s\mathcal{N}_M(Q) = \{x \in M : xQx^* \subset Q \quad \text{and} \quad x^*Qx \subset Q\}$$

Stable normalizers behave well under taking corners:

$$s\mathcal{N}_{qMq}(qQq)'' = q(s\mathcal{N}_M(Q)'')q \quad \text{for every } q \in \mathcal{P}(Q)$$

This is no longer true for usual normalizers.

Close encounters of the third kind

Thanks to the work of Connes-Tomita-Takesaki from the 70s, to any type III von Neumann algebra M, one associates the triple

$$(\mathrm{c}(M), \theta, \mathrm{Tr})$$

called the **noncommutative flow of weights**:

- $\mathrm{c}(M)$ is a semifinite von Neumann algebra with trace Tr
- For every faithful state $\varphi \in M_*$, we have $\mathrm{c}(M) = M \rtimes_{\sigma^\varphi} \mathbf{R}$

$$\mathrm{L}_\varphi(\mathbf{R}) := \lambda_\varphi(\mathbf{R})'' \subset \mathrm{c}(M) \quad \text{and} \quad \sigma_t^\varphi(x) = \lambda_\varphi(t) x \lambda_\varphi(t)^*$$

- For every subalgebra $P \subset M$ with expectation, we have $\mathrm{c}(P) \subset \mathrm{c}(M)$ with Tr-preserving conditional expectation
- For every $s \in \mathbf{R}$, $\mathrm{Tr} \circ \theta_s = \exp(-s) \, \mathrm{Tr}$
- $\mathrm{c}(M) \rtimes_\theta \mathbf{R} \cong M \mathbin{\overline{\otimes}} \mathbf{B}(\mathrm{L}^2(\mathbf{R})) \cong M$

Free AW factors have the CMAP

CMAP = Complete Metric Approximation Property

Theorem (H-Ricard 2010)

All free Araki-Woods factors $M = \Gamma(H_{\mathbf{R}}, U)''$ have the **CMAP***:*

There is a sequence of normal finite rank completely bounded maps $\varphi_n : M \to M$ such that

$$\varphi_n \to \mathrm{id}_M \ \sigma\text{-strongly pointwise} \quad \text{and} \quad \lim_n \|\varphi_n\|_{\mathrm{cb}} = 1$$

The sequence $(\varphi_n)_n$ is constructed explicitly using **Wick calculus**.

By general theory, the continuous core $\mathrm{c}(M) = M \rtimes_{\sigma^\varphi} \mathbf{R}$ of any free Araki-Woods factor also has the CMAP.

Partial action of the stable normalizer

Let $A \subset (M, \tau)$ be any inclusion of tracial von Neumann algebras. Denote by $\mathrm{ctr}_A : A \to \mathcal{Z}(A)$ the center-valued trace.

For every $x \in s\mathcal{N}_M(A)$, denote by $z_x^\ell \in \mathcal{Z}(A)$ (resp. $z_x^r \in \mathcal{Z}(A)$) the support projection of $\mathrm{ctr}_A(xx^*)$ (resp. $\mathrm{ctr}_A(x^*x)$).

Denote by $\alpha_x : \mathcal{Z}(A)z_x^r \to \mathcal{Z}(A)z_x^\ell$ the unique isomorphism determined by $xa = \alpha_x(a)x$ for all $a \in \mathcal{Z}(A)z_x^r$.

The main difficulty is that α_x **may not be trace preserving** and may even be of type III!

Let Δ_x be the Radon-Nikodym derivative between τ and $\tau \circ \alpha_x$. Observe that $\Delta_x = \mathrm{ctr}_A(xx^*)\alpha_x(\mathrm{ctr}_A(x^*x)^{-1})$ is affiliated with $\mathcal{Z}(A)z_x^\ell$.

Weak compactness of stable normalizers

Define $s\mathcal{N}_M(A)^\circ$ to be the subset of all the elements $x \in s\mathcal{N}_M(A)$ for which there exists $\delta > 0$ such that

$$\mathrm{ctr}_A(x^*x) \geq \delta z_x^r \quad \text{and} \quad \mathrm{ctr}_A(xx^*) \geq \delta z_x^\ell$$

We have that $(s\mathcal{N}_M(A)^\circ)'' = s\mathcal{N}_M(A)''$.

Our key technical result extends Ozawa-Popa's **weak compactness** criterion for normalizers to stable normalizers.

