Hochschild Homology and De Rham Cohomology of Stratifolds

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Preface

This work originated in 1999 when my advisor Professor Dr. Matthias Kreck had the idea that a generalization of Alain Connes' result about the Hochschild and cyclic homology of the algebra of smooth functions on a smooth manifold could also hold in the concept of his previously defined stratifolds. These stratifolds, which he invented and since then have gone through various stages of development, are some kind of singular spaces. We will introduce stratifolds in detail in chapter 1 but should mention so far, that they belong to the class of stratified spaces. Roughly spoken a stratified space is a space which is decomposed into smooth manifolds, the so called strata. To such spaces one can associate some kind of algebra of smooth functions. The most naive way is to say, that a function on a stratified space is smooth, if the restriction to any of the strata is smooth. It will turn out, that this is not enough for our purposes, but it gives us a first idea. In section 1.3 we will introduce the algebra $C^{\infty}(X)$ of smooth functions on a stratifold in detail. The analytical properties of this algebra is where the concept of stratifolds differs from other concepts of singular spaces. The standpoint of this work is the analytical one, that is we consider the algebra $C^{\infty}(X)$ as our starting point and all other constructions and methods will evolve from it.Nevertheless, we keep things as geometric as possible.

One can say, that from the analytic standpoint the theory of smooth manifolds is quite well developed. This means, that there are concepts like differential forms, differential operators, geometric constructions like curvature and connections and so on. Since around 1960 when stratified spaces first appeared in the literature (see [Whitney] and [Thom]) people tried to generalize these concepts to stratified spaces. In the context of differential forms and de Rham cohomology one should mention Verona first of all (see [Verona71]). He introduced differential forms and proved some kind of de Rham theorem for stratified spaces which come together with some kind of tubular neighbourhoods around the strata. His approach differs from ours in the sense that he considers differential forms strata by strata, satisfying certain compatibility conditions, whereas our start point is the space X itself and the algebra $C^\infty(X)$ of smooth functions on X. In his work about intersection homology (see [Brasselet91]) Brasselet used the ideas of Verona to give a description of intersection homology of so called pseudo manifolds in terms of differential forms with certain extra conditions. Pseudo manifolds are closely connected to what we call locally coned stratifolds. A good summary of the actual state of research on stratified spaces in general has been given by Pflaum in his “Habilitationsschrift” (see [Pflaum]). His work is mostly based on so called Whitney stratified spaces. How these spaces are related to stratifolds is a work in progress by Anna Grinberg. Pflaum also tackles the problem of Hochschild homology for these spaces and gives some partial answers. In a quite different context, namely the context of rational homotopy theory, differential forms on simplicial complexes have been introduced by Quillen [Griffiths]. Quillen also proves a de Rham theorem in this context. In his work [Teleman98] Teleman claims (but doesn't proof) that Quillen's ideas together with his result about localization of the Hochschild complex will work to generalize Connes' result mentioned in the beginning to simplicial complexes. Our two main results concern the de Rham cohomology and the Hochschild homology of stratifolds. The first one can be summarized as follows.

Theorem 1. *Let X be a compact stratifold. Then there is a natural isomorphism*

$$H^n_{dR}(X) \to Hom(H_n(X), \mathbb{R})$$

for all $n \in \mathbb{N}$ given by integration of differential forms on classes in the integral homology of X.

We will prove this theorem in chapter 4, where we also prove an analogous statement when X is noncompact. Our second theorem has a more analytic character and is a generalization of [Connes87].

Theorem 2. *Let X be a locally coned stratifold. Then there is a natural topological isomorphism*

$$\bar{\Omega}^n_{\bar{C}^\infty(X)} \to HH_n(\bar{C}^\infty(X))$$

for all $n \in \mathbb{N}$.

In Theorem 2 the left hand side stands for some completed version of differential forms on a stratifold whereas the right hand side stands for the continuous Hochschild homology of a completed version of $C^\infty(X)$. These objects will be constructed in chapters 5 and 6. We will prove Theorem 2 in chapter 7. Since in the manifold case Hochschild and cyclic homology is closely connected to what is called index theory, we hope that this result is one step forward in generalizing this theory to some classes of singular spaces.

Since I was always fascinated by the interactions between analysis, algebra and topology I must thank my advisor Prof. Dr. Matthias Kreck that he gave me the right task as a theme for my doctoral thesis. His idea about how smooth functions on stratifolds should look like showed all its strength when proving Theorem 2. I must also thank Prof. Bruce Blackadar from the University of Nevada, Reno who gave me advice on some of the more analytical parts of this work. Also I thank Prof. Don Pfaff and his wife for giving me accommodation during my stay in Reno. From the department of mathematics in Heidelberg I thank Anna Grinberg for many mathematical and nonmathematical discussions. From the department of mathematics in Mainz I thank Frank Baldus. Also I thank Anna Warzecha for our interesting discussions, our nice walks in the Odenwald and some other things. Of course I have to thank my parents too.

Kapitel 1

Introduction to Stratifolds

In this chapter we will introduce a class of topological spaces, we call stratifolds. These spaces have been invented by Matthias Kreck in 1998 to serve as the right objects, to give a very concrete geometric description of ordinary integral homology as a bordism theory. Since then there have been various versions of these objects. The one we use here, in particular the version of stratifolds with boundary is the one which suits our purposes best.

We will present our version of stratifolds and study some basic properties of this class of spaces. Stratifolds are in some kind constructed similar to CW-complexes, but have a much finer structure. Constructions known from differential topology can be generalized to a certain class of stratifolds. In fact, special classes of them form bordism categories. In one case the associated homology theory is ordinary integral homology. This description of singular homology is due to Kreck and will be of major importance in the later chapters. Our main source for this chapter is [Kreck00].

1.1 The Class of c-Manifolds

Roughly spoken, stratifolds will be obtained by gluing together a couple of smooth manifolds. To do this gluing process in a nice and organized way and also for structural properties of the associated algebra of smooth functions on a stratifold the introduction of a certain class of manifolds, we call **c-manifolds** has been proven successful.From the collar theorem in differential topology (see [Hirsch],page 113) it follows, that every smooth manifold W with boundary possesses a collar, which is given by an embedding

$$\gamma : \partial W \times [0, \epsilon) \to W$$

for some $\epsilon > 0$. In general many choices of the map γ are possible. Two such collars γ_1,γ_2 on W will be called **equivalent**, if there is an open neighbourhood U of ∂W in W such that γ_1 and γ_2 coincide on $\gamma_1^{-1}(U) = \gamma_2^{-1}(U) \subset \partial W \times [0, \epsilon)$. We denote the equivalence class of a collar γ with $[\gamma]$. As one could expect, the "c" in "c-manifold" stands for collared manifold.

Definition 1.1.1. *A c-manifold is a pair $(W, [\gamma])$, where W is a smooth manifold with boundary and $[\gamma]$ is an equivalence class of a collar on W.*

Often we write W when in fact we mean $(W, [\gamma])$ and just speak of a manifold when we really mean c-manifold. In situations, when emphasize is made on, that the manifold in question is not treated as a c-manifold, we will speak of a manifold in the **naive sense**. Also all constructions which are based on manifolds in the naive sense will be referred to as that.

Two c-manifolds $(W_1, [\gamma_1])$ and $(W_2, [\gamma_2])$ will be called diffeomorphic and treated as equal, if there is a diffeomorphism $f : W_1 \to W_2$, such that the induced collar $f_*\gamma_1 := f \circ \gamma_1 \circ (f^{-1}_{|\partial W_2} \times id)$ and γ_2 are equivalent. This notion of diffeomorphacy of c-manifolds seems very natural, though it is very strict and not so well suited for our purposes. We will define a category of c-manifolds using the following definition of smooth functions on a c-manifold as a start point.

Definition 1.1.2. *Let $(W, [\gamma])$ be a c-manifold. By definition a map*

$$g : W \to \mathbb{R}$$

belongs to $C^\infty(W, [\gamma])$ if g is smooth on W and there exists an open neighbourhood U of ∂W in $\partial W \times [0, \epsilon)$ such that the following diagram commutes

$$\begin{tikzcd} U \subset \partial W \times [0,\epsilon) \arrow[r, "g\circ\gamma"] \arrow[dr, "p"'] & \mathbb{R} \\ & \partial W \arrow[u, "g"'] \end{tikzcd}$$

Here p denotes the projection on the first coordinate.

In other words a function on $(W, [\gamma])$ is smooth, if it is smooth in the naive sense and has the property that in a small neighbourhood of the boundary it is constant along the collar, i.e. in direction of the paths $t \mapsto \gamma(x, t)$ for $x \in \partial W$. It should be clear that $C^\infty(W, [\gamma])$ indeed only depends on the equivalence class of the collar. Later we will often write $C^\infty(W)$ instead of

$C^\infty(W,[\gamma])$ to shorten the notation. There should be no misunderstandings, since if we treat W as a manifold in the naive sense we use the symbol $C^\infty_{naive}(W)$.

Of course the algebra $C^\infty(W,[\gamma])$ differs from the algebra $C^\infty_{naive}(W)$. This will show best when studying the local situation. The following proposition gives an answer to that.

Proposition 1.1.1. *Let $(W,[\gamma])$ denote a c-manifold, let $x \in W$ be a point and and let $\mathcal{O}_{W,x} = \varinjlim_{x\in U} C^\infty(U,[i^U_*\gamma]))$ denote the algebra of germs at x. Here U runs through the open neighbourhoods of $x \in W$ and each U is considered as a c-manifold itself using the inclusion map $i^U : U \to W$ and the induced collar $i^U_*\gamma$. Let n denote the dimension of W. Then there are two cases.*

1. *If x lies in the interior W° of W we have $\mathcal{O}_{W,x} \cong \mathcal{O}_{R^n,0}$.*
2. *If x lies in the boundary ∂W of W we have $\mathcal{O}_{W,x} \cong \mathcal{O}_{R^{n-1},0}$.*

Proof. In the first case, choosing local coordinates will prove that $\mathcal{O}_{W,x} \cong \mathcal{O}_{R^n,0}$. In the second case one can choose coordinates around x as follows. Take as a first coordinate the coordinate t which is given by $\gamma(y,t) \mapsto t$ $\forall (y,t) \in \partial W \times [0,\epsilon)$ in a small neighbourhood of x and for the remaining $n-1$ coordinates $x_1,...x_{n-1}$ take a set of coordinates $y_1,...y_{n-1}$ of ∂W defined in a neighbourhood of x in ∂W and define $x_i(\gamma(y,t)) := y_i(y)$. Using these coordinates the condition on $g : W \to \mathbb{R}$ to belong to $C^\infty(W,[\gamma])$ is not to depend on t for t small. On the other side, there is no restriction on the other $n-1$ coordinates. This of course shows $\mathcal{O}_{W,x} \cong \mathcal{O}_{R^{n-1},0}$.

□

Let us now introduce smooth maps between c-manifolds

Definition 1.1.3. *Let $(W_1,[\gamma_1])$ and $(W_2,[\gamma_2])$ be c-manifolds and let*

$$f : W_1 \to W_2$$

be a map. We say that f is smooth if for any $g \in C^\infty(W_2,[\gamma_2])$ the composition $g \circ f : W_1 \to \mathbb{R}$ lies in $C^\infty(W_1,[\gamma_1])$. We denote the set of these functions by $C^\infty((W_1,[\gamma_1]),(W_2,[\gamma_2]))$.

We can now setup our **category of c-manifolds** as follows. Objects are c-manifolds and morphisms are smooth maps between c-manifolds. In this context an isomorphism between two c-manifolds is a smooth map

$f : (W_1, [\gamma_1]) \to (W_2, [\gamma_2])$ such that the inverse map f^{-1} exists and is contained in $C^\infty((W_2, [\gamma_2]), (W_1, [\gamma_1]))$. The c-manifolds $(W_1, [\gamma_1])$ and $(W_2, [\gamma_2])$ are then called **isomorphic**. We should mention that there is a real difference between isomorphisms and diffeomorphisms of c-manifolds. An isomorphism allows some kind of reparametrization in direction along the collar which a diffeomorphism doesn't. We admit, that the name diffeomorphism in this context might be a little bit confusing, since for most topologists a diffeomorphism is a smooth map which has a smooth inverse. In our sense this corresponds exactly to an isomorphism. We should keep that in mind.

1.2 Stratifolds

In this section we introduce stratifolds in its most general form. We should remind again that the word manifold here stands for c-manifold. Though this is not of importance in this section, it will be crucial in the next one.

Definition 1.2.1. *Let X be a topological space, $\Sigma \subset X$ be a subspace and R be a manifold. Let $R^\circ = R - \partial R$ denote the interior part of R. Let $\varphi : R \to X$ be a map, such that*

$$\varphi(R^\circ) \subset X - \Sigma$$

$$\varphi(\partial R) \subset \Sigma$$

and φ induces a homeomorphism

$$R \cup_\varphi \Sigma \approx X.$$

We call $X - \Sigma$ the **regular part** *of X and Σ the* **singular part** *of X. We refer to φ as a* **singular chart** *of the pair (X, Σ). The pair (X, Σ) is called a* **singular space***.*

Let us approach our first definition of a stratifold.

Definition 1.2.2. *A topological space X together with proper maps*

$$\varphi_i : R_i \to X,$$

where i runs through an index set $I \subset \mathbb{N}$, is called a **stratifold** *if these data satisfy the following conditions :*

1. *For any $i \in I$ the space R_i is a manifold of dimension i.*

2. *For any pair $i \neq j \in I$*

$$\varphi_i(R_i^\circ) \cap \varphi_j(R_j^\circ) = \emptyset$$

and $X = \bigcup_{i \in I} \varphi_i(R_i^\circ)$.

3. *For any $i \in I$ φ_i is a singular chart of the pair (X_i, X_{i-1}), where $X_i = \bigcup_{j \in \{0,..i\} \cap I} \varphi_j(R_j^\circ)$ and X_i is closed in X.*

Though the charts belong to the definition of a stratifold, we do most times only speak of the stratifold X, keeping the charts in mind.

We call

$$dim(X) = sup\{i \in I | R_i \neq \emptyset\} \tag{1.1}$$

the **dimension** of X. If the dimension of X is n, we refer to $\Sigma = X_{n-1}$ as the **singular part** of X and $X - \Sigma \approx R_n^\circ$ as to the **regular part**. Clearly (X, Σ) becomes a singular space with singular chart φ_n. More general, we call

$$S_i = X_i - X_{i-1} \tag{1.2}$$

the i-th **stratum** of X, and it is clear, that by choosing these sets as strata X becomes a **stratified space**. Clearly

$$S_i \approx R_i^\circ$$

and sometimes the R_i will be referred to as the **full strata** of X. We should also mention, that under this definition stratifolds of infinite dimensions are allowed, and some of our results are also valid in this case. X_i is called the **i-skeleton** of X and clearly is itself a stratifold and will be considered as this throughout the whole work. We should mention that in bordism theory of stratifolds a different definition of dimension has been used by Kreck, defining the dimension of X as $sup(I)$.We will denote this dimension as $Dim(X)$. Clearly we have that

$$dim(X) \leq Dim(X).$$

If more than one stratifold occurs at the same time, we use symbols like $R_i(X)$,$S_i(X)$ etc. to denote the corresponding data.

From the construction of stratifolds, it should arise, that they are built similar to CW complexes. Instead of cells, we attach arbitrary manifolds. In particular, any CW complex can be given the structure of a stratifold, by choosing all of the R_i as discs. The attaching maps then induce charts. On the other hand, any manifold, hence any of the strata R_i can be given a CW structure. These structures can be used to define a CW structure on the stratifold. Though, there is no canonical way to do this, and it's completely unclear, how this CW structure corresponds to the structure as a stratifold. Other questions, like triangulation of stratifolds and piecewise linear structures have to be seen in the same context and so far, haven't been tackled.

As topological spaces stratifolds will turn out to be paracompact. This will follow from the existence of a partition of unity (see Corollary 1.7.1). Moreover they are locally compact, even in the infinite case. This follows similar as in the case of CW complexes, since we consider the weak topology corresponding to the decomposition into strata. The empty set $\emptyset$ will be considered as a stratifold of any specified dimension. Let us give some less trivial examples.

Example 1.2.1. *1. Given a manifold M without boundary and let m be its dimension. We get a stratifold of dimension m by choosing*

$$I = \{m\}, R_m = M$$

and $\varphi : R_m \to M$ as the identity. This is the way we consider manifolds as stratifolds if nothing else is said. Clearly $dim(M) = m$ is the same as the dimension of M, if M is considered as a manifold.

2. Given two topological manifolds W, S of dimension r respectively s, where $s < r$ and a proper map $f : \partial W \to S$. Then the topological space

$$X = W \cup_f S$$

is considered as a stratifold by choosing $I = \{s, r\}$,

$$R_r = W, R_s = S$$

and φ_r respectively φ_s as the natural projections of W respectively S on the quotient space X. Clearly $dim(X) = r$. If the map $f : \partial W \to S$ is surjective, X can be considered as a manifold with singularities in the set S, hence the notation S for singularities. The whole concept of stratifolds is a generalization of this.

