Von Neumann's Problem and Extensions of Non-Amenable Equivalence Relations

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Outline

Topics

1. Probability measure preserving (pmp) group actions up to orbit equivalence
2. Measure theoretic versions of von Neumann's problem

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2. Measure theoretic versions of von Neumann's problem

$\rightsquigarrow$ Results of joint work with Lewis Bowen and Adrian Ioana

Preliminaries

Notation

- (X, μ) – a standard probability space
- G – a lcsc unimodular group (if countable, write Γ)
- $G \curvearrowright X$ – a group action s.t.
  1. the map $\begin{cases} G \times X \to X \\ (g, x) \mapsto gx \end{cases}$ is Borel, and
  2. $G \curvearrowright X$ is **probability measure preserving (pmp)**, i.e.

$$\mu = \mu \circ g \text{ for all } g \in G$$

Preliminaries

Definitions

A pmp action $G \curvearrowright X$ is

- **ergodic** if for measurable $E \subset X$

$$[\mu(gE \setminus E) = 0 \ \forall \ g \in G] \implies \mu(E) \in \{0, 1\}$$

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Definition

The **orbit equivalence relation** $\mathcal{R}(G \curvearrowright X)$ is
$$(x, y) \in \mathcal{R}(G \curvearrowright X) \iff y = gx \text{ for some } g \in G$$

Orbit Equivalence

Definition

$G \curvearrowright (X, \mu)$ and $H \curvearrowright (Y, \nu)$ are **orbit equivalent (OE)** if $\exists$ measure space isomorphism $\Theta : X \to Y$ with

$$(x, x') \in \mathcal{R}(G \curvearrowright X) \iff (\Theta(x), \Theta(x')) \in \mathcal{R}(H \curvearrowright Y)$$

Basic Question

How many OE-distinct actions does a given G have?

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Basic Question

How many OE-distinct actions does a given G have?

Theorem (Dye '59, Ornstein-Weiss '80, Connes-Feldman-Weiss '81)

1. All ergodic actions of countable amenable Γ are OE.
2. All properly ergodic actions of amenable non-discrete G are OE.

Actions of non-amenable groups

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Question (Von Neumann's Problem)

Do all countable non-amenable Γ contain $\mathbb{F}_2$ as a subgroup?

Theorem (Ol'shanskii '80)

No.

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Theorem (Gaboriau-Lyons '07)

Any non-amenable Γ has $\mathbb{F}_2$ as a "measurable subgroup," i.e. $\exists$ free ergodic actions $\Gamma \curvearrowright (X, \mu)$, $\mathbb{F}_2 \curvearrowright (X, \mu)$ with

$$\mathcal{R}(\mathbb{F}_2 \curvearrowright X) \subset \mathcal{R}(\Gamma \curvearrowright X)$$

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G non-amenable	Bowen-H-Ioana '15

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Measured Equivalence Relations

Notation

- $\mathcal{R} \subset X \times X$ – a measurable equivalence relation on X
- $[\mathcal{R}] = \{g \in \text{Aut}(X) \mid (x, gx) \in \mathcal{R} \text{ a.e. } x \in X\}$
- $[x]_{\mathcal{R}}$ – the equivalence class of $x \in X$
- $\mathcal{R}$ will always be **countable**, i.e. $[x]_{\mathcal{R}}$ is countable for a.e. $x \in X$

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Definitions

$\mathcal{R}$ is

1. **pmp** if $\mu = \mu \circ g$ for all $g \in [\mathcal{R}]$
2. **ergodic** if for measurable $E \subset X$

$$[\mu(gE \setminus E) = 0 \;\; \forall \;\; g \in [\mathcal{R}]] \implies \mu(E) \in \{0, 1\}$$

Construction of $L(\mathcal{R})$

1. For $g \in [\mathcal{R}]$, define $u_g \in \mathcal{U}(L^2(\mathcal{R}))$ by

$$[u_g f](x,y) = f(g^{-1}x, y) \quad \text{for} \quad f \in L^2(\mathcal{R})$$

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2. We consider $L^\infty(X) \subset \mathrm{B}(L^2(\mathcal{R}))$ by defining
$$[af](x,y) = a(x)f(x,y) \quad \text{for} \quad a \in L^\infty(X), f \in L^2(\mathcal{R})$$

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3. Then define

$$L(\mathcal{R}) = \overline{\{\sum_{\text{finite}} a_g u_g : g \in [\mathcal{R}], a_g \in L^\infty(X)\}}^{\text{WOT}} \subset \mathrm{B}(L^2(\mathcal{R}))$$