Theorem (BHV 2015)

Let (M, τ) be any tracial von Neumann algebra with the CMAP and $A \subset M$ any amenable subalgebra.
Then there exists a net of vectors $\xi_n \in \mathrm{L}^2(A \mathbin{\overline{\otimes}} A^{\mathrm{op}})_+$ such that

1. $\lim_n \|(a \otimes 1^{\mathrm{op}})\xi_n - (1 \otimes a^{\mathrm{op}})\xi_n\|_2 = 0$ *for all* $a \in A$
2. $\lim_n \|(x \otimes 1^{\mathrm{op}})\xi_n(x^* \Delta_x^{1/2} \otimes 1^{\mathrm{op}}) - (1 \otimes x^{\mathrm{op}})\xi_n(1 \otimes (x^*)^{\mathrm{op}})\|_2 = 0$ *for all* $x \in s\mathcal{N}_M(A)^\circ$
3. $\lim_n \langle (x \otimes 1^{\mathrm{op}})\xi_n, \xi_n \rangle = \tau(x)$ *for all* $x \in M$

Strategy for proving strong solidity of free AW factors

Let $M = \Gamma(H_{\mathbf{R}}, U)''$ be any free Araki-Woods factor. Let $Q \subset M$ be any diffuse amenable subalgebra with expectation. We have to show that $\mathcal{N}_M(Q)''$ is amenable.

Since M is **solid**, $Q \vee Q' \cap M$ is also diffuse amenable with expectation. Thus, up to replacing Q by $Q \vee Q' \cap M$, we may assume that $Q' \cap M = \mathcal{Z}(Q)$.

Then the inclusion $\mathrm{c}(Q) \subset \mathrm{c}(M)$ is canonical and we have

$$\mathrm{c}(\mathcal{N}_M(Q)'') \subset \mathcal{N}_{\mathrm{c}(M)}(\mathrm{c}(Q))''$$

For every finite projection $q \in \mathrm{c}(Q)$, we have

$$q\left(\mathcal{N}_{\mathrm{c}(M)}(\mathrm{c}(Q))''\right) q \subset \left(s\mathcal{N}_{q\mathrm{c}(M)q}(q\mathrm{c}(Q)q)\right)''$$

We show that $\left(s\mathcal{N}_{q\mathrm{c}(M)q}(q\mathrm{c}(Q)q)\right)''$ is amenable using our weak compactness criterion with Popa's **deformation/rigidity** theory.

Classification of a family of non almost periodic free Araki-Woods factors

(joint with D. Shlyakhtenko and S. Vaes)

arXiv:1605.06057

Isomorphism preserving the free quasi-free states

If $U = U^{\mathrm{ap}} \oplus U^{\mathrm{wm}}$ and $A = A^{\mathrm{ap}} \oplus A^{\mathrm{wm}}$, denote by Λ_a the subgroup of $(\mathbf{R}, +)$ generated by the eigenvalues of $\log A^{\mathrm{ap}}$.

Our main result is:

Theorem (HSV 2016)

Let $U : \mathbf{R} \curvearrowright H_{\mathbf{R}}$ and $V : \mathbf{R} \curvearrowright L_{\mathbf{R}}$ be any representations. Assume that $U^{\mathrm{ap}} \neq 0$ and $\Lambda_a \neq 0$.

If $\Gamma(H_{\mathbf{R}}, U)'' \cong \Gamma(L_{\mathbf{R}}, V)''$, then there exists an isomorphism preserving the free quasi-free states

$$(\Gamma(H_{\mathbf{R}}, U)'', \varphi_U) \cong (\Gamma(L_{\mathbf{R}}, V)'', \varphi_V)$$

The assumption that $U^{\mathrm{ap}} \neq 0$ and $\Lambda_a \neq 0$ ensures in particular that M^{φ_U} is a nonamenable II_1 factor.

Joint measure class as an invariant

Let $U : \mathbf{R} \curvearrowright H_{\mathbf{R}}$ and write $U_t = A^{\mathrm{i}t}$ where A is the generator.

Choose a finite symmetric Borel measure μ on $\mathbf{R}$ such that the measure class $\mathcal{C}(\mu)$ coincides with the spectral measure of $\log A$.

Define the **joint measure class**

$$\mathcal{C}(\bigvee \mu^{*k}) := \bigcap_{k \geq 1} \mathcal{C}(\mu^{*k})$$

We have

$$\mathcal{C}(\bigvee \mu^{*k}) = \mathcal{C}(\log \Delta_{\varphi_U})$$

Corollary (HSV 2016)

The joint measure class $\mathcal{C}(\bigvee \mu^{*k})$ is an isomorphism invariant for the family of free Araki-Woods factors $\Gamma(H_{\mathbf{R}}, U)''$ satisfying $U^{\mathrm{ap}} \neq 0$ and $\dim H_{\mathbf{R}}^{\mathrm{ap}} \geq 2$.

Definition of the set $\mathcal{S}_{\mathrm{nap}}$

Let $U : \mathbf{R} \curvearrowright H_{\mathbf{R}}$ and write $U = U^{\mathrm{ap}} \oplus U^{\mathrm{wm}}$ and $A = A^{\mathrm{ap}} \oplus A^{\mathrm{wm}}$. Denote by

- Λ_a the subgroup generated by the eigenvalues of $\log A^{\mathrm{ap}}$ and by δ_{Λ_a} a finite atomic measure on $\mathbf{R}$ with Λ_a as set of atoms.
- μ_c a **continuous** finite symmetric Borel measure on $\mathbf{R}$ such that $\mathcal{C}(\mu_c)$ is the spectral measure of $\log A^{\mathrm{wm}}$.