3. *If in the last example we choose S to consist only of points, we speak of* **manifolds with isolated singularities**. *In algebraic geometry many people are interested in the resolution of such singularities. The resolution of isolated singularities in the world of topological stratifolds has been studied in [Grinberg00].*

At the end of this chapter we should try to give at least one motivation for the name that has been chosen to denote our class of spaces. The name stratifold just seems right to indicate that this class of spaces consists of stratified spaces where special emphasize has been made on the role of the strata and the way how they are glued (folded) together.

1.3 Smooth Structures on Stratifolds

We will now assign an extra structure to stratifolds, which we call smoothness. This structure will help us, to carry over constructions known from differential topology of manifolds to the world of stratifolds.Since so far, we have only allowed smooth manifolds for the strata, the reader might think, we already have something, we could call smooth stratifold. This, let's say smoothness on the strata, turns out to be unsatisfying. The strata can be glued together in a very wild way, so the right notion of smoothness should reflect, that the gluing process is done in a fairly nice and smooth way. This will lead us to the definition of a smooth stratifold.

Let X be a stratifold. We call a function

$$f : X \to \mathbb{R}$$

smooth, if for any $i \in I$ the composition

$$R_i \xrightarrow{\varphi^i} X \xrightarrow{f} \mathbb{R}$$

defines an element in $C^\infty(R_i)$, where $C^\infty(R_i)$ is the algebra of functions on R_i defined in Definition 1.1.2. Clearly these maps build an $\mathbb{R}$ algebra, which we denote with $C^\infty(X)$.

Definition 1.3.1. *A stratifold X is called a* **smooth stratifold** *if for any $i \in I$ the image of the induced map*

$$\varphi_i^* : C^\infty(X_{i-1}) \to C(\partial R_i)$$

is contained in $C^\infty(\partial R_i)$. *If* X *and* Y *both are smooth topological stratifolds we call a map* $f : X \to Y$ *smooth if the image of the induced map*

$$f^* : C^\infty(Y) \to C(X)$$

is contained in $C^\infty(X)$. *We denote the set of these functions with* $C^\infty(X, Y)$.

The condition on X to be a smooth stratifold can now simply be restated as that for any $i \in I$ the restriction of the chart φ_i to ∂R_i lies in $C^\infty(\partial R_i, X_{i-1})$ in other words, the attaching maps are smooth.

Example 1.3.1. *If we require* W, S *and* $f : \partial W \to S$ *as in Example 1.2.1 to be smooth, we end up with a smooth stratifold since the map*

$$\varphi_m^* : C^\infty(X_{m-1}) \to C(\partial R_m)$$

is precisely the map induced by f.

From this point on, we will only consider smooth stratifolds and usually omit the word smooth in front of stratifold. When we write stratifold, we always mean smooth stratifold.

The following **category of stratifolds** is the category we work with. Objects are stratifolds and morphisms are smooth maps between stratifolds. In this context an **isomorphism** between stratifolds is a smooth map which has a smooth inverse.In this case the algebras of smooth functions are isomorphic. Hence isomorphic stratifolds are indistinguishable by the methods presented in this work, and will therefore be treated as equal. We should mention, that other categories of stratifolds so far appeared in different contexts, as for example in [Grinberg00] and [Minatta01]. We should mention one special case, since it occurs in the definition of locally coned stratifolds. We call two stratifolds X and Y **diffeomorphic**, if there is a homeomorphism $f : X \to Y$ which is induced by diffeomorphisms $f_i : R_i(X) \to R_i(Y)$ of c-manifolds on all full strata. As in the case of c-manifolds, there is a real difference between diffeomorphisms and isomorphisms. Nevertheless, a diffeomorphism is always an isomorphism in our category. We will keep that in mind.

We close this section by studying the local picture, in equal how smooth functions on a stratifold X behave in a small neighbourhood of some point

$x \in X$. The following proposition states that the algebra of **germs** of functions at some point $x \in X$, which we briefly define as

$$\mathcal{O}_{X,x} = \varinjlim_{x \in U} C^\infty(U) \tag{1.3}$$

is completely determined by restricting these functions to the stratum S_k. Here U runs through the open neighbourhoods of $x \in X$ and the limit is taken by restriction. It is not clear at this point, what exactly we mean with $C^\infty(U)$ for an open subset $U \subset X$. Briefly, we can say, that U inherits the structure of a stratifold, so that we can speak of $C^\infty(U)$. How this is done in more detail is presented in section 1.4. We nevertheless think it might be helpful for understanding how $C^\infty(X)$ is built up, to state the following proposition at this point.

Proposition 1.3.1. *Let X be a stratifold and $x \in S_k$ be a point in the k-stratum of X. Then the map induced by restriction*

$$\mathcal{O}_{X,x} \to \mathcal{O}_{S_k,x}$$

is an isomorphism.

In other words, the proposition says that we can somehow interpret the algebra $C^\infty(X)$ as built up of the algebras $C^\infty(S_k)$ put together in a nice way. As we will see in the proof, the reason for this to be true lies in the use of the concept of c-manifolds (compare Proposition 1.1.1) .

Proof. Let $i \geq k$. We will first show

$$\mathcal{O}_{X_{i+1},x} \cong \mathcal{O}_{X_i,x},$$

where the map is given by restriction. To see this let $f \in \mathcal{O}_{X_{i+1},x}$ be an element in the kernel of the restriction map. That is f is defined on some open neighbourhood $U \subset X_{i+1}$ and $f_{|U \cap X_i} = 0$. Since $S_{i+1} \cap S_i = \emptyset$ we have

$$\varphi_{i+1}^{-1}(U \cap X_i) \subset \partial R_{i+1}.$$

Here φ_{i+1} denotes the i-th chart of the stratifold X. Since $U \cap X_i$ is open in X_i we have that $\varphi_{i+1}^{-1}(U \cap X_i)$ is open in ∂R_{i+1}. Since $f \circ \varphi_{i+1} \in C^\infty(R_{i+1})$ must approach the boundary constant along the collar in a small neighbourhood V of ∂R_{i+1} in R_{i+1} it vanishes on an open neighbourhood $\tilde{U}$ of $\varphi_{i+1}^{-1}(U \cap X_i)$ in R_{i+1} such that the image $\varphi_{i+1}(\tilde{U})$ of $\tilde{U}$ is an open neighbourhood of x in X_{i+1} on which f vanishes. This proves injectivity for $\mathcal{O}_{X_{i+1},x} \to \mathcal{O}_{X_i,x}$. A similar argument, where we extend a function given on an open subset of ∂R_{i+1} on an open subset of R_{i+1} constant along the collar will prove surjectivity of the latter map. Clearly we have that

$$\mathcal{O}_{X,x} = \varinjlim_{k \leq i} \mathcal{O}_{X_i,x}.$$

Since all maps in the direct limit are isomorphisms we have

$$\mathcal{O}_{X,x} = \mathcal{O}_{X_k,x}$$

Since the k-stratum S_k is open in the k-skeleton X_k, we also have

$$\mathcal{O}_{X_k,x} \cong \mathcal{O}_{S_k,x},$$

where the isomorphism is again given by restriction. Hence the proposition follows. □

1.4 Substratifolds

Let X be a stratifold with strata $R_k(X)$ and charts φ_k. For a subset $A \subset X$ we can consider the sets

$$\varphi_k^{-1}(A) \subset R_k(X).$$

Let's assume that each of the $\varphi_k^{-1}(A)$ is a submanifold of $R_k(X)$. Then we can define

$$R_j(A) = \coprod_{dim(\varphi_k^{-1}(A))=j} \varphi_k^{-1}(A)$$

to get a stratification of A. We also get maps

$$\psi_j : R_j(A) \to A$$

by restricting the charts φ_k to the corresponding components of $R_j(A)$. We call these data the induced data on A. The following is our definition of a substratifold.

Definition 1.4.1. *With the notation from above $A \subset X$ is called a* **substratifold***, if the $\varphi_k^{-1}(A)$ are submanifolds of the $R_k(X)$ and A together with the induced data has the structure of a stratifold.*

The following examples are easy, nevertheless they are important.

Example 1.4.1.
1. *Any open subset U of a stratifold X can, and will throughout this work be considered as a substratifold of X.*
2. *For any number $i \in \mathbb{N}$ the i-skeleton X_i is a substratifold of X. We have a natural sequence*

$$\emptyset = X_{-1} \subset X_0 \subset ... \subset X_n = X$$

where each inclusion means as a substratifold.

1.5 Stratifolds with Boundary

As in the world of manifolds there is also a concept of stratifolds with boundary. In fact, there is more than one concept of stratifolds with boundary. All concepts of course have in common that they are generalizations of the concept of stratifolds in the way that a stratifold with boundary the empty set is the same as a stratifold. Furthermore they have in common, that some categories of stratifolds with boundary form bordism categories. The latter fact will be exploited later. For this work the following concept is suited best. It was also the original concept (see [Kreck99]).

Definition 1.5.1. *Let $(X, \Sigma, \partial X)$ be a triple of topological spaces, such that $\Sigma \subset X$ and $\partial X \subset X$, and let R be a manifold with boundary ∂R smoothly decomposed as*

$$\partial R = \partial^+ R \cup \partial^- R,$$

such that

$$\partial(\partial^+ R) = \partial^+ R \cap \partial^- R = \partial(\partial^- R).$$

Furthermore let $\varphi : R \to X$ be a map which satisfies the following conditions

1. *$\varphi(R^\circ) = X - (\Sigma \cup \partial X)$*
2. *$\varphi(\partial^+ R) \subset \Sigma$*
3. *$\varphi(\partial^- R) \subset \partial X$.*

If φ induces a homeomorphism

$$X = R \cup_{\varphi_{|\partial^+ R}} \Sigma$$

we call $(X, \Sigma, \partial X)$ a **singular space with boundary** *and φ a singular chart of $(X, \Sigma, \partial X)$.*

Now we proceed similar as in the definition of stratifolds without boundary as follows.

Definition 1.5.2. *A pair of topological spaces $(X, \partial X)$ together with proper maps*

$$\varphi_i : R_i \to X,$$

where i runs through an index set I is called a **stratifold with boundary** *if the following conditions hold :*

1. *For any $i \in I$ the space R_i is a manifold of dimension i with boundary ∂R_i smoothly decomposed as $\partial R_i = \partial^+ R_i \cup \partial^- R_i$ similar to the decomposition in Definition 1.5.1.*

2. *For any $i \in I$ we have $\varphi_i(\partial^- R_i) \subset \partial X$ and ∂X together with charts $\psi_i := \varphi_{i+1|\partial^- R_{i+1}}$ and $R_i(\partial X) = \partial^- R_{i+1}$ is a stratifold.*

3. *For any pair $i \neq j \in I$ we have $\varphi(R_i^\circ \cup (\partial^- R_i)^\circ) \cap \varphi(R_j^\circ \cup (\partial^- R_j)^\circ) = \emptyset$ and $X = \bigcup_{i \in I} \varphi(R_i^\circ \cup (\partial^- R_i)^\circ)$.*

4. *For any $i \in I$ the map φ_i is a singular chart of the singular space with $(X_i, X_{i-1}, (\partial X)_i)$ where $X_i = \bigcup_{j \in \{0,..,i\} \cap I} \varphi(R_i^\circ \cup (\partial^- R_i)^\circ)$ and X_i is closed in X.*

5. *The maps $\varphi_{i|\partial^+ R_i} : \partial^+ R_i \to X_{i-1}$ and $\varphi_{i|\partial^- R_i} : \partial^- R_i \to \partial X$ are smooth.*

We consider all stratifolds with boundary the empty set as stratifolds. A **substratifold** of a stratifold with boundary is defined in pure analogy to Definition 1.4.1 , namely a set $A \subset X$ such that A with the induced data is a stratifold with boundary. It is then clear that the sets $X - \partial X$ and ∂X are substratifolds of X. The following example shows that any stratifold can be realised as the boundary of a stratifold with boundary namely the cone over the stratifold.

Example 1.5.1. *Let X be a stratifold without boundary. We give the cone CX over X the structure of a stratifold with boundary as follows. Let $I = [0,1]$ be the closed unit interval.*

$$R_0(CX) = pt$$

$$R_k(CX) = R_{k-1}(X) \times I$$

$$\partial^+ R_k(CX) = R_{k-1}(X) \times \{0\} \cup \partial R_{k-1}(X) \times I$$

$$\partial^- R_k = R_{k-1}(X) \times \{1\}$$

$$\psi_k : R_k(CX) = R_{k-1}(X) \times I \to CX$$

$$(x,t) \mapsto [\varphi_{k-1}(x), t],$$

where the φ_k denote the charts of X. It can be verified that CX together with these data defines a stratifold with boundary, which we call the **cone** *over X. It is clear from the construction that $\partial(CX) = X$.*

The example above has major consequences for bordism theories based on stratifolds (see section 1.9). If in the construction above, we exchange $I = [0,1]$ by the half open interval $[0,1)$ the "-" part of ∂R_k in the definition above vanishes and we get a stratifold without boundary. We denote this stratifold with cX and call it the **open cone** of X. We have

$$cX = CX - \partial CX = CX - X.$$

1.6 Locally Coned Stratifolds

In this section we will introduce locally coned stratifolds. They are in close connection to so called **pseudo manifolds**, see for example [Borel87] for the definition. The idea behind the definition is, that locally each singularity has a neighbourhood which is a cone over a stratifold of smaller dimension. More precisely we have the following definition.

Definition 1.6.1. *We call an n dimensional stratifold* **locally coned***, if for each $k \in I$ and $x \in S_k$ there exists a neighbourhood U_x in X and a stratifold L_x of dimension $n-k-1$ together with a diffeomorphism of stratifolds*

$$U_x \cong B^k \times cL_x,$$

where cL_x denotes the open cone over the stratifold L_x and B^k the open unit ball in euclidean k-space. L_x will be referred to as the **link** *at x and U_x will be called a* **cone neighbourhood** *of x.*

Replacing cL_x in the definition above by a product of cones yields to a class of stratifolds which is called **locally product coned** stratifolds.

It can easily be seen that the stratifolds L_x occurring in the definition above, are also locally coned (see [Weber01]). Our results concerning de Rham cohomology of stratifolds are valid for general stratifolds, whereas our results on Hochschild homology of stratifolds are only valid for locally coned, or more general locally product coned stratifolds. The reason for this is, that when we know the algebra $C^\infty(X)$, we know the algebra $C^\infty(cX)$ almost as well. Hence the local situation is far easier and obtainable by inductive methods, than in the general case of a stratifold. Another nice aspects of locally coned stratifolds is that the introduction of some nice conditions on the links can also yield to interesting new homology theories. This can also be found in [Weber01].

1.7 Some Properties of Stratifolds

In this section, we establish some properties of stratifolds which will be useful in later chapters.

Lemma 1.7.1. *Let X be a stratifold and $x \in X$. Let U be an open neighbourhood of x. Then there is a smooth function $\rho : X \to [0, \infty)$, such that $supp(\rho) \subset U$ and $\rho(x) > 0$.*

Proof. This is Lemma 4.2. of [Kreck00]. The proof presented there works also in the case of infinite dimensional stratifolds. □

The existence of a smooth partition of unity, subordinated to a certain open covering is established by the following corollary.

Corollary 1.7.1. *Let X be a stratifold and $(U_j)_{j \in J}$ be an open covering of X. Then there is a* **smooth partition of unity** *$(f_j)_{j \in J}$ subordinated to the covering, in equal $f_j \in C^\infty(X)$ such that*

$$\sum_{j \in J} f_j \equiv 1_X$$

with $supp(f_j) \subset U_j$ and $\forall x \in X$ the set $\{j \in J | f_j(x) \neq 0\}$ is finite.

Proof. see [Kreck00]. □

As a consequence of the existence of partitions of unity we get the following result.

Corollary 1.7.2. *Let X be a stratifold, then X is paracompact as a topological space.*

1.8 Stratifolds and Orientation

Since we have seen, that any stratifold is the boundary of its cone, the bordism category of all stratifolds is not particularly interesting. To get something more interesting, we will introduce orientations on stratifolds. Before we proceed, the reader should be reminded at the difference between $dim(X)$ and $Dim(X)$ (see (1.1) and page 7).

Definition 1.8.1. *A stratifold X of dimension $Dim(X) = n$ is said to be* **$\mathbb{Z}/2$- oriented** *if $R_{n-1}(X) = \emptyset$. We say that X is* $\mathbb{Z}$- **oriented**, *if in addition the top stratum $R_n(X)$ is an oriented manifold.*

The fact that the second highest stratum of an oriented stratifold is empty will be crucial, when proving a generalisation of Stokes' Theorem in chapter 4 .

1.9 Bordism Theory based on Stratifolds

One intention of the construction of stratifolds was to give a concrete bordism like description of singular homology. In this section we briefly outline the construction of Kreck given in [Kreck99].

Let X, Y be $\mathbb{Z}$ oriented, compact stratifolds of some given dimension n. We say X and Y are **bordant**, if there is a $\mathbb{Z}$ oriented $n+1$ dimensional compact stratifold W with boundary

$$\partial W = X + (-Y),$$

where $-Y$ denotes Y with orientation reversed. We call W a **bordism** between X and Y.

