NCGOA Spring Institute

A product formula for Pinsker factors with applications to completely positive entropy

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- Conceptually, the definition of sofic entropy is very close to Voiculescu's microstates free entropy dimension.
- It has connections to deformation/rigidty theory, through classification of Bernoulli shifts.
- I will also give applications to actions on compact groups by automorphisms and talk about how this connects with functional analysis/operator theory.

Beginning of Sofic Entropy: Sofic Groups

A countable, discrete group G is *sofic* if there is a trace-preserving embedding $L(G) \to \prod_{k \to \omega} M_k(\mathbb{C})$ which takes $G \to \prod_{k \to \omega} S_k$ (here S_k is the group of permutations.)

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Note:

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$$\frac{1}{d_k}|\{1 \leq j \leq d_k : \sigma_k(gh)(j) \neq \sigma_k(g)\sigma_k(h)(j)\}|,$$

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We call σ_k as above a sofic approximation.

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- Closed under wreath products (H.-Sale).

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One way to say this is as follows:consider $\Phi \colon A^{d_k} \to \mathrm{Prob}(A^G)$ given by $\Phi(a) = (\phi_a)_*(u_{d_k})$.Set $\widetilde{\mu_k} = \Phi_*(\mu_k)$. We then say μ_k *models* μ if $\widetilde{\mu_k} \to \delta_\mu$ weak*.

Definition of Sofic Entropy II

More concretely, μ_k models μ if and only if for every wk* neighborhood $\mathcal{O}$ of μ in $\mathrm{Prob}(A^\Gamma)$ we have

$$\mu_k(\{a : (\phi_a)_*(u_{d_k}) \in \mathcal{O}\}) \to 1.$$

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Define the *sofic entropy* of $G \curvearrowright (X, \mu)$ by

$$h_{(\sigma_i)_i,\mu}(X, G) = \sup_{\{\mu_k\}} \limsup_{k\to\infty} \frac{H(\mu_k)}{d_k},$$

where the supremum is over all sequences μ_k with μ_k modeling μ. Here

$$H(Y, \nu) = -\sum_{y\in Y} \nu(\{y\}) \log \nu(\{y\})$$

(interpreting $0 \log 0 = 0$) if (Y, ν) is atomic.

Entropy in the Presence

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Up to isomorphism, we can think of this as the following situation: we have finite sets A, B with a surjective map $\pi \colon A \to B$ so that:

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We then set

$$h_{(\sigma_k)_k, \mu}(P : L^{\infty}(X, \mu), G) = \sup_{\{\mu_k\}} \limsup_{k \to \infty} \frac{H((\pi^{d_k})_* \mu)}{d_k}.$$

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We call this the entropy of $G \curvearrowright P$ in the presence of $G \curvearrowright L^\infty(X, \mu)$. It is monotone increasing in the first variable and monotone decreasing in the second variable.

Entropy in the Presence

For our purposes, we will need a modified version for inclusions $G \curvearrowright (P,\mu) \subseteq G \curvearrowright L^{\infty}(X,\mu)$.
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The Outer Pinsker algebra

Given a pmp action $G \curvearrowright (X, \mu)$, we let P be the largest von Neumann subalgebra of $L^{\infty}(X, \mu)$ so that $h_{(\sigma_k)_k,\mu}(P : L^{\infty}(X, \mu), G) = 0$.

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Theorem

Suppose that $G \curvearrowright (X,\mu), G \curvearrowright (Y,\nu)$ are weak mixing actions, then

$$\Pi_{(\sigma_k)_k,G}(X \times Y, \mu \otimes \nu) = \Pi_{(\sigma_k)_k,G}(X,\mu)\overline{\otimes}\Pi_{(\sigma_k)_k,G}(Y,\nu).$$

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Sketch of Proof of Pinsker Product Formula

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This is because if $G \curvearrowright (Y,\nu)$ is any action, then there is an extension $G \curvearrowright (Z,\zeta)$ (i.e. $G \curvearrowright L^\infty(Y,\nu) \subseteq G \curvearrowright L^\infty(Z,\zeta)$) so that $\mathrm{Aut}(G \curvearrowright (Z,\zeta))$ acts ergodically on (Z,ζ).

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For example, take $(Z,\zeta) = (Y,\nu)^{\mathbb{Z}}$ with G acting diagonally.Then the Bernoulli action of $\mathbb{Z}$ on (Z,ζ) commutes with the G-action and is ergodic.

Sketch of Proof Continued

We now use the following general fact: if $G \curvearrowright^{\alpha} (N, \tau)$ is an ergodic trace-preserving action, if Q is any von Neumann algebra and $Q \otimes 1 \subseteq P \subseteq Q \overline{\otimes} N$ and $(1 \otimes \alpha_g)(P) = P$ for every $g \in G$, then $P = Q_0 \overline{\otimes} N$ for some subalgebra $Q_0 \subseteq P$.

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the reverse inclusion is obvious and this proves the theorem.

Actions on Compact Groups

Corollary

Let X be a compact group with Haar measure m_X. Suppose that $G \curvearrowright (X, m_X)$ is ergodic. Then there exists a unique, closed, normal subgroup Y of X so that

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Applications I: Specific Algebraic Actions

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Given $A \in M_{k,n}(\mathbb{Z}(G))$, let X_A be the Pontryagin dual of $\mathbb{Z}(G)^{\oplus n}/\mathbb{Z}(G)^{\oplus k}A$.

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Suppose that $A \in M_n(\mathbb{Z}(G)) \cap GL_n(L(G))$. Then $G \curvearrowright (X_A, m_{X_A})$ has trivial Pinsker algebra. Thus the restriction of the action of G to any G-invariant subalgebra has positive entropy.

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Theorem (Bowen-Seward)

The map ϕ is injective if G is sofic and $f \in \mathbb{Z}(G) \cap L(G)^\times$.

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and set $\Psi(\theta) = \eta(\theta) - L(\theta)$. Note that Ψ actually lands in $Y_f = \{y \in [-C, C]^G : y * f^* = 0\}$. Set $\nu = \Psi_* m_{X_f}$.

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Define $\mathcal{Y}\colon Y_f \to [-C, C]$ by $\mathcal{Y}(y) = y(1)$. Note that the G-translates of $\mathcal{Y}$ generate $L^\infty(Y_f, \nu)$ as a von Neumann algebra. Set $\mathcal{H} = \overline{\mathrm{Span}(G\mathcal{Y})}^{\|\cdot\|_2}$. We claim that $\mathrm{Hom}_G(\mathcal{H}, \ell^2(G)) = \{0\}$.

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By our previous cpe result, we know that Ψ is almost surely constant. It follows from this that Φ is injective modulo null sets.