Definition ($\mathcal{S}_{\mathrm{nap}}$)

Denote by $\mathcal{S}_{\mathrm{nap}}$ the set of all representations $U : \mathbf{R} \curvearrowright H_{\mathbf{R}}$ such that $U^{\mathrm{ap}} \neq 0$, $\Lambda_a \neq \{0\}$ and $\mu_c * \mu_c \prec \mu_c$.

Example

Let $0 < \lambda < 1$. Then

$$U = \mathrm{L}^2_{\mathbf{R}}(\mathbf{R}) \oplus \begin{pmatrix} \cos(\log(\lambda)t) & -\sin(\log(\lambda)t) \\ \sin(\log(\lambda)t) & \cos(\log(\lambda)t) \end{pmatrix} \in \mathcal{S}_{\mathrm{nap}}$$

Complete classification of $\Gamma(H_{\mathbf{R}}, U)''$ for $U \in \mathcal{S}_{\mathrm{nap}}$

Theorem (HSV 2016)

The set of free Araki-Woods factors

$$\left\{\Gamma(H_{\mathbf{R}}, U)'' : U \in \mathcal{S}_{\mathrm{nap}}\right\}$$

is exactly classified, up to isomorphism, by the subgroup Λ_a and the measure class $\mathcal{C}(\mu_c * \delta_{\Lambda_a})$.

- The **non-isomorphism** part follows from our previous corollary and uses the assumption that $\mu_c * \mu_c \prec \mu_c$ and $\dim H_{\mathbf{R}}^{\mathrm{ap}} \geq 2$.
- The **isomorphism** part relies on Shlyakhtenko's matrix models and uses the assumption that $U^{\mathrm{ap}} \neq 0$ and $\Lambda_a \neq \{0\}$.

The class $\mathcal{S}_{\mathrm{nap}}$ is very large and provides many non-isomorphic free Araki-Woods factors having the same Connes invariants.

Stable inner conjugacy of faithful normal states

Our key technical result is a deformation/rigidity criterion describing when two normal states are **stably inner conjugate**.

Theorem (HSV 2016)

Let $P \subset M$ be any inclusion of von Neumann algebras with expectation $\mathrm{E}_P : M \to P$ and $\psi \in M_*$ any faithful state.

The following conditions are equivalent:

1. *We have $\mathrm{L}_\psi(\mathbf{R}) \preceq_{\mathrm{c}(M)} \mathrm{c}(P)$ in the sense of Popa's intertwining-by-bimodules.*
2. *There exist a faithful positive functional $\varphi \in P_*$ and a nonzero partial isometry $v \in M$ such that $p = v^*v \in M^{\varphi \circ \mathrm{E}_P}$, $q = vv^* \in M^\psi$ and*

$$\psi q = v(\varphi \circ \mathrm{E}_P)v^*$$

Stable inner conjugacy of faithful normal states

Corollary (HSV 2016)

Let M be any von Neumann algebra and $\varphi, \psi \in M_*$ any faithful states. The following conditions are equivalent:

1. *We have $\mathrm{L}_\psi(\mathbf{R}) \preceq_{\mathrm{c}(M)} \mathrm{L}_\varphi(\mathbf{R})$ in the sense of Popa's intertwining-by-bimodules.*
2. *There exist $\lambda > 0$ and a nonzero partial isometry $v \in M$ such that $p = v^*v \in M^\varphi$, $q = vv^* \in M^\psi$ and*

$$\psi q = \lambda \, v\varphi v^*$$

If the above conditions hold, we have that

$$\mathrm{Ad}(v) : (pMp, \varphi_p) \to (qMq, \psi_q)$$

is a state preserving isomorphism.

Free AW factors with amenable centralizers

Definition

We say that a von Neumann algebra M has all its **centralizers amenable** if M^ψ is amenable for every faithful state $\psi \in M_*$.

Corollary (HSV 2016)

Let $U : \mathbf{R} \curvearrowright H_{\mathbf{R}}$ be any representation. Write $U = U^{\mathrm{ap}} \oplus U^{\mathrm{wm}}$. Then $\Gamma(H_{\mathbf{R}}, U)''$ has all its centralizers amenable iff $\dim H_{\mathbf{R}}^{\mathrm{ap}} \leq 1$.

The above corollary was previously known only when U^{wm} is moreover **mixing** (H 2008).

All free Araki-Woods factors have **trivial bicentralizer** (H 2008). When $\dim H_{\mathbf{R}}^{\mathrm{ap}} \leq 1$ and $M = \Gamma(H_{\mathbf{R}}, U)''$, by Haagerup's result, there exists a faithful state $\psi \in M_*$ such that

$$M^\psi \cong R \quad \text{and} \quad (M^\psi)' \cap M = \mathbf{C}1$$