Now let Z be a topological space. We consider classes $[f : X \to Z]$, where two classes $[f : X \to Z]$ and $[g : Y \to Z]$ are called **equivalent** if X and Y are bordant via a bordism W and there is a map $h : W \to Z$ such that $h_{|X} = f$ and $h_{|Y} = g$. This indeed defines an equivalence relation. The set of equivalence classes of these objects becomes a group with addition induced by the topological sum.

In the situation above we let n run through the natural numbers and end up with a functor

$$\mathcal{H}_* : Top \to GrAb$$

from the category of topological spaces to the category of graded abelian groups. Moreover it can be shown, that this functor is a **homology theory**. This remains valid if one reduces the category of stratifolds involved by assuming some nice extra conditions, for example on the strata of some given dimensions etc. In this context interesting new questions in the study of homology theories arise. Some of them have been studied by Lecibyll in [Lecibyll00]. In our case the question which homology theory arises from this construction is completely answered by the following theorem.

Theorem 1.9.1. *The homology theory given by bordism of $\mathbb{Z}$ oriented stratifolds is naturally isomorphic to ordinary integral homology. That means there is a natural equivalence between the functors $\mathcal{H}_*$ and $H\mathbb{Z}_*$ where the latter means singular homology with integer coefficients.*

Proof. [Kreck99] □

Kapitel 2

Tools from Sheaf Theory

In this chapter we present some of the basic concepts of sheaf theory. The reader who is familiar with things like sheaf theoretic cohomology can skip this chapter or may look up things later. By brutal force we could have avoided the use of sheaf theory entirely, but we think it makes proofs more elegant. Since this chapter only presents methods and tools, we skip almost any proof, but say exactly where it can be found in the book of Bredon [Bredon97]. The reader who wants more detailed information about sheaves should also consult this book.

2.1 Basic Definitions

Throughout this chapter X denotes a topological space.

Definition 2.1.1. *A* **presheaf** *A of abelian groups on X is a contravariant functor from the category of open subsets of X and inclusions as morphisms to the category of abelian groups. This means to any open subset $U \subset X$ there is associated an abelian group $A(U)$ and if $V \subset U$ is another open subset of X, then there is a restriction map*

$$r_{U,V} : A(U) \to A(V),$$

such that whenever $W \subset V \subset U$ are three open subsets of X the equation

$$r_{U,W} = r_{V,W} \circ r_{U,V}$$

holds.

To simplify the notation we often write $s_{|V}$ instead of $r_{U,V}(s)$ for the restriction of an element $s \in A(U)$ to a subset $V \subset U$.

The following examples are fundamental in the sense, that all sheaves or presheaves occurring in this work, will be based on these.

Example 2.1.1. *1. Let G be any group and X a topological space. Then we can associate G to any open subset of X, in equal $\mathcal{G}(U) = G$ for all open subsets U of X. This clearly defines a presheaf on X which we call the constant sheaf $\mathcal{G}$ with value G on X.*

2. Let X be a stratifold. If we consider an open subset $U \subset X$ as a sub-stratifold of X according to Example 1.4.1, we can build $C^\infty(U)$. The association

$$U \mapsto C^\infty(U)$$

for all open subsets U of X is a presheaf on X. The restriction maps are given by restriction of functions. We denote this presheaf $\mathcal{O}_X$ and refer to it as the **structure sheaf** *on X.*

The presheaves in the example above have more structure, than it is required for presheaves. Indeed, they are sheaves. This is our next definition.

Definition 2.1.2. *A presheaf A over a topological space X is called a* **sheaf***, if it satisfies the following two conditions*

1. *If $U = \bigcup_\alpha U_\alpha$ is an open covering of an open subset $U \subset X$, and $s, t \in A(U)$ are elements, such that $s_{|U_\alpha} = t_{|U_\alpha}$ for each of the U_α, than $s = t$.*

2. *If under the conditions above there are given $s_\alpha \in A(U_\alpha)$ for each of the U_α such that $s_{\alpha_{|U_\alpha \cap U_\beta}} = s_{\beta_{|U_\alpha \cap U_\beta}}$ for all indices α and β, then there is an element $s \in A(U)$ such that $s_{|U_\alpha} = s_\alpha$ for all α .*

It is clear from the definition, that the constant sheaf and the structure sheaf $\mathcal{O}_X$ of a stratifold are indeed sheaves. On the other side there is a canonical way to construct sheaves out of presheaves. This process is called **sheafification**. For this let A be a presheaf on X. For each $x \in X$ define the **stalk** of A at x to be

$$A_x = \varinjlim_{x \in U} A(U) \tag{2.1}$$

where U runs through the open neighbourhoods of X. This group contains the local structure of A at the point x. An Element of A_x is given by the equivalence class of some $s \in A(U)$. We denote this class with s_x. We give $\bigcup_{x \in X} A_x$ the topology generated by the open sets

$$\{s_x \in A_x | x \in U, s \in A(U)\}, \ \forall\, U \subset X \text{ open.}$$

We denote this space with $\mathcal{A}$. It comes together with a continuous map

$$\pi : \mathcal{A} \to X,$$

which is the projection on the base point of the corresponding stalk. For an open subset $U \subset X$ let us denote the sections of $\mathcal{A}$ over U as $\Gamma(U, \mathcal{A})$. Then it is clear, that the association $U \mapsto \Gamma(U, \mathcal{A})$ is a sheaf on X. Sometimes we denote this sheaf by

$$Sheaf(U \mapsto A(U)).$$

The process of sheafification also shows, how one can imagine sheafs geometrically as topological spaces. This is the content of the next proposition.

Proposition 2.1.1. *Let A be a presheaf on X. Then the associated topological space $\mathcal{A}$ has the following properties.*

1. *$\pi : \mathcal{A} \to X$ is a local homeomorphism.*
2. *Each of the $\mathcal{A}_x = \pi^{-1}(x)$ is an abelian group, and will be called the* **stalk** *of A at $x \in X$.*
3. *The group operations on the stalks are continuous. This means that the map*

$$- : \{(\alpha, \beta) \in \mathcal{A} \times \mathcal{A} | \pi(\alpha) = \pi(\beta)\} \to \mathcal{A},$$

$$(\alpha, \beta) \mapsto \alpha - \beta$$

is continuous.

If A is already a sheaf the groups $A(U)$ and $\Gamma(U, \mathcal{A})$ are naturally isomorphic. So in the case we are starting with a sheaf, sheafification yields to nothing new.

The reader should be warned, that as topological spaces, sheaves in general have no particularly good topological properties.For example they usually lack to be Hausdorff. It follows from the proposition above, that the fibres are always discreet. As a consequence, in some cases continuous sections can look very obscure.

In the following calligraphic letters always correspond to the related roman letters, though according to the proposition above we often identify a sheaf A with its associated topological space $\mathcal{A}$. If $Y \subset X$ denotes an arbitrary subspace, we can restrict the sheaf $\mathcal{A}$ on Y which is

$$\mathcal{A}_{|Y} = \pi^{-1}(Y) \to Y. \tag{2.2}$$

To get a category of sheafs, we have to say, what a morphism of sheaves is.

Definition 2.1.3. 1. *Let A and B be presheaves on X. A* **morphism** *of presheaves*

$$h : A \to B$$

is a collection of group homomorphisms

$$h_U : A(U) \to B(U)$$

defined for all open subsets $U \subset X$, which are compatible with the restriction maps. In the language of category theory $h : A \to B$ is a **natural transformation** *between the functors A and B.*

2. *Let A and B be sheaves on X. A morphism*

$$h : A \to B$$

is a continuous map $h : \mathcal{A} \to \mathcal{B}$, such that

$$h(\mathcal{A}_x) \subset \mathcal{B}_x$$

for all $x \in X$ and the restrictions of h to the stalks are group homomorphisms.

These two definitions are related in the way, that a morphism of presheaves $h : A \to B$ induces a morphism of sheaves $h : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are constructed out of A and B by the process of sheafification. This is done by passing to direct limits. On the other side, any morphism of sheaves $h : \mathcal{A} \to \mathcal{B}$ induces a morphism on presheaves, by passing to sections.

We will now proceed by defining subsheaves and quotient sheaves as well as images and kernels. The category of sheaves in fact will turn out to be an abelian category and methods from homological algebra can be applied.

Definition 2.1.4. 1. *A* **subsheaf** *$\mathcal{A}$ of a sheaf $\mathcal{B}$ on X is an open subspace of $\mathcal{B}$, such that $\mathcal{A}_x = \mathcal{A} \cap \mathcal{B}_x$ is a subgroup of $\mathcal{B}_x$ for all $x \in X$. It is then clear, that $\mathcal{A}$ is a sheaf on X with its induced structure.*

2. *Let $\mathcal{A}$ be a subsheaf of $\mathcal{B}$. We define the* **quotient sheaf** $\mathcal{B}/\mathcal{A}$ *as the sheafification of the presheaf which associates to an open subset $U of X$ the abelian group $\mathcal{B}(U)/\mathcal{A}(U)$.*

Definition 2.1.5. 1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a morphism of sheaves. We define the* **kernel** *of h to be*

$$ker(h) := \{\alpha \in \mathcal{A} | h(\alpha) = 0\}.$$

This is a subsheaf of $\mathcal{A}$. On the other side, it is clear that the **image** *of h*

$$im(h) = \{h(\alpha) | \alpha \in \mathcal{A}\} \subset \mathcal{B}$$

is a subsheaf of $\mathcal{B}$.

2. *We call a sequence*

$$\mathcal{A} \xrightarrow{f} \mathcal{B} \xrightarrow{g} \mathcal{C}$$

of morphisms of sheaves **exact** *, if $im(f) = ker(g)$.*

Given a sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of morphisms of presheaves, sheafification yields to a corresponding sequence of sheafs. It can be seen, that this sequence is exact, if and only if the sequence of presheaves above is exact on stalks. Since passing to direct limits is exact this for example is the case, if for each open subset $U \subset X$ the sequence

$$A(U) \xrightarrow{f_U} B(U) \xrightarrow{g_U} C(U)$$

is exact. The latter condition though is not a necessary condition for a sequence of sheaves to be exact.

Now let X and Y be two topological spaces and $f : X \to Y$ be a map. Let $\mathcal{A}$ be a sheaf on X and $\mathcal{B}$ be a sheaf on Y. Given these data, one can construct two new sheaves as follows.

Definition 2.1.6. 1. *In the situation above we define the* **direct image** *of $\mathcal{A}$ under f to be the sheaf on Y which associates to each open subset $U \subset X$ the abelian group $\mathcal{A}(f^{-1}(U)$. The two conditions in Definition 2.1.2 can be easily verified. We denote this sheaf with $f\mathcal{A}$.*

2. *We define the* **inverse image** *of $\mathcal{B}$ under f to be the sheaf on X, given by the pullback*

$$f^*\mathcal{B} = \{(x, b) \in X \times \mathcal{B} | f(x) = \pi(b)\}.$$

It is easy to check, that this is indeed a sheaf.

2.2 Supports

In this section we discuss supports of sections in sheaves. For most of our purposes, we can restrict our attention to arbitrary in equal closed supports or compact supports. In two cases though the situation is more delicate and we will give a general treatment here. Let X be an arbitrary topological space.

Definition 2.2.1. *A family of* **supports** *on X is a family ϕ of closed subsets of X such that*

1. *Any closed subset of a member of ϕ is again a member of ϕ.*
2. *The family ϕ is closed under finite unions.*

A family of supports ϕ is called **paracompactifying** if each element of ϕ is paracompact and has a closed neighbourhood which is in ϕ. The two most important families of supports are the family of closed and the family of compact supports on X. In general it is unclear, whether these systems are paracompactifying. For the family of compact supports, a sufficient condition on the space is to be locally compact. Since stratifolds are by construction locally compact and by Corollary 1.7.2 paracompact and furthermore closed subsets of paracompact spaces are paracompact the following proposition holds.

Proposition 2.2.1. *Let X be a stratifold. Then the family of closed as well as the family of compact supports are paracompactifying.*

Let $Y \subset X$ be a subset and ϕ a system of supports on X. Then we define a system of supports on Y

$$\phi_{|Y} = \{K \subset Y | K \in \phi\}. \tag{2.3}$$

Now let $\mathcal{A}$ be a sheaf on X and $s \in \mathcal{A}(X)$ be a **global section**. We define the **support** of s as

$$supp(s) = \{x \in X | s_x \neq 0\} \tag{2.4}$$

and denote this set with $|s|$. This set is already closed, since it's complement is open, which can be verified easily. Now let ϕ be a family of supports. The global sections of $\mathcal{A}$ with supports in ϕ are denoted with $\Gamma_\phi(\mathcal{A})$. If ϕ is the family of compact supports we also write $\Gamma_c(\mathcal{A})$ for the section with compact supports. If ϕ is the family of all closed subsets of X we simply write $\Gamma(\mathcal{A})$. The functor

$$\mathcal{A} \to \Gamma_\phi(\mathcal{A})$$

from the category of sheaves to the category of abelian groups is left exact. In general this functor is not right exact. In fact the right derived functors lead to sheaf theoretic cohomology, which will be treated in Section 2.3 .

Definition 2.2.2. *Let ϕ be a family of supports. A sheaf $\mathcal{A}$ on X is called ϕ-***soft***, if the restriction map $\mathcal{A}(X) \to \mathcal{A}(K)$ for any $K \in \phi$ is surjective. Here $\mathcal{A}(K)$ is defined as*

$$\mathcal{A}(K) = \varinjlim_{K \subset U} \mathcal{A}(U).$$

The importance of the following proposition will only show up in the next chapter. Nevertheless we think it might be helpful to state it at this place, since it also delivers a good example for a soft sheaf.

Proposition 2.2.2. *Let X be a stratifold. And let ϕ be either the family of all closed subsets or the family of all compact subsets of X. Then the structure sheaf $\mathcal{O}_X$ of X is ϕ-soft.*

Proof. Let $A \subset X$ be a closed subset and $f \in \mathcal{O}_X(K)$ be defined on an open neighbourhood U of A. Since by Corollary 1.7.2 as a space X is paracompact, we can find an open neighbourhood $V \subset U$ of A such that the closure $\bar{V}$ of V is still contained in U. This follows for example from [Bredon97] page 20 applied to the open covering $X = U \cup (X - A)$ of X. Now by applying Corollary 1.7.1 we can find a partition of unity subordinated to the open covering of X given by

$$X = U \cup (X - \bar{V}).$$

In this way we get smooth functions α and β on X, such that

$$\alpha + \beta \equiv 1,$$

$$supp(\alpha) \subset U, supp(\beta) \subset X - \bar{V}.$$

From this, it is clear that $\alpha_{|V} \equiv 1$ and hence

$$\bar{f} = f \cdot \alpha$$

is a well defined smooth function on X such that $\bar{f}_{|V} = f_{|V}$, wich shows the claimed surjectivity in the definition above. $\square$

Proposition 2.2.3. *Let ϕ be a paracompactifying family of supports and let*

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$$

be an exact sequence of sheaves. Suppose that $\mathcal{A}'$ is soft. Then the sequence of global sections with support in ϕ

$$0 \to \Gamma_\phi(\mathcal{A}') \to \Gamma_\phi(\mathcal{A}) \to \Gamma_\phi(\mathcal{A}'') \to 0$$

is also exact.

Proof. [Bredon97] page 67. $\square$

Similar to sheafs with values in the category of graded abelian groups one can consider sheafs with values in the category of rings, algebras etc. . Almost any algebraic construction can be carried over to sheaves. We start with the notion of a module.

Definition 2.2.3. *Let $\mathcal{A}$ be a sheaf of rings on X. We call a sheaf $\mathcal{B}$ over X a* **module** *over $\mathcal{A}$, if for each open set $U \subset X$ the abelian group $\mathcal{B}(U)$ is equipped with a module structure over $\mathcal{A}(U)$ such that the restriction maps are module homomorphisms.*

Example 2.2.1. *Let M be a smooth manifold. Then the structure sheaf $\mathcal{O}_M$ is a sheaf of rings. The sheaf Ω^n_M of differential forms on M is a module over $\mathcal{O}_M$.*

The following proposition is very important, when we consider the sheaf of differential forms on a stratifold. For a proof see [Bredon97] page 69.