Extensions of $\mathcal{R}$

Definition

$\tilde{\mathcal{R}}$ on $(\tilde{X}, \tilde{\mu})$ is an **extension** of $\mathcal{R}$ on (X, μ) if $\exists$ Borel $p : \tilde{X} \to X$ with

1. $\mu = \tilde{\mu} \circ p^{-1}$
2. $p|_{[x]_{\tilde{\mathcal{R}}}}$ injective for a.e. $x \in \tilde{X}$
3. $p([x]_{\tilde{\mathcal{R}}}) = [p(x)]_{\mathcal{R}}$ for a.e. $x \in \tilde{X}$

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Proposition (Popa '05)

In this case $\exists$ τ-preserving $L(\mathcal{R}) \hookrightarrow L(\tilde{\mathcal{R}})$ with

$$L(\tilde{\mathcal{R}}) = \{L(\mathcal{R}) \cup L^\infty(\tilde{X})\}''$$

$$\begin{array}{ccc} & & L(\tilde{\mathcal{R}}) \\ L(\mathcal{R}) & & | \\ | & & L^\infty(\tilde{X}) \\ L^\infty(X) & & \end{array}$$

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$$L(\mathcal{R}) \subset L(\tilde{\mathcal{R}}), \quad L^\infty(X) \subset L(\mathcal{R}), \quad L^\infty(X) \subset L^\infty(\tilde{X}) \subset L(\tilde{\mathcal{R}})$$

$$\mathcal{N}_{L(\mathcal{R})}(L^\infty(X)) \hookrightarrow \mathcal{N}_{L(\tilde{\mathcal{R}})}(L^\infty(\tilde{X}))$$

Finding $\mathbb{F}_2$

Theorem (Gaboriau-Lyons '07)

Any non-amenable Γ has $\mathbb{F}_2$ as a "measurable subgroup," i.e. $\exists$ free ergodic actions $\Gamma \curvearrowright (X, \mu)$, $\mathbb{F}_2 \curvearrowright (X, \mu)$ with

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Theorem (Bowen-H-Ioana '15)

For any non-amenable ergodic $\mathcal{R}$ on (X, μ) there is an ergodic **extension** $\tilde{\mathcal{R}}$ on $(\tilde{X}, \tilde{\mu})$ and free ergodic action $\mathbb{F}_2 \curvearrowright \tilde{X}$ with

$$\mathcal{R}(\mathbb{F}_2 \curvearrowright \tilde{X}) \subset \tilde{\mathcal{R}}$$

$\leadsto$ Proof uses techniques of Gaboriau-Lyons

Finding $\mathbb{F}_2$

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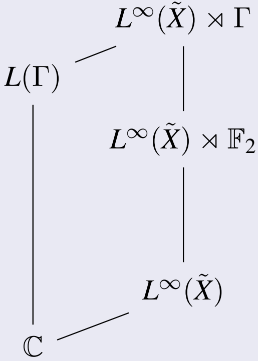

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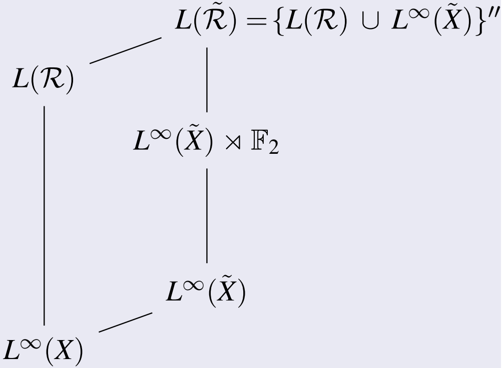

Uncountably many extensions of non-amenable $\mathcal{R}$

Theorem (Bowen-H-Ioana '15)

Any non-amenable ergodic $\mathcal{R}$ has uncountably many ergodic extensions $\{\tilde{\mathcal{R}}_i\}_{i\in I}$ s.t.

$$L(\tilde{\mathcal{R}}_i)\,\overline{\otimes}\,\mathrm{B}(\ell^2(\mathbb{N})) \not\cong L(\tilde{\mathcal{R}}_j)\,\overline{\otimes}\,\mathrm{B}(\ell^2(\mathbb{N})) \quad \text{for} \quad i \neq j$$

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Any nonamenable lcsc unimodular group G admits uncountably many free ergodic pmp actions $\{G \overset{\alpha_i}{\curvearrowright} X_i\}_{i\in I}$ s.t.