Proposition 2.2.4. *Let ϕ be a paracompactifying family of supports, then any module over a ϕ-soft sheaf is again a ϕ-soft sheaf.*

2.3 Sheaf Theoretic Cohomology Theory

Sheaf theoretic cohomology is a very important tool in algebraic geometry and some other parts of mathematics. For a given sheaf we give an explicit resolution to define its cohomology. Thus we avoid homological algebra terms and don't bother to define things like injective sheafs etc. Throughout this section let ϕ be a family of supports

Let $\mathcal{A}$ be a sheaf on X. For an open subset $U \subset X$ let

$$C^0(U, \mathcal{A}) := \{f : U \dashrightarrow \mathcal{A} | \pi \circ f = id\} \tag{2.5}$$

denote the set of not necessarily continuous sections from U into $\mathcal{A}$.Such not necessary continuous sections are called **serrations**. An alternative way is to say

$$C^0(U, \mathcal{A}) = \prod_{x \in U} \mathcal{A}_x.$$

The association

$$U \to C^0(U, \mathcal{A})$$

defines a sheaf on X which we denote with $\mathcal{C}^0(X, \mathcal{A})$. Since each continuous section can also be considered as a serration we have an inclusion

$$\mathcal{A}(U) \to C^0(U, \mathcal{A}) = \mathcal{C}^0(X, \mathcal{A})(U)$$

and hence a natural monomorphism

$$\epsilon : \mathcal{A} \to \mathcal{C}^0(X, \mathcal{A}).$$

We define

$$\mathcal{Z}^1(X, \mathcal{A}) = coker(\epsilon : \mathcal{A} \to \mathcal{C}^0(X, \mathcal{A})).$$

In this way we get an an exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{\epsilon} \mathcal{C}^0(X, \mathcal{A}) \xrightarrow{\partial} \mathcal{Z}^1(X, \mathcal{A}) \longrightarrow 0\ .$$

Inductively we define

$$\mathcal{C}^n(X, \mathcal{A})) = \mathcal{C}^0(X, \mathcal{Z}^n(X, \mathcal{A})) \tag{2.6}$$

$$\mathcal{Z}^{n+1}(X, \mathcal{A}) = \mathcal{Z}^1(X, \mathcal{Z}^n(X, \mathcal{A})). \tag{2.7}$$

In this way we get exact sequences of the form

$$0 \longrightarrow \mathcal{Z}^n(X, \mathcal{A}) \xrightarrow{\epsilon} \mathcal{C}^n(X, \mathcal{A}) \xrightarrow{\partial} \mathcal{Z}^{n+1}(X, \mathcal{A}) \longrightarrow 0 .$$

By splicing these sequences together we get a long exact sequence

$$0 \longrightarrow \mathcal{A} \xrightarrow{\epsilon} \mathcal{C}^0(X, \mathcal{A}) \xrightarrow{d} \mathcal{C}^1(X, \mathcal{A}) \xrightarrow{d} \mathcal{C}^2(X, \mathcal{A}) \xrightarrow{d} \cdots ,$$

where $d = \epsilon \circ \partial$. It is an easy exercise to show that this sequence is exact. So we end up with what we call the **canonical resolution** of the sheaf $\mathcal{A}$. Any other exact sequence of the form above, where $\mathcal{C}^i(X, \mathcal{A})$ is replaced by some sheaves $\mathcal{L}^i$ is called a **resolution** of $\mathcal{A}$.

Let us proceed with constructing a chain complex from this resolution. We define

$$C^n_\phi(X, \mathcal{A}) := \Gamma_\phi(\mathcal{C}^n(X, \mathcal{A})) \tag{2.8}$$

and can now present the definition of **sheaf theoretic cohomology groups** of a space X with coefficients in the sheaf $\mathcal{A}$.

Definition 2.3.1. *Let X be a topological space and let $\mathcal{A}$ be a sheaf over X. The cohomology groups of X with coefficients in the sheaf $\mathcal{A}$ and supports in ϕ are defined as*

$$H^n_\phi(X, \mathcal{A}) = \frac{ker(d : C^n_\phi(X, \mathcal{A}) \to C^{n+1}_\phi(X, \mathcal{A}))}{im(d : C^{n-1}_\phi(X, \mathcal{A}) \to C^n_\phi(X, \mathcal{A}))}.$$

In general we suppress the index ϕ if ϕ denotes the system of closed supports.

In fact the homology groups above are the right derived functors of the left exact functor Γ_ϕ and to define sheaf cohomology, we could have chosen any injective resolution of $\mathcal{A}$ instead of the canonical resolution. The resulting cohomology groups would have been the same. This would require more homological algebra though, so we stay with this very concrete definition.

Since Γ_ϕ is left exact, we have an exact sequence

$$0 \to \Gamma_\phi(\mathcal{A}) \to \Gamma_\phi(\mathcal{C}^0(X, \mathcal{A})) \to \Gamma_\phi(\mathcal{C}^1(X, \mathcal{A})),$$

so by definition of the cohomology groups and (2.8) there is a natural isomorphism

$$\Gamma_\phi(\mathcal{A}) \cong H^0_\phi(X, \mathcal{A}).$$

Hence we see that the cohomology classes of dimension 0 are precisely the global sections of the sheaf with support in ϕ.

From the definition of $C^n_\phi(X, \mathcal{A})$ it can easily be seen, that as a functor on sheaves $C^n_\phi(X, -)$ is exact. So if we start with an exact sequence of sheaves

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0,$$

we get a long exact sequence of cohomology groups

$$... \to H^p_\phi(X, \mathcal{A}') \to H^p_\phi(X, \mathcal{A}) \to H^p_\phi(X, \mathcal{A}'') \to H^{p+1}_\phi(X, \mathcal{A}')...,$$

from which things like the **Mayer-Vietoris sequence** and **excision** can be followed.

The most famous example for sheaf theoretic cohomology is probably the Čech-cohomology of a space X, which in case X is a nice space coincides with the singular cohomology.

Example 2.3.1. *Let X be a topological space and G an abelian group.Let $\mathcal{G}$ be the constant sheaf with value G on X. Then $H^p_\phi(X, \mathcal{G})$ are called the Čhech-cohomology groups of X.*

2.4 Acyclic Sheaves

Acyclic sheaves over a space X are objects with trivial cohomology. More precisely we say :

Definition 2.4.1. *Let $\mathcal{A}$ be a sheaf over some space X and ϕ a family of supports. We call $\mathcal{A}$ ϕ-**acyclic** if*

$$H^p_\phi(X, \mathcal{A}) = 0, \forall p > 0.$$

The following proposition will be fundamental in chapter 4 when proving de Rham's theorem for stratifolds. There it will be applied on the complex of sheaves Ω^*_X of differential forms on a stratifold X.

Proposition 2.4.1. *Let X be a topological space and*

$$0 \longrightarrow \mathcal{A} \xrightarrow{\epsilon} \mathcal{L}^0 \xrightarrow{d} \mathcal{L}^1 \xrightarrow{d} \mathcal{L}^2 \xrightarrow{d} \cdots$$

be a resolution of $\mathcal{A}$ by ϕ-acyclic sheaves, then there is a natural isomorphism

$$H^p(\Gamma_\phi(\mathcal{L}^*), d) \cong H^p_\phi(X, \mathcal{A}).$$

Proof. The proof is very easy and follows directly from the last part of section 2.2. The proof can also be found in the book of Bredon [Bredon97] page 47. $\square$

For our purposes the following proposition is of major importance. The proof can be found in [Bredon97] page 68 but uses the concept of flabby sheaves, which we don't introduce here.

Proposition 2.4.2. *Let ϕ be a paracompactifying system of supports and X be a topological space. Furthermore let $\mathcal{A}$ be a ϕ-soft sheaf over X. Then $\mathcal{A}$ is ϕ-acyclic.*

2.5 Relative Sheaf Cohomology

As in almost any cohomology theory, there is also a relative version of sheaf theoretic cohomology. Let Y be a subspace of the topological space X and let

$$i : Y \to X$$

be the inclusion. This induces a morphism of sheaves

$$i^* : \mathcal{C}^*(X, \mathcal{A}) \to i\mathcal{C}^*(Y, \mathcal{A}_{|Y}),$$

where the right hand side denotes the direct image under i.We define a new complex of sheaves as

$$\mathcal{C}^*(X, Y, \mathcal{A}) = ker\ i^*.$$

From this we get a chain complex defining

$$C^*_\phi(X, Y, \mathcal{A}) = \Gamma_\phi(\mathcal{C}^*(X, Y, \mathcal{A})), \tag{2.9}$$

for any family of supports ϕ, where the differential is given by the the restriction of the differential on $\mathcal{C}^*_\phi(X, \mathcal{A})$. The relative version of homology is now defined as follows.

Definition 2.5.1. *Let $\mathcal{A}$ be a sheaf over the topological space X, ϕ a family of supports on X, and let $Y \subset X$ be a subspace. Then $\forall n \in \mathbb{N}$ we define*

$$H^n_\phi(X, Y, \mathcal{A}) = \frac{ker(d : C^n_\phi(X, Y, \mathcal{A})) \to C^{n+1}_\phi(X, Y, \mathcal{A}))}{im(d : C^{n-1}_\phi(X, Y, \mathcal{A})) \to C^n_\phi(X, Y, \mathcal{A}))}.$$

As usual there are some strong relations between absolute and relative cohomology, though in general one has to be careful on the choice of systems of supports. We will only use the following exact sequence of pairs and some version of excision in sheaf cohomology, which we will state afterward.

Proposition 2.5.1. *Let ϕ be a paracompactifying family of supports. Under the assumptions above there is a long exact sequence of cohomology groups*

$$... \to H^p_\phi(X, Y, \mathcal{A}) \to H^p_\phi(X, \mathcal{A}) \to H^p_\phi(Y, \mathcal{A}_{|Y}) \to H^{p+1}_\phi(X, Y, \mathcal{A}) \to ...$$

Proof. The proof is easy, nevertheless uses the concept of flabby sheaves. It can be found in [Bredon97] page 84. ☐

Proposition 2.5.2. *If in addition to the assumptions above, the space Y is a closed subspace of X, then there is a natural isomorphism*

$$H^p_\phi(X, Y, \mathcal{A}) \cong H^p_{\phi_{|X-Y}}(X - Y, \mathcal{A}).$$

Proof. The proof of this statement can be found in [Bredon97] on page 87. ☐

Kapitel 3

Constructions from Algebra

In this short chapter we present some of the algebraic tools we use in the later chapters. Probably most of the readers are well acquainted to things like localization or the local global principle. Not so well known are algebraic differential forms. Throughout the chapter R denotes a commutative ring. In general we do not suppose that this ring R has a unit. If so we denote this unit with 1_R. If R has also the structure of an algebra over a field k, we switch symbols and denote it with A. Again we do not suppose that A has a unit but we concentrate on the case, where the underlying field k has characteristic zero. Later A will be the algebra $C^\infty(X)$ of smooth functions on a stratifold X or some related algebra and k will be the real or complex numbers.

3.1 Localization

Let $S \subset R$ be a multiplicative subset of the ring R, that is S is closed under multiplication. Let M be a module over R. We define the **localization** of M at S, denoted M_S, as the set of equivalence classes

$$(m, s) \in M \times S$$

under the equivalence relation

$$(m, s) \sim (m', s'),$$

whenever there is an element $t \in S$ such that

$$t \cdot (s'm - sm') = 0.$$

The equivalence class of (m, s) will be denoted with $\frac{m}{s}$. Such equivalence classes build an abelian group with addition given by

$$\frac{m}{s} + \frac{m'}{s'} = \frac{s'm + sm'}{ss'}.$$

Application on $M = R$ will give us a ring R_S with multiplication

$$\frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}.$$

M_S then becomes a module over R_S in a natural way. Furthermore let

$$\phi : M \to N$$

be a homomorphism of R-modules, then we get an homomorphism of R_S-modules

$$\phi_S : M_S \to N_S,$$

$$\frac{m}{s} \mapsto \frac{\phi(m)}{s}.$$

It is not hard to see that

$$M_S \cong R_S \otimes_R M.$$

In practice localisations are often done at maximal ideals. For example let $P \subset R$ be a maximal ideal of R, then

$$S = R - P$$

is a multiplicative subset of R. We call $M_P = M_S$ the localization of M at P.

The following lemma gives a nice example of localization and in addition shows how localization helps to understand the local situation, for example in case of a stratifold X.

Lemma 3.1.1. *Let X be a compact stratifold and $C^\infty(X)$ its algebra of smooth functions. Every maximal ideal P in $C^\infty(X)$ is of the form*

$$P = ker(ev_x : C^\infty(X) \to \mathbb{R}),$$

where ev_x denotes the evaluation map at some point $x \in X$. Furthermore

$$C^\infty(X)_P \cong \mathcal{O}_{X,x}.$$

Proof. Clearly for any $x \in X$ we have that $ker(ev_x)$ is a maximal ideal. Now let P be any maximal ideal in $C^\infty(X)$. Assume that P is not of the kind described in the proposition. Then for any $x \in X$ there exists $f_x \in P$ such that $f_x(x) \neq 0$. Clearly we can assume $f_x(x) > 0$. It is not hard to see, that by compactness of X one can use a finite number of these functions and paste them together with a partition of unity to get a function $f \in P$ such that $f(x) > 0 \ \forall x \in X$. But then we have that $1/f$ is a well defined function in $C^\infty(X)$ and since P is an ideal $1 = (1/f) \cdot f \in P$, which is a contradiction to P being a maximal ideal. For the second assertion let f/g denote an element in $C^\infty(X)_P$. Then there is a neighbourhood U of x such that $g_{|U}$ has no zeroes. Hence the function f/g is well defined on U and we can consider its equivalence class $[f/g] \in \mathcal{O}_{X,x}$. The association $f/g \mapsto [f/g]$ is clearly surjective. It is also injective. If f/g maps to zero, then for some open neighbourhood V of x we have $f_{|V} = 0$. It follows from Proposition 1.7.1 that we can find $\rho \in C^\infty(X)$ such that $\rho(x) \neq 0$ and $supp(\rho) \subset V$. Hence in $C^\infty(X)_P$ we have

$$f/g = 1/\rho \cdot \rho \cdot f/g = 1/\rho \cdot (\rho \cdot f)/g = 0$$

since $\rho \cdot f$ is identical zero. □

In some cases, the localization of a module is much easier to handle, because it has some good properties. For example it might turn out that some localization of a module is a free module over the localized ring. The following definition is a special case of this.

Definition 3.1.1. *Let M be a module over the ring R. We call M a locally free module, if for any maximal ideal P of R the localization M_P is free over R_P.*

In our case it will turn out that certain modules of differential forms will be locally free over the algebra of smooth functions on a stratifold.

3.2 The Local Global Principle

The local global principle is somehow a bridge between the local and the global pictures. If we know a ring localized at any maximal ideal P it should somehow be possible to identify the ring itself. In some sense this is what the following proposition states.

Proposition 3.2.1. *Let $\phi : M \to N$ be a homomorphism of R-modules such that for any maximal ideal P of R the localized map $\phi_P : M_P \to N_P$ is a mono-, respectively epi-, respectively isomorphism , then ϕ itself is a mono-, respectively epi-, respectively isomorphism.*

Proof. The Proof of this proposition can be found in [Eisenbud95] page 68. □

3.3 Adjunction of a Unit

Some of our algebras doesn't come with a unit element, in equal an element $1_A \in A$ which satisfies $1_A \cdot a = a \cdot 1_A = a \; \forall a \in A$. For any k algebra A we can consider

$$A_+ := A \oplus k \tag{3.1}$$

as a vectorspace over k and define a multiplication on this vectorspace via

$$(a, r) \cdot (b, s) := (ab + rb + sa, rs). \tag{3.2}$$

Together with this multiplication A_+ is a commutative k-algebra with unit given by the element $(0, 1)$. We get the following short exact sequence of k-algebras

$$0 \longrightarrow A \xrightarrow{\varphi} A_+ \xrightarrow{\epsilon} k \longrightarrow 0\ ,$$

where the maps are given as

$$\varphi(a) := (a, 0), \forall a \in A$$

$$\epsilon(a, r) := r, \forall (a, r) \in A_+.$$

In general this sequence is not a split exact sequence of algebras. In the case we already started with a unital algebra there is a splitting given by the map

$$\delta : A_+ \to A$$

$$\delta(a, r) := a + r \cdot 1_A.$$

In this case we have an isomorphism of algebras $A_+ = A \oplus k$. Identifying a and $(a, 0)$ for all $a \in A$ we consider A as a subset of A_+. Using this identification we have that A is a maximal ideal in A_+.

3.4 Derivations

Since derivations play a major role in the following chapters, we briefly recount the definition and some elementary properties. For this let M be a bimodule over A which is also a vectorspace over k such that multiplication with elements of A and of k is associative. If A has a unit, M inherits the k-vectorspace structure from A using the map

$$r \mapsto r \cdot 1_A, \forall r \in k.$$

Definition 3.4.1. *We denote the set of k-linear maps*

$$D : A \to M,$$

which satisfy the **Leibniz rule**

$$D(ab) = (Da)b + a(Db)$$

with $Der(A, M)$ and call it derivations of A with values in M. In case $M = A$ is the regular module over A, we write $Der(A) = Der(A, A)$.

In the unital case, we always assume that the unit 1_A acts as the identity on M. The Leibniz rule then implies that $D(1_A) = 0$, hence because of k-linearity

$$D(\lambda \cdot 1_A) = 0, \forall \lambda \in k. \tag{3.3}$$

Later when we deal with Hochschild and Cyclic Homology, topological algebras will occur. It then makes sense to speak of continuous derivations. We don't use an extra symbol, but say so, if we require derivations to be continuous. Anyway, in the end it will turn out that in the case we are interested in, that is $A = C^\infty(X) = M$, there are no non-continuous derivations. The situation there is similar to the manifold case, where any derivation can be represented by a smooth vectorfield, hence is continuous.

3.5 Differential Forms for Algebras

In this section we generalize the concept of differential forms, as known in the world of smooth manifolds, to arbitrary commutative algebras. In this section, for a matter of simplicity we assume all algebras to be unital. Unlike for Hochschild homology there is not much about differential forms for nonunital algebras in the literature. Nevertheless nonunital versions of the stuff presented in this section are possible, though a little bit technical. There are also versions working in the noncommutative case (see [Loday91], page 82).

Definition 3.5.1. *Let A be a commutative, unital algebra over k. We denote with $F(A)$ the free A-module generated in symbols da $\forall a \in A$ and with $R(A)$ the submodule which is generated by the elements of the form*

$$d(ab) - adb - bda \ \forall a, b \in A.$$

We define the A-module of differential 1-forms or **Kaehler differentials** *on A as*

$$\Omega^1_A = F(A)/R(A).$$

This A module has a universal property which is closely connected to what we have done in the previous section.

Proposition 3.5.1. *Let M be a bimodule over A and let $D : A \to M$ be any M valued derivation on A. Furthermore let $d : A \to \Omega^1_A$ denote the map, which associates to $a \in A$ the class of da in Ω^1_A. Then there is unique A-linear map $f : \Omega^1_A \to M$ such that the following diagram commutes*

$$\begin{tikzcd} A \arrow[r, "D"] \arrow[d, "d"] & M . \\ \Omega^1_A \arrow[ur, "f"] & \end{tikzcd}$$

Hence there is an isomorphism

$$Der(A, M) \cong Hom_A(\Omega^1_A, M).$$

Proof. This is clear from the construction of Ω^1_A. □

We will now give a second construction of Ω^1_A. Its strength will show up, when defining topological versions of the stuff presented in this section (see chapter 5). Let us consider the multiplication map

$$A \otimes A \to A$$

$$a \otimes b \to ab.$$

We denote the kernel of this map with I and consider it as a module over A. Let us also consider the ideal I^2 and finally the quotient I/I^2. This will be our candidate for Ω^1_A. Clearly, as an A-module I is generated by elements of the form

$$1_A \otimes a - a \otimes 1_A \ \forall a \in A. \tag{3.4}$$

Let us denote classes of elements of the form (3.4) modulo I^2 as

$$[1_A \otimes a - a \otimes 1_A].$$

The following proposition gives answer on how this is related with Ω^1_A.

Proposition 3.5.2. *Let A be a unital commutative algebra. Then there is a natural isomorphism*

$$\Omega^1_A \to I/I^2$$

$$da \mapsto [1_A \otimes a - a \otimes 1_A].$$

Proof. [Loday91] page 26. ☐

We can now define higher differential forms by using the exterior algebra construction.

Definition 3.5.2. *For a unital commutative algebra A, let*

$$\Omega^n_A = \Lambda^n_A \Omega^1_A$$

be the A module of differential n- forms over A.

This module has a universal property, which can simply be obtained by composing the two universal properties of the exterior product construction and differential 1-forms and has something to do with alternating forms on derivations. We don't go into this in detail.

As the following proposition shows. The process of building differential forms is compatible with the process of localization. This fact proves very useful in calculations.

Proposition 3.5.3. *Let A be a unital commutative algebra and P be a maximal ideal in A. Then there is a natural isomorphism of modules over A_P*

$$\Omega^n_{A_P} \cong (\Omega^n_A)_P.$$

Proof. Since the process of building alternating algebras is compatible with localization, we can assume $n = 1$. In this case, it is not hard to see, that $(\Omega^n_A)_P$ has the universal property of Proposition 3.5.1. for A_P from which the proposition follows. See also [Weibel95] page 307 . ☐

That algebraically defined differential forms are in fact a generalization of the concept of differential forms on manifolds as one can find in [Bredon91] on page 261 for example, is the content of the following proposition.

Proposition 3.5.4. *Let M be a compact smooth manifold and $A = C^\infty(M)$ denote the algebra of smooth functions on M. Furthermore let $\Omega^n(M)$ denote the module of differential n-forms on M Then there is a natural isomorphism*

$$\Omega^n_{C^\infty(M)} \cong \Omega^n(M).$$

Proof. Clearly there is a well defined map

$$\Omega^n_{C^\infty(M)} \to \Omega^n(M)$$

$$f_0 df_1 ... df_n \mapsto f_0 df_1 ... df_n,$$

where the left hand expressions is understood as an algebraic differential form, whereas the right hand expression stands for the alternating n-fold product of the n differential one forms $df_1, ... df_n$ and the smooth function f_0. This map is an isomorphism. To check this, using the local global principle (Proposition 3.2.1) together with Proposition 3.5.3 and Lemma 3.1.1 one can assume that $M = \mathbb{R}^k$. In this case the proposition is clearly true. $\square$

Kapitel 4

de Rham Theory of Stratifolds

In this chapter we will generalize concepts like vectorfields and de Rham cohomology which are well known in the world of smooth manifolds to the world of stratifolds. The main result of this chapter is the generalization of de Rham's theorem to the case of stratifolds, namely that the de Rham cohomology with compact support of a stratifold X is naturally isomorphic to its real valued singular cohomology with compact support. We present a very concrete and geometric isomorphism, which is given by integrating differential forms over homology classes.

4.1 Tangentspaces

Let X be a stratifold. As before, we denote with $\mathcal{O}_X$ the structure sheaf of X. Let $x \in X$ be a point and $\mathcal{O}_{X,x}$ the stalk of $\mathcal{O}_X$ at x. The following definition of the tangent space of X at the point x has to be seen in complete analogy to the case of a smooth manifold.

Definition 4.1.1. *We define the* **tangent space** *of the stratifold X at some point $x \in X$ as*

$$T_xX = Der(\mathcal{O}_{X,x}, \mathbb{R}).$$

Clearly T_xX is a vectorspace over the real numbers. From Proposition 1.3.1 it follows, that if x lies in the k-th stratum S_k of X, we have

$$\mathcal{O}_{X,x} \cong \mathcal{O}_{S_k,x},$$

where $\mathcal{O}_{S_k}$ denotes the structure sheaf of the k stratum. The isomorphism is given by restriction of germs to the k-stratum. We follow that

$$Der(\mathcal{O}_{S_k,x}, \mathbb{R}) \cong Der(\mathcal{O}_{X,x}, \mathbb{R})$$

induced by the inclusion of the k-stratum. So we have proven the following easy, but nevertheless important proposition.

Proposition 4.1.1. *Let X be a stratifold and $x \in S_k \subset X$. Then the inclusion of the k-th stratum induces an isomorphism*

$$T_x S_k \cong T_x X.$$

In particular the dimension of $T_x X$ is equal to k.

Clearly, if we have a smooth map $f : X \to Y$ between two stratifolds, we get an induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$. This map then induces a map

$$f_* : Der(\mathcal{O}_{X,x}, \mathbb{R}) \to Der(\mathcal{O}_{Y,f(x)}, \mathbb{R}),$$

which is the same as a map

$$f_* : T_x X \to T_{f(x)} Y. \tag{4.1}$$

We call f_* the tangential of f at the point x. It is clear that f_* is a vectorspace homomorphism.

In the proposition above no restrictions on the dimension of the stratifold are needed. So, even an infinite dimensional stratifold has finite dimensional tangentspaces. The dimension of the tangentspace depends on the stratum which contains the point. In anyway the dimension of the tangentspaces varies. This is one major difference to the world of manifolds. It makes it somehow difficult to define something as the tangentbundle by pasting together the tangentspaces in a certain way. We don't bother to define some alternative of the tangentbundle and come right away to vectorfields. In the manifold case, smooth sections of the tangentbundle are in one to one correspondence to the derivations of the algebra of smooth functions. This will be our start point in the next section.

4.2 Derivations and Vectorfields

Definition 4.2.1. *For a stratifold X we denote with $Der(X)$ the derivations of the algebra $C^\infty(X)$.*

So far nothing has been said about the topology of $C^\infty(X)$ and derivation here just means derivation, we don't require anything as continuous here. Later we will introduce a topology on $C^\infty(X)$ and in analogy to the manifold case it will become clear, that any derivation of $C^\infty(X)$ is automatically continuous.

We should begin with a property of $Der(X)$ we call **locality**, which means that for any $D \in Der(X)$ and $f \in C^\infty(X)$ the value of Df at some point $x \in X$ only depends on the behaviour of f in a small neighbourhood of x. We all know this behaviour of derivations from the world of manifolds, and the reason this is valid for stratifolds as well is Lemma 1.7.1

Proposition 4.2.1. *Let $D \in Der(X)$ and $f, g \in C^\infty(X)$. Let $x \in X$ be some point and U be an open neighbourhood of x in X such that $f_{|U} = g_{|U}$. Then*

$$Df(x) = Dg(x).$$

Proof. We have $(f-g)_{|U} \equiv 0$. According to Lemma 1.7.1 we can choose $\rho \in C^\infty(X)$ such that $\rho(x) = 1$ and $supp(\rho) \subset U$. Then $0 \equiv \rho \cdot (f-g)$ on the whole of X, hence

$$0 = D(\rho \cdot (f-g)) = D\rho \cdot (f-g) + \rho \cdot D(f-g).$$

Evaluation at x shows that $Df(x) = Dg(x)$.

$\square$

We will give an explicit description of $Der(X)$ in form of vectorfields on the strata R_k of the stratifold X. Let us denote the vectorfields on R_k with $\Gamma(R_k)$. We should remind the reader at this point, that since we are working with c-manifolds, for $x \in \partial R_k$ we have from Proposition 1.1.1 that

$$\mathcal{O}_{R_k,x} \cong \mathcal{O}_{\partial R_k,x}.$$

Clearly this isomorphism carries over when we consider derivations on $\mathcal{O}_{R_k,x}$. So we have a natural isomorphism

$$T_x R_k \cong T_x \partial R_k.$$

This isomorphism is given by forgetting the component orthogonal to the boundary. One might think that this is a loss of information. It isn't, when we consider vectorfields on R_k, since then, the component orthogonal to the boundary does have an impact on germs, when considered arbitrary close to the boundary, but not on the boundary. Since all our vectorfields are assumed to be smooth, the behaviour close to the boundary uniquely determines the behaviour on the boundary. Another point to keep in mind is that an arbitrary vectorfield on R_k in general won't deliver a derivation of $C^\infty(R_k)$.

This is because functions in $C^\infty(R_k)$ have to satisfy that extra condition to be constant along the collar close to the boundary. To get derivations from vectorfields on R_k, we must require some condition on the components orthogonal to the boundary, which guarantees the condition to be constant along the collar. This will be expressed in condition 2. of Definition 4.2.2 .

Definition 4.2.2. *Let X be a stratifold with charts φ_k and strata R_k for $k \in I$. We define*

$$\Gamma(X) \subset \{\gamma = (\gamma_k)_{k\in I} | \gamma_k \in \Gamma(R_k)\},$$

to be those sequences of vectorfields which satisfy the following two conditions.

1. *For any pair $x \in R_k$ and $y \in R_j$ such that $\varphi_k(x) = \varphi_j(y) = z \in X$*

$$\varphi_{k*}(\gamma_k(x)) = \varphi_{j*}(\gamma_j(y)) \in T_z X.$$

2. *For any k let $p_k : \partial R_k \times [0, \epsilon) \to \partial R_k$ denote the projection from the collar of R_k to the boundary. Then for any $y \in \partial R_k$ the function*

$$[0, \epsilon) \to T_y \partial R_k$$

$$t \mapsto p_{k*}\gamma(y, t)$$

is constant in a small neighbourhood of zero.

$\Gamma(X)$ has a natural structure as a module over $C^\infty(X)$ and will be called the module of **vectorfields** *on the stratifold X.*

We will now recognize Derivations on $C^\infty(X)$ as vectorfields on X. This is the content of the following theorem.

Theorem 4.2.1. *There is an isomorphism of modules over $C^\infty(X)$*

$$Der(X) \cong \Gamma(X).$$

Proof. Let $D \in Der(X)$ be a given derivation and let R_k denote the full strata of X. We define $\gamma_k \in R_k$ as follows. Let $x \in R_k$ and $f_{|x} \in \mathcal{O}_{R_k,x}$ be defined on an open subset U of R_k. The set $V := U \cap R_k^\circ$ is also open and because of the properties of the charts φ_k, we have that $\varphi_k(V)$ is an open subset of S_k.Then

$$g := f \circ \varphi^{-1}_{k|\varphi_k(V)}$$

is a smooth function defined on an open subset of S_k. As in the proof of Proposition 1.3.1 there is a unique way to extend g on an open subset of X. We can then further extend g to the whole of X. Define

$$\gamma_k(x)(f_{|x}) := (Dg)(\varphi_k(x)).$$

This is well defined by the property of locality of D and the explicit extension in a small neighbourhood of $\varphi_k(V)$. To check condition 2 from Definition 4.2.2, we identify $x = (y,t) \in \partial R_k \times [0,\epsilon)$ in a small neighbourhood of the boundary. Let $f \in \mathcal{O}_{\partial R_k, y}$ and let $p : \partial R_k \times [0,\epsilon)$ be the projection on the first coordinate. Then the function $f \circ p$ on R_k is constant in the second variable which means the germ $f_{|(y,t)}$ remains constant when t is changed. This allows us to take one single function $g \in C^\infty(X)$ as above which suits all these germs. We then get

$$p_*\gamma_k(y,t)(f_{|y}) = \gamma_k(y,t)(f \circ p_{|(y,t)}) = (Dg)(\varphi_k(y,t)).$$

The latter function is constant in t for small t, since D is a derivation on $C^\infty(X)$ and hence $Dg \in C^\infty(X)$. If $f_{1|x}$ and $f_{2|x}$ are two elements in $\mathcal{O}_{R_k,x}$ both defined on U, then we have extensions g_1 respectively g_2 and $g_1 \cdot g_2$ coincides with the canonical extension of $f_{1|x} \cdot f_{2|x}$ in a small neighbourhood of $\varphi_k(V)$. Hence

$$\gamma_k(x)(f_{1|x} \cdot f_{2|x}) = D(g_1 \cdot g_2)(\varphi_k(x)) = (Dg_1 \cdot g_2 + g_1 \cdot Dg_2)(\varphi_k(x)),$$

which shows that $\gamma_k(x)$ is a derivation, hence $\gamma_k(x) \in T_x R_k$. This construction gives us vectorfields γ_k on R_k. These vectorfields also satisfy the compatibility condition, which can be seen as follows. Let $z \in X$ and $f_{|z} \in \mathcal{O}_{X,z}$, $x \in R_k$ $y \in R_j$ such that $\varphi_k(x) = \varphi_j(y) = z$. Then by definition of γ_k and γ_j we have

$$\gamma_k(x)(f \circ \varphi_{k|x}) = (Df)(\varphi_k(x)) = (Df)(\varphi_j(y)) = \gamma_j(y)(f \circ \varphi_{j|y}).$$

Hence $\gamma_D = (\gamma_k)_{k\in I}$ is a well defined element in $\Gamma(X)$. The association $D \mapsto \gamma_D$ clearly is a homomorphism of modules over $C^\infty(X)$. On the other side let there be given a vectorfield $\gamma = (\gamma_k)_{k\in I} \in \Gamma(X)$ and let $f \in C^\infty(X)$ be a smooth function on X.Let $x \in X$ be a point and y an arbitrary point in the preimage of x under some φ_k. We define

$$D_\gamma f(x) := \gamma_k(x)(f \circ \varphi_{k|y}).$$

By the compatibility condition of the γ_k this value doesn't depend whether on the choice of $k \in I$ nor on the special choice of $y \in R_k$. The continuity of the function

$$D_\gamma f : x \mapsto D_\gamma f(x)$$

is clear from the topology of X as a quotient space (see Definition 1.2.1 and Definition 1.2.2). It remains to check that $D_\gamma f$ is smooth in our sense. To show this let φ_k be a chart of X. Then

$$x \mapsto (D_\gamma f \circ \varphi_k)(x) = D_\gamma f(\varphi_k(x)) = \gamma_k(x)(f \circ \varphi_{k|x})$$

clearly varies smoothly on $x \in R_k$. Let $x = (y,t) \in \partial R_k \times [0,\epsilon)$ be a point in the collar of R_k. Then

$$(y,t) \mapsto (D_\gamma \circ \varphi_k)(y,t) = \gamma_k(y,t)(f \circ \varphi_{k|(y,t)}),$$

doesn't depend on t for small t because the germ $f \circ \varphi_{k|(y,t)}$ remains constant when the second variable is changed for small t and γ_k satisfies condition 2 of Definition 4.2.2. Hence we have proven that $D_\gamma f \in C^\infty(X)$ and by a trivial argument $D_\gamma \in Der(C^\infty(X))$. By construction it is clear that the maps

$$\gamma \mapsto D_\gamma$$

$$D \mapsto \gamma_D$$

are inverse to each other. Hence the statement of the theorem follows. $\square$

To get a better feeling of how vectorfields or equally derivations on a stratifold look like, we should give an example.This is the most easy example one could think of, nevertheless reflects the situation very well.

Example 4.2.1. *We consider $S^1 = [0,1] \cup_\varphi pt.$ as a two strata stratifold, where $\varphi(0) = \varphi(1) = pt.$ In this case both conditions of Definition 4.2.2 are empty, hence $Vect(S^1) = Vect([0,1])$. The latter can then be identified with smooth functions on the unit interval. This has to be considered as a module over*

$$C^\infty(S^1) = \{f : [0,1] \to \mathbb{R}, f(0) = f(1), f\, constant\ around\ \{0,1\}\}.$$

So, as we can easily see, $Vect(S^1)$ is not finitely generated over $C^\infty(S^1)$. This situation carries over to any stratifold which has singularities. It makes life harder, when one is trying to use theorems of commutative algebra to establish results for vectorfields (or later differential forms), cause most of them only work in the finitely generated case.