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Theorem (Bowen-H-Ioana '15)

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Proof sketch:

1. Start with any free ergodic pmp $G \curvearrowright (X, \mu)$:

$$G \curvearrowright X$$
$$\Big\downarrow \text{ cross section}$$
$$\mathcal{R}$$

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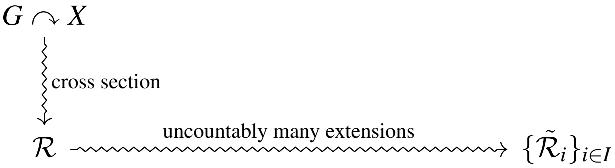

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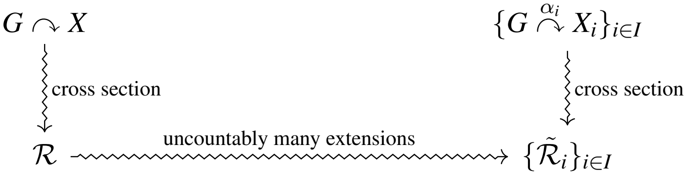

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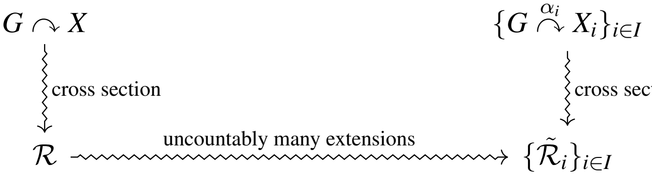

2. The $\{L^\infty(X_i) \rtimes G\}_{i \in I}$ are distinct since

$$L^\infty(X_i) \rtimes G \cong L(\tilde{\mathcal{R}}_i) \overline{\otimes} \mathrm{B}(\ell^2(\mathbb{N}))$$

by [Kyed-Petersen-Vaes '13].

Rigid Actions

Definition (Popa)

A free ergodic pmp action $\Gamma \curvearrowright (X, \mu)$ is **rigid** if the inclusion $L^\infty(X) \subset L^\infty(X) \rtimes \Gamma$ as relative property (T).

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Example (Popa)

- $\mathbb{Z}^2 \leq \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$ has relative (T) (Margulis), $\mathbb{F}_2 \leq \mathrm{SL}_2(\mathbb{Z})$ with finite index, and $\mathbb{Z}^2 \leq \mathbb{Z}^2 \rtimes \mathbb{F}_2$ has relative (T).
- $\hat{\mathbb{Z}}^2 = \mathbb{T}^2 \rightsquigarrow L(\mathbb{Z}^2) \cong L^\infty(\mathbb{T}^2, m^2)$ and

$$L(\mathbb{Z}^2) \subset L(\mathbb{Z}^2 \rtimes \mathbb{F}_2) \cong L^\infty(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2) \rtimes \mathbb{F}_2$$

  has relative (T).
- Hence $\mathbb{F}_2 \curvearrowright (\mathbb{T}^2, m^2)$ is rigid.

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Suppose $\mathcal{R}$ has $\mathcal{R}(\mathbb{F}_2 \curvearrowright X) \subset \mathcal{R}$.

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$$L^\infty(X)$$

Why is finding $\mathbb{F}_2$ useful?

Suppose $\mathcal{R}$ has $\mathcal{R}(\mathbb{F}_2 \curvearrowright X) \subset \mathcal{R}$. Take a rigid free action $\mathbb{F}_2 \overset{\alpha}{\curvearrowright} Y$:

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$L^\infty(X)$ $\qquad$ $L^\infty(Y)$ ←-- Relative (T) Subalgebra

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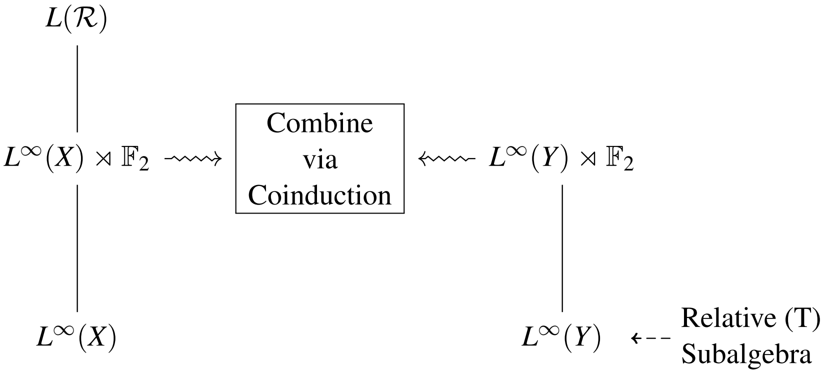