4.3 Differential Forms on Stratifolds

In section 3.5 we already introduced differential forms for arbitrary commutative unital algebras. This of course works for the algebra $A = C^\infty(X)$ for a stratifold X. On the other side, this construction is somehow abstract and ungeometric. For this reason we choose the following approach, which is modelled as close as possible on the manifold case. We will later show, that in the compact case both versions of differential forms coincide.

On a manifold M, a differential form is given as a smooth section

$$\omega : M \to \coprod_{x \in M} \Lambda^n T_x^* M = \coprod_{x \in M} Alt^n(T_x M, \mathbb{R}),$$

where the two right hand expressions have been given an appropriate topology and Alt^n denotes the alternating n-forms.. For a stratifold X we mentioned that it is not easy, to give $\coprod_{x \in X} \Lambda^n T_x^* X$ any natural topology. It is well known, that in the manifold case a not necessary continuous section of $\Lambda^n T^* M$ is a differential form if and only if it can locally be represented as a sum of forms

$$f_0 df_1 \wedge ... \wedge df_n,$$

where the f_i are smooth functions on M. This is the start point for our definition of differential forms on stratifolds. Usually we skip the $\wedge$ in our notation.

Given functions $f_0, ...f_n$ on a stratifold X, we define $f_0 df_1 ... df_n$ as the section

$$f_0 df_1 ... df_n : X \to \coprod_{x \in X} \Lambda^n T_x^* X = \coprod_{x \in X} Alt^n(Der(\mathcal{O}_{X,x}), \mathbb{R}) \tag{4.2}$$

$$(f_0 df_1 ... df_n)(x)(D_1, ..., D_n) = \sum_{\pi \in \Sigma_n} (-1)^{sign \pi} f_0(x) D_1 f_{\pi(1)|x} ... D_n f_{\pi(n)|x}, \tag{4.3}$$

where Σ_n denotes the permutations of $\{1, 2, ...n\}$ and $D_i \in Der(\mathcal{O}_{X,x}, \mathbb{R})$ are derivations. For comparison to the manifold case see for example [Bredon91] page 262. We can now define differential forms on stratifolds.

Definition 4.3.1. *Let X be a stratifold. A section*

$$\omega : X \to \coprod_{x \in X} \Lambda^n T_x^* X$$

is called a **differential n-form**, *if for any $x \in X$ there is an open neighbourhood $U \subset X$ and finitely many smooth functions f_i^j defined on U such that*

$$\omega_{|U} = \sum_j f_0^j df_1^j ... df_n^j$$

as defined in (4.2),(4.3) We denote with $\Omega^n(X)$ the set of differential n-forms and consider $\Omega^n(X)$ as a module over $C^\infty(X)$.

When working with non compact stratifolds, we have to pay attention on the supports of certain differential forms. This yields to the following definition.

Definition 4.3.2. *Let $\omega \in \Omega^n(X)$ be a differential n-form on a stratifold X. We call the closure of*

$$\{x \in X | \omega(x) \neq 0\}$$

the **support** *of X and denote it supp(ω). We further define*

$$\Omega_c^n(X) := \{\omega \in \Omega^n(X) | supp(\omega) \text{ is compact } \}$$

to be the module over $C^\infty(X)$ of differential n-forms on X with **compact support**.

By definition it is clear that $\Omega^0(X) = C^\infty(X)$ and $\Omega_c^0(X) = C_c^\infty(X)$ where the latter denotes smooth functions on X with compact support.The geometric meaning of higher order differential forms becomes clearer, when we study the local picture in form of sheaves in section 4.5.

4.4 Functorial Properties of Differential Forms

Since we defined the tangential g_* of a smooth map $g : X \to Y$ between stratifolds it is very easy to see that the association

$$X \mapsto \Omega^n(X)$$

is functorial. In fact this is completely analogous to the manifold case. The map g induces maps

$$g_* : T_x X \to T_{g(x)} Y.$$

Let $\omega \in \Omega^n(Y)$ then we define $g^*\omega \in \Omega^n(X)$ as

$$g^*\omega(x)(D_1, ..., D_n) = \omega(g(x))(g_*D_1, ..., g_*D_n).$$

If ω is locally represented by a sum

$$\omega_{|U} = \sum_j f_0^j df_1^j ... df_n^j,$$

so is $g^*\omega$ by

$$g^*\omega_{|g^{-1}(U)} = \sum_j (f_0^j \circ g) d(f_1^j \circ g) ... d(f_n^j \circ g).$$

4.5 Sheaves of Differential Forms

By the definition of differential forms, it is clear that the association

$$U \mapsto \Omega^n(U)$$

for open subsets U of a stratifold X defines a sheaf on X. We denote this sheaf by Ω^n_X and call it the sheaf of differential forms on X.

Proposition 4.5.1. *Let Φ be either the system of compact or closed supports. Then the sheaf Ω^n_X of differential n-forms on a stratifold X is Φ-soft. In particular it is Φ-acyclic.*

Proof. This is an application of Propositions 2.2.1, 2.2.2 and 2.2.4 on the module Ω^n_X over $\mathcal{O}_X$ □

We should now study the local picture in form of the germs $\Omega^n_{X,x}$ of the sheaf of differential n-forms at some point $x \in X$. The following proposition is a generalisation of Proposition 1.3.1.

Proposition 4.5.2. *Let S_k be the k-stratum of the stratifold X and $x \in S_k$. Then the inclusion*

$$i : S_k \to X$$

induces an isomorphism

$$\Omega^n_{X,x} \cong \Omega^n_{S_k,x},$$

where the right hand side denotes the germ of differential n-forms on the smooth manifold S_k.

Proof. The reason for this to be true is the fact that locally, differential forms are representable by the exterior product of differentials of smooth functions and smooth functions in a close neighbourhood of a stratum are identified by their restriction to the stratum.The latter is the content of Proposition 1.3.1 . To be more precise, we let $n = 1$, $x \in X$ be some point in the k-stratum S_k and $dg \in \Omega^1_{X,x}$ such that

$$i^* dg = d(g \circ i) = 0 \in \Omega^1_{S_k,x}.$$

Then since S_k is a smooth manifold, we have that g is constant in a small neighbourhood V of x in S_k. By Proposition 1.3.1 there is also a small neighbourhood U of x in X, such that g restricted to U is constant. Hence, by definition of dg and (3.3) we have that $dg_{|U} = 0$ so that $dg = 0 \in \Omega^1_{X,x}$ which proves injectivity in the case $n = 1$. Surjectivity is also clear from Proposition 1.3.1 . For general n the proposition follows from the fact, that

$$\Omega^n_{X,x} \cong \Lambda^n_{\mathcal{O}_{X,x}} \Omega^1_{X,x},$$

$$\Omega^n_{S_k,x} \cong \Lambda^n_{\mathcal{O}_{S_k,x}} \Omega^1_{S_k,x},$$

and last but not least

$$\mathcal{O}_{X,x} \cong \mathcal{O}_{S_k,x},$$

where in general $\Lambda^*_{\mathcal{R}}$ denotes the exterior algebra over the ring $\mathcal{R}$.

We should mention at this point, that the same technique used in Lemma 3.1.1 can be used to show that

$$\Omega^n_{X,x} = \Omega^n(X)_P, \tag{4.4}$$

where $P = ker(ev_x : C^\infty(X) \to \mathbb{R})$ indicates localization at P. This for example shows that for a compact stratifold X as a module over $C^\infty(X)$ we have that $\Omega^n(X)$ is locally free, and the local rank is given by $\binom{n}{k}$ for $x \in S_k$. Since $\Omega^n(X)$ is not finitely generated as a module over $C^\infty(X)$ one can not follow from this, that it is projective as a module over $C^\infty(X)$. In fact it is not, since in this case, the local rank would be constant.

$\square$

Since we know, how $\Omega^n(X)$ locally looks like, we can establish the generalization of Proposition 3.5.4 for stratifolds.

Proposition 4.5.3. *Let X be a compact stratifold and $A = C^\infty(X)$ the algebra of smooth functions on X. Then there is a natural isomorphism*

$$\Omega^n_{C^\infty(X)} \cong \Omega^n(X).$$

Proof. The proof is the same as in Proposition 3.5.4 using the local global principle, Proposition 4.5.2 and the remark above. $\square$

4.6 de Rham Cohomology of Stratifolds

To build a version of de Rham cohomology for stratifolds, absolute or sheaf theoretic, we have to define an operator d, which is known as exterior derivation in the manifold case. With our definition of differential forms, to define d becomes very easy.

Definition 4.6.1. *Let X be a stratifold and $\omega \in \Omega^n(X)$ a differential form such that locally*

$$\omega_{|U} = \sum_j f_0^j df_1^j ... df_n^j.$$

Define $d\omega \in \Omega^{n+1}(X)$ as the differential form which is locally represented as

$$(d\omega)_{|U} = \sum_j df_0^j df_1^j ... df_n^j.$$

That $d\omega$ is indeed a well defined differential form is clear from the definition. To show that $d\omega$ doesn't depend on the local representation is somehow technical, and only uses algebraic properties of $\Omega^n(X)$. This can be looked up in the book [Weibel95], page 349. So we get an operator

$$d : \Omega^n(X) \to \Omega^{n+1}(X). \tag{4.5}$$

We call this operator **exterior derivation** in analogy to the manifold case, where it can be defined via the same method used here. Since $d(1_X) = 0$, where 1_X denotes the constant function with value 1 on X, it follows from the definition of d that $d \circ d = 0$. hence we get a chain complex $(\Omega^*(X), d)$ which we call the **de Rham complex** of X.

The following lemma states that d is also well defined, when working with compact supports.

Lemma 4.6.1. *The exterior differential*

$$d : \Omega^n(X) \to \Omega^{n+1}(X)$$

maps $\Omega^n_c(X)$ *into* $\Omega^{n+1}_c(X)$.

Proof. Let $\Omega^n_c(X)$. We have to show that $supp(d\omega)$ is compact. Since we know that $supp(\omega)$ is compact, we can find a finite number of open sets $U_1, ..., U_k$ in X such that ω is locally representable on each of the U_j and

$$supp(\omega) \subset \bigcup_{i=1}^{k} U_i.$$

Since X is locally compact, we can choose each U_i to be relative compact. By definition of d, $d\omega$ is zero outside this union, hence

$$supp(d\omega) \subset \bigcup_{i=1}^{k} U_i \subset \bigcup_{i=1}^{k} \bar{U}_i.$$

This proves that $supp(d\omega)$ is compact. □

One can even show, that d decreases supports, but we won't need this. We are now able to define the de Rham cohomology groups.

Definition 4.6.2. *Let X be a stratifold. For $n \in \mathbb{N}$ we call*

$$H^n_{dR}(X) = \frac{ker(d : \Omega^n(X) \to \Omega^{n+1}(X))}{im(d : \Omega^{n-1}(X) \to \Omega^n(X))}$$

the **de Rham cohomology groups** *of X. We also define*

$$H^n_{dR,c}(X) = \frac{ker(d : \Omega^n_c(X) \to \Omega^{n+1}_c(X))}{im(d : \Omega^{n-1}_c(X) \to \Omega^n_c(X))}$$

to be the **de Rham cohomology groups with compact support**.

Both groups of course coincide if the stratifold X is compact. If X is non compact, we're mostly interested in the de Rham cohomology groups with compact support. We don't present any theorems for non compact supported cohomology in this case.

For our sheaf theoretic approach it is very useful to see the de Rham complex not only as a chain complex, but as a complex of sheaves. Since the exterior differential d as defined in (4.5) is clearly natural with respect to inclusions it induces a morphism of sheaves

$$d : \Omega_X^n \to \Omega_X^{n+1} \ \forall n \in \mathbb{N} \tag{4.6}$$

Again we have $d \circ d = 0$. Hence we get a complex of sheaves

$$0 \longrightarrow \mathbb{R} \xrightarrow{\epsilon} \mathcal{O}_X = \Omega_X^0 \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots$$

We call this complex according to the previous definition the **de Rham complex** of X. The following proposition can be seen as a generalisation of the Poincare lemma for manifolds.

Proposition 4.6.1. *Let X be a stratifold and Φ be either the family of compact supports or the family of closed supports then the de Rham complex is a resolution of the constant sheaf $\mathbb{R}$ by Φ-acyclic sheafs.*

Proof. By Proposition 4.5.1 the sheaves Ω_X^n are Φ-acyclic and we are left to show that the de Rham complex is exact. Exactness has to be checked on the stalks, so let $x \in X$ be some point. Then x lies in some stratum S_k and since by Proposition 4.5.2

$$\Omega_{X,x}^n \cong \Omega_{S_k,x}^n$$

the complex on stalks is precisely the complex

$$0 \longrightarrow \mathbb{R} \xrightarrow{\epsilon} \mathcal{O}_{S_k,x} = \Omega_{S_k,x}^0 \xrightarrow{d} \Omega_{S_k,x}^1 \xrightarrow{d} \Omega_{S_k,x}^2 \xrightarrow{d} \cdots$$

The exactness of this complex follows from the Poincare Lemma applied to the smooth manifold S_k. Hence we have proven the proposition. $\square$

The last proposition has the following immediate consequence which we state as a theorem because it calculates the de Rham groups.

Theorem 4.6.1. *Let X be a stratifold, then its de Rham cohomology groups with compact support are isomorphic to its real valued singular cohomology groups with compact support, i.e.*

$$H_{dR,c}^*(X) \cong H_c^*(X, \mathbb{R})$$

Proof. By proposition 4.6.1 above the de Rham complex is a resolution of the constant sheaf $\mathbb{R}$ by acyclic sheaves. According to Proposition 2.4.1 this resolution induces an isomorphism

$$H^*(\Gamma_c(\Omega_X^*)) \cong H_c^*(X, \mathbb{R}).$$

On the left side of the last equation we have the de Rham cohomology groups of X with compact support, whereas on the right side by definition we have the compact supported cohomology groups of X with coefficients in the constant sheaf X. The latter groups are for nice spaces, in particular for stratifolds, isomorphic to the compact supported singular cohomology groups with real coefficients. Hence, we have proven the theorem. □

In section 4.10. we give a geometrical meaning to this isomorphism which is given by integration. Integration of forms on stratifolds will be introduced in the next section.

4.7 Integration of Differential Forms on Stratifolds

We assume the reader is familiar with the process of integrating differential forms on manifolds. Integration of forms on the full strata $R_k(X)$ of X can be done using a Riemannian square density associated to a Riemannian metric on $R_k(X)$ as it is done in [Lang99],pages 466-470. The reader who doesn't know how to work with densities can also think of integration via a volume-form on the top stratum. Of course this only works for the top stratum of $\mathbb{Z}$ oriented stratifolds, but in the end, this will be the only case where we need integration. Nevertheless, here is the general version.

We let X be a stratifold and $\omega \in \Omega_c^k(X)$ be a differential form on X. Let $\varphi_k : R_k \to X$ be the chart of the k-dimensional stratum. Then $\varphi_k^* \omega$ which is defined by it's local representations

$$\varphi_k^* \omega_{|U} = \sum_j (f_0^j \circ \varphi_k) d(f_1^j \circ \varphi_k) ... d(f_k^j \circ \varphi_k)$$

is a differential form with compact support on the smooth manifold R_k.

Definition 4.7.1. *Let X be a stratifold and $\omega \in \Omega_c^k(X)$ a differential k-form on X, then we define*

$$\int_X \omega := \int_{R_k} \varphi_k^* \omega.$$

There is one major difference to the manifold case, that is, that integration of k forms which have smaller degree than the dimension of the stratifold may yield nontrivial results. This effect is indeed very interesting and can be used to define certain subcomplexes of differential forms which might lead to interesting new cohomology theories. But so far this is only speculation, and as we mentioned earlier in this work only integration over the top stratum plays a role.

In the following section we present a version of Stokes Theorem for differential forms on stratifolds.

4.8 Stokes' Theorem for Differential Forms on Stratifolds

When integration of differential forms is defined, a natural question is, whether there is a Stokes' like theorem such as

$$\int_X d\omega = \int_{\partial X} i^*\omega,$$

where $i : \partial X \to X$ denotes the inclusion. In general this will not be true. As the proof of the theorem below will show, such a formula can only hold in general, if the second highest stratum of X is empty. This condition is satisfied by $\mathbb{Z}$ and $\mathbb{Z}/2$ oriented stratifolds, which we are particularly interested in, since they are the building blocks of integral respectively $\mathbb{Z}/2$ homology.

Theorem 4.8.1. *Let X be a $\mathbb{Z}$ or $\mathbb{Z}/2$ oriented stratifold and let $n = Dim(X)$. Let $\omega \in \Omega_c^{n-1}(X)$ be a differential form on X and let $i : \partial X \to X$ denote the inclusion. Then we have*

$$\int_X d\omega = \int_{\partial X} i^*\omega.$$

Proof. Of course we will use Stokes' theorem for manifolds. By definition of the integral on the left side we have

$$\int_X d\omega = \int_{R_n} \varphi_n^* d\omega = \int_{R_n} d\varphi_n^*\omega = \int_{\partial R_n} j^*\varphi_n^*\omega$$

$$= \int_{\partial R_n^+} j^*\varphi_n^*\omega + \int_{\partial R_n^-} j^*\varphi_n^*\omega,$$

where $j : \partial R_n \to R_n$ denotes the inclusion and $\varphi_n : R_n \to X$ is the n-th chart of X. On the other side we have again by definition of the integral and the top stratum of ∂X that

$$\int_{\partial X} i^*\omega = \int_{\partial R_n^-} j^*\varphi_n^*\omega.$$

So we have proven the theorem, when we show that

$$\int_{\partial R_n^+} j^* \varphi_n^* \omega = 0.$$

Since by the orientability assumption on X we have $X_{n-1} = X_{n-2}$ and for dimensional reasons (see Proposition 4.5.2)

$$\omega_{|X_{n-1}} = \omega_{|X_{n-2}} = 0.$$

We also have that $i^*\omega = \omega_{|X_{n-1}}$ and hence the theorem follows from the following commutative diagram

$$\begin{tikzcd} \partial R_n^+ \arrow[r, "\varphi_n"] \arrow[dr, "\varphi_n"'] & X \\ & X_{n-1} \arrow[u, "i"'] \end{tikzcd} .$$

□

4.9 Relative de Rham Cohomology

As in almost any cohomology theory there is also a relative version of de Rham cohomology of stratifolds. This relative version can be compared to the absolute one by a long exact sequence, similar to the exact sequence of pairs known from singular cohomology. This will be done in this section.

Let X be a stratifold and $Y \subset X$ be a substratifold. The inclusion map

$$i : Y \to X$$

induces a morphism of sheaves over X

$$i^* : \Omega_X^n \to i\Omega_Y^n,$$

where $i\Omega_Y^n$ denotes the direct image of Ω_Y^n under i (see Definition 2.1.6). This map is given by restriction.

Lemma 4.9.1. *Let Y be a closed substratifold of X. Then $i^* : \Omega_X^n \to i\Omega_Y^n$ is surjective.*

Proof. We have to prove surjectivity of the maps on stalks

$$i_x^* : \Omega^n_{X,x} \to (i\Omega^n_Y)_x$$

for all $x \in X$. This is clear for $x \in X - Y$ since then $(i\Omega^n_Y)_x = 0$. So let us assume that $x \in Y$. There is a stratum $S_k(X)$ of X containing x. Let $S_i(Y) = S_k(X) \cap Y$ be the corresponding stratum of Y which contains x. By definition of substratifold in section 1.4. $S_i(Y)$ is a submanifold of $S_k(X)$. By choosing a tubular neighbourhood for example, we can see that the induced map

$$i_x^* : \Omega^n_{S_k(X),x} \to \Omega^n_{S_i(Y),x}$$

is surjective. By Proposition 3.4.2 we also have

$$\Omega^n_{X,x} \cong \Omega^n_{S_k(X),x}$$

$$\Omega^n_{Y,x} \cong \Omega^n_{S_i(Y),x}$$

from which the proposition follows. $\square$

We define a new sheaf on X by

$$\Omega^n_{X,Y} := kern(i^* : \Omega^n_X \to i\Omega^n_Y). \tag{4.7}$$

This sheaf is given by the association

$$U \mapsto ker(i^* : \Omega^n_X(U) \to \Omega^n_Y(U \cap Y)).$$

In particular we have

$$\Omega^n_{X,Y}(X) = ker((i^* : \Omega^n(X) \to \Omega^n(Y)).$$

Clearly the differential d on Ω^*_X induces a differential also denoted by d on $\Omega^*_{X,Y}$, so that we get a complex of sheaves over X.

The definition of relative de Rham cohomology is as follows.

Definition 4.9.1. *Let X be a stratifold and $Y \subset X$ be a substratifold. We define the* **relative de Rham groups with compact support** *of the pair (X,Y) as*

$$H^k_{dR,c}(X,Y) := \frac{ker(d : \Gamma_c(X, \Omega^k_{X,Y}) \to \Gamma_c(X, \Omega^{k+1}_{X,Y}))}{im(d : \Gamma_c(X, \Omega^{k-1}_{X,Y}) \to \Gamma_c(X, \Omega^k_{X,Y}))}.$$

Since $\Omega^n_{X,Y}$ is a module over $\mathcal{O}_X$ it is also a Φ soft sheave for Φ either the family of closed or compact supports. In this context we will only use the latter system. This fact enables us to prove the following proposition.

Proposition 4.9.1. *Let X be a stratifold and $Y \subset X$ be a closed substratifold, then there exist a long exact sequence of de Rham cohomology groups*

$$.. \to H^k_{dR,c}(X,Y) \to H^k_{dR,c}(X) \to H^k_{dR,c}(Y) \to H^{k+1}_{dR,c}(X,Y) \to ...$$

Proof. Consider the short exact sequence of sheaves over X

$$0 \longrightarrow \Omega^n_{X,Y} \longrightarrow \Omega^n_X \xrightarrow{i^*} i\Omega^n_Y \longrightarrow 0\,.$$

Since $\Omega^n_{X,Y}$ is soft it follows from Proposition 2.2.3 that we have an exact sequence

$$0 \longrightarrow \Gamma_c(X,\Omega^n_{X,Y}) \longrightarrow \Gamma_c(X,\Omega^n_X) \xrightarrow{i^*} \Gamma_c(X, i\Omega^n_Y) \longrightarrow 0$$

for all n. These sequences add up to a short exact sequence of chain complexes. By application of a fundamental lemma of homological algebra this sequence induces the sequence from the proposition. □

Besides the long exact sequence above, there is another way to compare the relative groups with the absolute ones. This is in general known as **excision**. Again let X be a stratifold and Y be a closed substratifold. Then $X-Y$ is an open subset of X, and by Example 1.4.1 a stratifold itself. Hence we can compare the relative de Rham cohomology groups of the pair (X,Y) with the absolute ones of the stratifold $X-Y$. The following proposition says that they are isomorphic.

Proposition 4.9.2. *Let X be a stratifold and $Y \subset X$ be a closed substratifold, then we have a natural isomorphism*

$$H^k_{dR,c}(X,Y) \cong H^k_{dR,c}(X-Y),\ \forall k.$$

Proof. We apply Proposition 2.5.2 to the case $\mathcal{A} = \Omega^k_{X,Y}$ and $p = 0$.Then it follows that

$$\Gamma_c(X,\Omega^k_{X,Y}) \cong \Gamma_c(X-Y,\Omega^k_{X,Y}).$$

Since $i\Omega^k_{Y|X-Y} = 0$ we have

$$\Omega^k_{X,Y|X-Y} \cong \Omega^k_{X|X-Y}.$$

Since also $\Omega^k_{X|X-Y} \cong \Omega^k_{X-Y}$ we get a natural isomorphism

$$\Gamma_c(X,\Omega^k_{X,Y}) \cong \Gamma_c(X-Y,\Omega^k_{X-Y})$$

from which the proposition follows. □

4.10 de Rham's Isomorphism for Stratifolds

In this section we assume that X is a compact stratifold. We have already stated by using sheaf cohomology that de Rham cohomology of X is the same as its singular cohomology with real coefficients. This has been done by a more or less abstract isomorphism. In this section, we will see, that similar as in the world of manifolds there is a nice geometric description of this isomorphism, given by integrating forms over cycles and identifying

$$H^*(X, \mathbb{R}) \cong Hom(H_*(X), \mathbb{R}).$$

Instead of using singular simplices as representatives for cycles in integral homology we use singular stratifolds and the description of integral homology as a bordism theory as presented in section 1.9. This approach is far better suited for our situation.

Assume we have an element in $H_n(X)$ represented by a singular stratifold

$$f : Y \to X,$$

where Y is a stratifold with $Dim(Y) = n$. Without loss of generality we can assume that f is smooth. Let $\omega \in \Omega^n(X)$ be a differential form on X. Then we can define

$$\omega(f) := \int_Y f^*\omega,$$

where the integral on the right side is defined as in the previous section. We will now establish that the association

$$\omega \mapsto (f \mapsto \omega(f))$$

induces a well defined homomorphism

$$H^*_{dR}(X) \to Hom(H_*(X), \mathbb{R}).$$

To show this we have to verify that this map doesn't depend on the various choices made above.This follows from the following two lemmas.

Lemma 4.10.1. *Let $f_1 : Y_1 \to X$ respectively $f_2 : Y_2 \to X$ represent the same classes in $H_n(X)$ and let $\omega \in \Omega^n(X)$ be a cycle, in equal $d\omega = 0$. Then with the definition above $\omega(f) = \omega(g)$.*

Proof. Since f_1 and f_2 represent the same classes in homology, there is a bordism $g : W \to X$. This means $\partial W = Y_1 \sqcup -Y_2$ and $g_{|Y_1} = f_1$ respectively $g_{|Y_2} = f_2$. From our version of Stokes' theorem it follows that

$$0 = \int_W g^* d\omega = \int_W dg^*\omega = \int_{\partial W} i^* g^* \omega = \int_{Y_1} f_1^*\omega - \int_{Y_2} f_2^*\omega$$

$$= \omega(f_1) - \omega(f_2).$$

□

Lemma 4.10.2. *Let $d\omega \in \Omega^n(X)$ be a coboundary and let $f : Y \to X$ represent an element in $H_n(X)$. Then $d\omega(f) = 0$.*

Proof. Since $\partial Y = \emptyset$, Stokes' Theorem implies

$$d\omega(f) = \int_Y f^* d\omega = \int_Y df^*\omega = \int_{\partial Y} f^*\omega = 0.$$

□

It is clear that the map defined above is indeed a homomorphism. We call this homomorphism **de Rham homomorphism** and denote it by

$$\rho : H^*_{dR}(X) \to Hom(H_*(X), \mathbb{R}). \tag{4.8}$$

The next Theorem is a geometrical version of Theorem 4.6.1.

Theorem 4.10.1. *The de Rham homomorphism ρ of (4.8) is an isomorphism.*

Proof. We use the de Rham Theorem for smooth manifolds as one can find it in [Bredon97] for example. Let $n = dim(X)$. The theorem follows via induction on the skeleta of X, by applying the pair sequence on the pair (X, X_{n-1}) and identifying $H^k_{dR}(X, X_{n-1})$ via Proposition 4.9.2 with the ordinary k-th compact supported de Rham cohomology group of the smooth manifold $X - X_{n-1}$ from the five lemma and the commutative diagram

$$\begin{array}{ccccccccc}
H^{k-1}_{dR}(X^{n-1}) & \to & H^k_{dR}(X^n, X^{n-1}) & \to & H^k_{dR}(X^n) & \to & H^k_{dR}(X^{n-1}) & \to & H^{k+1}_{dR}(X^n, X^{n-1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{k-1}(X^{n-1}, \mathbb{R}) & \to & H^k(X^n, X^{n-1}, \mathbb{R}) & \to & H^k(X^n, \mathbb{R}) & \to & H^k(X^{n-1}, \mathbb{R}) & \to & H^{k+1}(X^n, X^{n-1}, \mathbb{R})
\end{array}$$

□

Kapitel 5

Some Constructions on Topological Vectorspaces and Algebras

In this chapter we present some more or less elementary things from analysis, which are crucial to understand the part on Hochschild homology of stratifolds. Since the analysis of topological vectorspaces over the real numbers is not particularly well developed, from now on we work over the complex numbers. This means that from now on, whenever we write $C^\infty(X)$ for a stratifold or $C^\infty_{naive}(M)$ for a manifold treated in the naive sense, we mean complex valued functions. Those can be obtained by simply tensoring the real valued versions with $\mathbb{C}$. All information in this chapter has been taken either from the book "Topological Vectorspaces, Distributions and Kernels" [Treves] or the book "The Homology of Banach and Topological Algebras" [Helemskii].

5.1 Fréchetspaces

All vectorspaces here are considered over the complex numbers. A **topological vectorspace** is simply a vectorspace together with a topology which is compatible with the linear structure, that is addition and scalar multiplication are continuous. In addition to the properties of a topological vectorspace a **topological algebra** has a continuous multiplication. Most times we consider Hausdorff topological vectorspaces and algebras. In chapter 6 though, when we discuss Hochschild homology, we will see, that in general the Hochschild homology groups lack the Hausdorff property. A topological vectorspace E is called **metrizable** if there exists a metric on E which generates the topology. Any metrizable topological vectorspace possesses a translation invariant metric. We then think of this space as equipped with such a metric. E is called **complete**, if any Cauchy sequence in E converges in E. There is a process called **completion** which constructs a complete topological vectorspace $\bar{E}$ out of a topological vectorspace E, together with a topological embedding

$$E \to \bar{E}$$

with dense image. Since we only consider metrizable vectorspaces, we don't have to bother about filters. A topological vectorspace is called **locally convex** if there is a basis of neighbourhoods of $0 \in E$ consisting of convex sets. A seminorm p on E is a norm, which lacks the property of definiteness, in equal there might be vectors $x \neq 0 \in E$ such that $p(x) = 0$. Any seminorm p on E induces a topology on E. We are now ready to define Fréchetspaces.

Definition 5.1.1. *A Fréchetspace is a topological vectorspace E which is complete, metrizable and locally convex.*

Let us discuss the following for our purposes fundamental example. Let Ω be an open subset of $\mathbb{R}^n$ and denote with $x_1, ..., x_n$ the canonical coordinates. For a multi-index $I = (i_1, ..., i_n)$ of nonnegative integers, we shall write

$$\frac{\partial^I}{\partial x^I} = (\frac{\partial}{\partial x_1})^{i_1} ... (\frac{\partial}{\partial x_n})^{i_n}.$$

Let's denote with $|I| = i_1 + ... + i_n$ the length of I which is the same as the order of the differential operator $\frac{\partial^I}{\partial x^I}$. Let us now consider the vectorspace $C^\infty(\Omega)$ of complex valued smooth functions on Ω. For any integer $m \in \mathbb{N}$ and any compact subset K of Ω we define a seminorm by setting

$$|f|_{m,K} = sup_{|I| \leq m}(sup\{|\frac{\partial^I f}{\partial x^I}(x)|, x \in K\}). \tag{5.1}$$

These seminorms induce a locally convex topology on $C^\infty(\Omega)$. Convergence in this topology means uniform convergence on compact subsets in all derivatives. Hence it is not hard to see, that this space is complete. By choosing a countable subfamily $\{p_n\}$ of the family of seminorms above, such that the family $\{p_n\}$ still generates the topology on $C^\infty(\Omega)$, we can define a metric on $C^\infty(\Omega)$ by setting

$$d(f, g) = \sum_{n \in \mathbb{N}} \frac{p_n(f - g)}{2^n(1 + p_n(f - g))}. \tag{5.2}$$

This indeed defines a metric on $C^\infty(\Omega)$ and it is not so hard to see, that the topology defined by the metric is the same as the topology defined by the seminorms. So we have more or less shown, that $C^\infty(\Omega)$ is a Fréchet space. More details can be found in [Treves] pages 86-89. For a smooth manifold M treated in the naive sense we can endow the vectorspace $C^\infty_{naive}(M)$ of all smooth functions with a Fréchetspace structure using local charts in the same way, as it was done above. In the following, when we speak of $C^\infty_{naive}(M)$ we mean smooth functions on M together with this Fréchetspace structure.

In general to any locally convex topological vectorspace one can construct a family of seminorms which generates the topology. See [Treves] pages 62-63 for example. We need this fact in the next section, when defining tensorproducts on topological vectorspaces.

In the topological context, two topological vectorspaces are considered as equal, if there is a topological isomorphism between the two of them. In general it is not so easy to decide, given a continuous bijective linear map, whether it is a topological isomorphism or not, or equivalently, whether its algebraically defined inverse is continuous. In the world of Fréchetspaces things are easier, since we have the following proposition, which is also known as the **open mapping theorem**. We will use that proposition several times in chapters 6 and 7. For a proof see [Treves] page 170.

Proposition 5.1.1. *Let E and F be Fréchetspaces and $\alpha : E \to F$ a continuous linear and bijective map. Then α is a topological isomorphism, in equal α^{-1} is continuous.*

5.2 Tensorproducts of Topological Vectorspaces

Let us denote with E and F two locally convex topological vectorspaces. We will define two kind of tensorproducts $E \otimes F$, namely the ϵ- and the π- tensorproduct. The latter is also called the **projective** tensorproduct. We denote with E'_σ and F'_σ the continuous duals of E and F together with its weak topologies. Weak topology means, that a sequence of continuous linearforms on E converges, if and only if it converges point wise. We do now consider the vectorspace $B(E'_\sigma, F'_\sigma)$ of continuous bilinear forms on E'_σ respectively F'_σ. We give $B(E'_\sigma, F'_\sigma)$ a topology by embedding it in a slightly larger space. This space will be denoted with $\mathcal{B}_\epsilon(E'_\sigma, F'_\sigma)$ and consists of the bilinear forms which are continuous in each variable provided with the topology of uniform convergence on sets which are products of an equicontinuous subset of E' with an equicontinuous subset of F'. Clearly

$$B(E'_\sigma, F'_\sigma) \subset \mathcal{B}_\epsilon(E'_\sigma, F'_\sigma).$$

This inclusion induces a topology on $B(E'_\sigma, F'_\sigma)$. Let us now consider the algebraic tensorproduct $E \otimes F$ and the algebraic isomorphism

$$E \otimes F \cong B(E'_\sigma, F'_\sigma).$$

This isomorphism induces a topology on $E \otimes F$ which we call the ϵ-topology. We denote the space $E \otimes F$ together with this topology as $E \otimes_\epsilon F$ and denote its completion with

$$E \hat{\otimes}_\epsilon F. \tag{5.3}$$

The latter space is a complete, locally convex vectorspace and is called the ϵ-tensorproduct of E and F.