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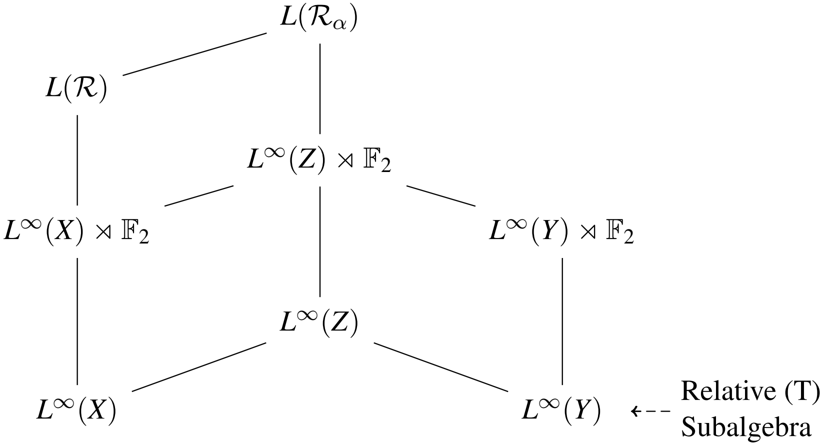

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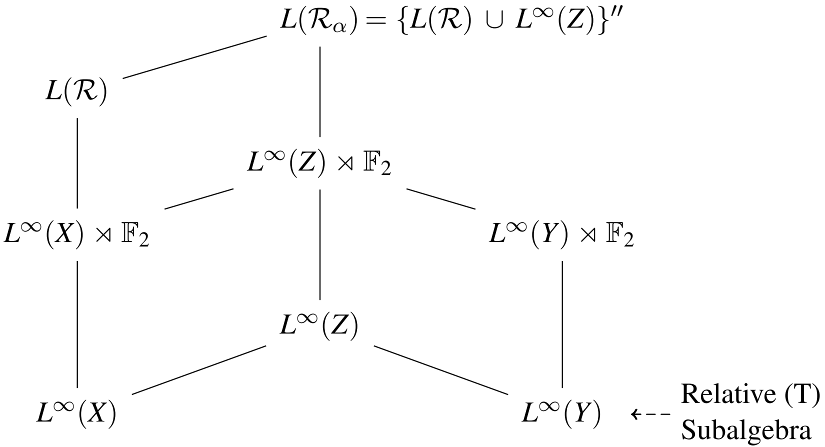

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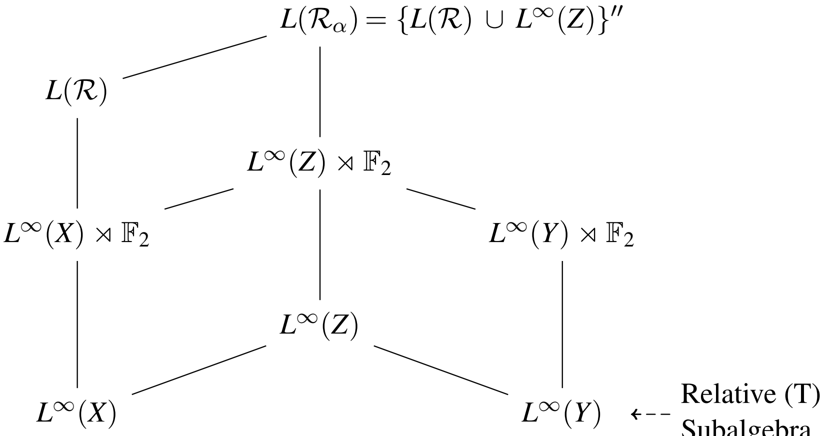

$$L^\infty(Y)' \cap L(\mathcal{R}_\alpha) = L^\infty(Z)$$

A Separability Argument

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1. Suppose $B \subset Q$ has relative (T). Let $\delta > 0$ and finite $F \subset Q$ s.t. for any Q-Q bimodule $\mathcal{H}$ and $\xi_0 \in \mathcal{H}$ with $\|x\xi_0 - \xi_0 x\| < \delta\|\xi_0\|$ for all $x \in F$, there is $\xi \in \mathcal{H}$ with $b\xi = \xi b$ for all $b \in B$.

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2. Find uncountable many $\{M_i\}_{i\in I}$ such that $Q \overset{\theta_i}{\hookrightarrow} M_i$.
3. Suppose $\exists$ uncountable $J \subset I$ and M s.t. $M_i \cong M$ for all $i, j \in J$.
4. For any $i, j \in J$ define an Q-Q bimodule action on $L^2(M)$ by

$$x \cdot \xi \cdot y = \theta_i(x)\xi\theta_j(y) \quad \text{for} \quad x, y \in Q, \xi \in L^2(M)$$

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5. Note that $\|x \cdot \hat{1} - \hat{1} \cdot x\| = \|\theta_i(x) - \theta_j(x)\|_2$ and since J is uncountable, $\exists\, i, j \in J$ s.t. $\|\theta_i(x) - \theta_j(x)\|_2 < \delta$ for all $x \in F$.

A Separability Argument

1. Suppose $B \subset Q$ has relative (T). Let $\delta > 0$ and finite $F \subset Q$ s.t. for any Q-Q bimodule $\mathcal{H}$ and $\xi_0 \in \mathcal{H}$ with $\|x\xi_0 - \xi_0 x\| < \delta\|\xi_0\|$ for all $x \in F$, there is $\xi \in \mathcal{H}$ with $b\xi = \xi b$ for all $b \in B$.
2. Find uncountable many $\{M_i\}_{i \in I}$ such that $Q \overset{\theta_i}{\hookrightarrow} M_i$.
3. Suppose $\exists$ uncountable $J \subset I$ and M s.t. $M_i \cong M$ for all $i,j \in J$.
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7. Use knowledge of $B' \cap M_i$ and $B' \cap M_j$.

Extension + Coinduction

Starting point: Any non-amenable II_1 factor M with Cartan $A \subset M$.

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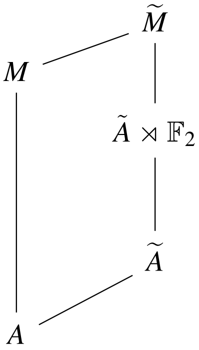

Extension + Coinduction

Starting point: Any non-amenable II_1 factor M with Cartan $A \subset M$.
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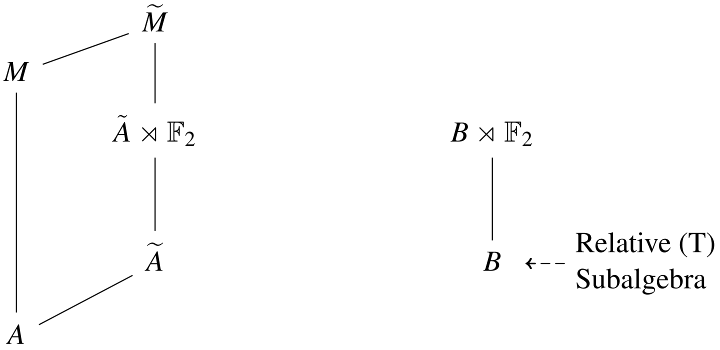

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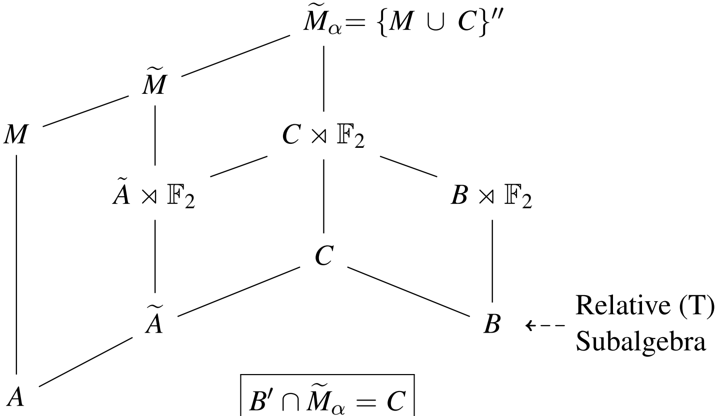

$$B' \cap \widetilde{M}_\alpha = C$$

Open Questions

Question (Gaboriau-Lyons)

Does every non-amenable ergodic $\mathcal{R}$ contain $\mathcal{R}(\mathbb{F}_2 \curvearrowright X)$ for some free ergodic action of $\mathbb{F}_2$?

Question

Does every non-amenable II_1 factor contain $L(\mathbb{F}_2)$?

Question

Let $N \subset M$ with $M = \mathcal{N}(N)''$ and $N' \cap M \subset N$. The notion of extension $\tilde{N} \subset \tilde{M}$ of $N \subset M$ was defined in [Popa-Shlyakhtenko-Vaes '15].

When can we find an extension with $\tilde{N} \subset \tilde{N} \rtimes \mathbb{F}_2 \subset \tilde{M}$?