There is another way to define a natural topology on $E \otimes F$ using seminorms. This construction will result in the so called π- or projective tensorproduct. For given seminorms p and q on E respectively F we define a seminorm $p \otimes q$ on $E \otimes F$ as follows. For $\Theta \in E \otimes F$ let

$$(p \otimes q)(\Theta) = inf\{\sum_j p(x_j)q(y_j) | \Theta = \sum_j x_j \otimes y_j\} \tag{5.4}$$

where the infimum is taken over all finite sets of pairs (x_j, y_j) such that

$$\Theta = \sum_j x_j \otimes y_j.$$

Now let $p_i, i \in I$ respectively $q_j, j \in J$ be families of seminorms generating the topologies of E respectively F. By the construction above we get a family of seminorms $p_i \otimes q_j$. This family then induces a locally convex topology on $E \otimes F$, which is called the π- or projective topology. $E \otimes F$ together with this topology will be denoted as $E \otimes_\pi F$. Its completion will be denoted with

$$E \hat{\otimes}_\pi F \tag{5.5}$$

and is called the π- or projective tensorproduct of E and F. It is a complete, locally convex topological vectorspace.

The methods above also work in the case, where more then two vectorspaces are involved. The projective tensorproduct has the following universal property (see proposition 4.9, chapter 2 in [Helemskii]).

Proposition 5.2.1. *Let E,F,G be Fréchetspaces and $\alpha : E \times F \to G$ a continuous, bilinear map. Let*

$$i : E \times F \to E\hat{\otimes}_\pi F$$

$$(e, f) \to e\hat{\otimes} f$$

the canonical map, then there exists a unique continuous homomorphism $\hat{\alpha}$: $E\hat{\otimes}_\pi F \to G$ such that the following diagram commutes

$$\begin{tikzcd} E \times F \arrow[r, "\alpha"] \arrow[d, "i"] & G \\ E\hat{\otimes}_\pi F \arrow[ur, "\hat{\alpha}"] & \end{tikzcd}$$

The following proposition gives an explicit description of the elements in the projective tensorproduct of two Fréchetspaces. This shows up to be useful in calculations.

Proposition 5.2.2. *Let E and F be two Fréchetspaces and $\Theta \in E\hat{\otimes}_\pi F$ be an element in the projective tensorproduct. Then Θ has the form*

$$\Theta = \sum_{n=0}^{\infty} \lambda_n x_n \hat{\otimes} y_n,$$

where (λ_n) is a sequence of real respectively complex numbers such that $\sum_{n=0}^{\infty} |\lambda_n| < 1$ and (x_n) and (y_n) are zero sequences in E respectively F.

Proof. This is Theorem 45.1 on page 459 in [Treves]. □

5.3 Nuclear spaces

In general the ϵ- and the projective tensorproduct doesn't coincide. On the other hand there is a large class of topological vectorspace where they do coincide. These spaces are called nuclear spaces. More precisely we have the following definition.

Definition 5.3.1. *A locally convex topological vectorspace E is said to be* **nuclear** *if for every locally convex topological vectorspace F the canonical map*

$$E\hat{\otimes}_\pi F \to E\hat{\otimes}_\epsilon F$$

is a topological isomorphism.

For nuclear spaces E and F we simply write $E\hat{\otimes}F$ meaning any of the two isomorphic tensorproducts above.

Let us list some of the properties of nuclear spaces.

1. A locally convex topological vectorspace E is nuclear, if and only if its completion $\bar{E}$ is nuclear.
2. A linear subspace of a nuclear space is nuclear.
3. The quotient of a nuclear space modulo a closed linear subspace is nuclear.
4. Any product of nuclear spaces is nuclear.
5. A countable topological direct sum of nuclear spaces is nuclear.
6. A Hausdorff projective limit of nuclear spaces is nuclear.
7. A countable inductive limit of nuclear spaces is nuclear.
8. If E and F are nuclear , then $E\hat{\otimes}F$ is also nuclear.

5.4 Further Examples

At this point we should at least give some examples of nuclear spaces and some applications of the tensorproducts discussed above. Others will follow. In the last section we introduced a topology on the algebra $C^{\infty}_{naive}(M)$ of smooth functions on a manifold M, which made it into a Fréchetalgebra. For reasons of simplicity we assume that M has no boundary, so $C^{\infty}(M) = C^{\infty}_{naive}(M)$. It is not so easy to see, but nevertheless true, that $C^{\infty}(M)$ is nuclear. Let E be any Fréchetspace. Then it follows from [Treves] Theorem 44.1 on page 449, that the natural map

$$C^{\infty}(M) \otimes E \to C^{\infty}(M, E),$$

$$f \otimes e \mapsto (x \mapsto f(x) \cdot e),$$

where f denotes a smooth complex valued function on M and e an arbitrary vector in E, induces a topological isomorphism

$$C^{\infty}(M, E) \cong C^{\infty}(M)\hat{\otimes}_{\epsilon}E.$$

Since $C^\infty(M)$ is a nuclear space, the right side is isomorphic to $C^\infty(M)\hat{\otimes}_\pi E$. In particular for a second manifold M' let's choose $E = C^\infty(M')$ and we have a natural series of isomorphisms

$$C^\infty(M \times M') \cong C^\infty(M, C^\infty(M')) \cong C^\infty(M)\hat{\otimes}C^\infty(M'), \tag{5.6}$$

where the right side denotes either one of the tensorproducts. The map on the left side is given by fixing the first coordinate in the product. The case when M will be replaced by a stratifold will be dealt with in section 5.7.

5.5 Tensorproducts and Alternating Products of Fréchetmodules over Fréchetalgebras

In the algebraic case, tensorproducts do not only work in the case of vectorspaces over a field, but also in the case of modules over some ring. The situation in the Fréchet world is similar. For a matter of simplicity, we assume that all Fréchetspaces in this section are also nuclear, so we don't have to worry which tensorproduct we take.

Definition 5.5.1. *Let A be a Fréchetalgebra and M be a Fréchetspace, which is also a module over A, such that addition and multiplication with elements of A is continuous, then we call M a Fréchetmodule over A*

Now let M_1, M_2 and N be Fréchetmodules over the Fréchetalgebra A, and let

$$\alpha : M_1 \times M_2 \to N$$

be a continuous A-bilinear map. By the universal property of the tensorproduct of Fréchetspaces, α induces a continuous map $\tilde{\alpha} : M_1\hat{\otimes}M_2 \to N$ such that the following diagram commutes

$$\begin{tikzcd} M_1 \times M_2 \arrow[r, "\alpha"] \arrow[d, "i"] & F \\ M_1\hat{\otimes}M_2 \arrow[ur, "\tilde{\alpha}"'] & \end{tikzcd},$$

where i is the canonical map from the product into the tensorproduct (see Proposition 5.2.1). Since α is not only bilinear, but A-bilinear, we see that elements like

$$am_1\hat{\otimes}m_2 - m_1\hat{\otimes}am_2 \ \forall a \in A, m_1 \in M_1, m_2 \in M_2$$

are contained in the kernel of $\tilde{\alpha}$. Hence if we define $M_1 \hat{\otimes}_A M_2$ to be the quotient of $M_1 \hat{\otimes} M_2$ by the closure of the module generated by elements of the form above, we get an A-linear, continuous map $\hat{\alpha} : M_1 \hat{\otimes}_A M_2 \to F$. We define

$$j : M_1 \times M_2 \to M_1 \hat{\otimes}_A M_2$$

as the composition of i with the natural map onto the quotient. From what we have done so far, it is clear that our construction satisfies the following universal property.

Proposition 5.5.1. *Let $M_1 \hat{\otimes}_A M_2$ be as defined above and $\alpha : M_1 \times M_2 \to N$ be a continuous A-bilinear map, where N is another Fréchetmodule over the Fréchetalgebra A. Then there is a unique continuous A-linear map $\hat{\alpha}$: $M_1 \hat{\otimes}_A M_2 \to N$ such that the following diagram commutes*

$$\begin{tikzcd} M_1 \times M_2 \arrow[r, "\alpha"] \arrow[d, "j"] & F \\ M_1 \hat{\otimes}_A M_2 \arrow[ur, "\hat{\alpha}"] & \end{tikzcd} .$$

Using this kind of tensorproduct, we are able to define alternating products, and hence an exterior algebra. M still denotes a Fréchetmodule over a Fréchetalgebra A. We can then build the n-fold tensorproduct $M^{\hat{\otimes}_A^n}$ and divide out the closure of the submodule which is generated by elements of the form

$$m_1 \hat{\otimes} ... \hat{\otimes} m_i \hat{\otimes} ... \hat{\otimes} m_j \hat{\otimes} ... \hat{\otimes} m_n - m_1 \hat{\otimes} ... \hat{\otimes} m_j \hat{\otimes} ... \hat{\otimes} m_i \hat{\otimes} ... \hat{\otimes} m_n .$$

The result is again a Fréchetmodule over A and will be denoted with $\bar{\Lambda}_A^n M$. From the construction it is clear that it satisfies the following universal property.

Proposition 5.5.2. *Let $\bar{\Lambda}_A^n M$ be as defined above and let*

$$\alpha : M \times ... \times M \to N$$

be a continuous multilinear alternating map, where N denotes another Fréchetmodule over the Fréchetalgebra A. Then there is a unique continuous A-linear map $\hat{\alpha} : \bar{\Lambda}_A^n M \to N$ making the following diagram commutative

$$\begin{tikzcd} M \times ... \times M \arrow[r, "\alpha"] \arrow[d, "i"] & N \\ \bar{\Lambda}_A^n M \arrow[ur, "\hat{\alpha}"] & \end{tikzcd} .$$

Here i is defined as the composition of the natural map into the tensorproduct and the projection onto the quotient.

As in the purely algebraic case, one can show, that if M and N are both Fréchetmodules over A, there is a natural topological isomorphism

$$\bar{\Lambda}^n_A(M \oplus N) \cong \sum_{p+q=n} \bar{\Lambda}^p_A M \hat{\otimes} \bar{\Lambda}^q_A N. \tag{5.7}$$

5.6 Differential Forms for Nuclear Fréchetalgebras

In this section we will modify the ideas presented in section 3.5. to suit the case of a topological algebra, or more precisely a unital commutative nuclear Fréchet algebra. The modifications are necessary to compare differential forms with Hochschild homology, as we do in chapters 6 and 7. As in the algebraic case, nonunital and noncommutative versions of the ideas presented in this section are possible.

To start with, let J denote the kernel of the multiplication map

$$A \hat{\otimes} A \to A$$

$$a \hat{\otimes} b \to ab.$$

Clearly J is an A-Fréchet bimodule. Let $\bar{J}^2$ denote the closure of the submodule J^2. Let us define

$$\bar{\Omega}^1_A = J/\bar{J}^2. \tag{5.8}$$

An application of the properties of nuclear Fréchetspaces listed in section 5.3 shows that $\bar{\Omega}^1_A$ is a Fréchet bimodule over A. There is a canonical map i of A into $\bar{\Omega}^1_A$ given by

$$a \mapsto [a \hat{\otimes} 1 - 1 \hat{\otimes} a] =: da.$$

As in section 3.5. it can be seen that this map is a derivation and $\bar{\Omega}^1_A$ has the following universal property.

Proposition 5.6.1. *Let $\bar{\Omega}^1_A$ be as defined above and M any Fréchet bimodule over A. Let further $D : A \to M$ be a continuous derivation. Then there exists* *a unique continuous A-linear map $f : \bar{\Omega}^1_A \to M$, which makes the following diagram commutative*

$$\begin{tikzcd} A \arrow[r, "D"] \arrow[d, "i"] & M . \\ \bar{\Omega}^1_A \arrow[ur, "f"] & \end{tikzcd}$$

Let us now define higher differential forms. Applying the methods of the previous section, we are able to make the following definition.

Definition 5.6.1. *Let A be a unital commutative nuclear Fréchetalgebra. We define the module of differential n-forms over A to be*

$$\bar{\Omega}^n_A := \bar{\Lambda}^n \bar{\Omega}^1_A.$$

This is a Fréchetmodule over A.

Using the description of Ω^1_A in Proposition 3.5.2 and Definition 3.5.2 we get a natural map

$$\Omega^n_A \to \bar{\Omega}^n_A.$$

The following proposition states, that $\bar{\Omega}^n_A$ can be considered as the completion of Ω^n_A.

Proposition 5.6.2. *Let A be a unital commutative nuclear Fréchetalgebra. Then the natural map $\Omega^n_A \to \bar{\Omega}^n_A$ is injective and has dense image.*

Proof. Without loss of generality we can assume $n = 1$. Let us denote the kernel of the multiplication map $A \otimes A \to A$ with I and let J be the kernel of the multiplication map $A \hat{\otimes} A \to A$. Clearly $I \subset J$. In fact, J is the closure of I in $A \hat{\otimes} A$. From this it follows that the image is dense. Further $I^2 = I \cap \bar{J}^2$, from which injectivity follows. $\square$

The following identity will later be useful to identify differential forms on Stratifolds which are products.

Proposition 5.6.3. *Let A and B be unital commutative nuclear Fréchetalgebras. Then there is a natural topological isomorphism*

$$\bar{\Omega}^n_{A \hat{\otimes} B} \cong \sum_{p+q=n} \bar{\Omega}^p_A \hat{\otimes} \bar{\Omega}^q_B.$$

Proof. Let us first show, that there is a canonical isomorphism

$$\bar{\Omega}^1_{A\hat{\otimes}B} \cong \bar{\Omega}^1_A\hat{\otimes}B \oplus A\hat{\otimes}\bar{\Omega}^1_B.$$

Clearly there is a derivation on $A\hat{\otimes}B$ with values in the right hand side given by

$$d(a\hat{\otimes}b) = da\hat{\otimes}b + a\hat{\otimes}db.$$

The universal property of the left hand side hence gives us a well defined map

$$\bar{\Omega}^1_{A\hat{\otimes}B} \to \bar{\Omega}^1_A\hat{\otimes}B \oplus A\hat{\otimes}\bar{\Omega}^1_B$$

$$d(a\hat{\otimes}b) \mapsto da\hat{\otimes}b + a\hat{\otimes}db.$$

This map has an inverse given by

$$da\hat{\otimes}b \mapsto d(a\hat{\otimes}1)(1\hat{\otimes}b)$$

$$a\hat{\otimes}db \mapsto (a\hat{\otimes}1)(1\hat{\otimes}db).$$

Clearly all these maps are continuous. We do now use the identity at the end of the last section and get

$$\bar{\Omega}^p_{A\hat{\otimes}B} \cong \bar{\Lambda}^n_{A\hat{\otimes}B}(\bar{\Omega}^1_A\hat{\otimes}B \oplus A\hat{\otimes}\bar{\Omega}^1_B) \cong \sum_{p+q=n} \bar{\Omega}^p_A\hat{\otimes}\bar{\Omega}^q_B.$$

□

5.7 The Case of a Stratifold

In this section we consider the algebra $C^\infty(X)$ of smooth complex valued functions on a stratifold as defined in section 1.3 in more detail. At this point, we should remind the reader, that for a c-manifold W with boundary, the algebra $C^\infty(W)$ as defined in section 1.1 slightly differs from what is classical known to be the algebra of smooth functions on W. To distinguish these two algebras, we write $C^\infty_{naive}(W)$, when we treat W in the naive sense, in equal, when we make no conditions along the collar. We already know that for each stratum R_k of X the algebras $C^\infty_{naive}(R_k)$ are nuclear Fréchet algebras. We consider

$$C^\infty(X) \subset \prod_k C^\infty_{naive}(R_k)$$

as a subalgebra. In this way $C^\infty(X)$ inherits a locally, convex, metrizable topology. Unfortunately $C^\infty(X)$ lacks one desirable property. It is not complete. This effect is due to the fact, that for a c-manifold W with boundary the algebra $C^\infty(W)$ is incomplete as a subspace of $C^\infty_{naive}(W)$. For example it is not hard to construct a sequence in $C^\infty(W)$ which converges to a function, which is not constant along the collar for any given neighbourhood of the boundary. On the other side any limit f of functions in $C^\infty(W)$ has the property, that

$$(\frac{\partial}{\partial t})^k f_{|\partial W} = 0, \ \forall k > 0$$

where t is the collar parameter. This means that all derivatives of f orthogonal to the boundary at the boundary are zero. That this is indeed true follows from the fact, that one can change the order of differentiation and building the limit if the convergence is strong enough. At this point it is not hard to see, that the completion $\bar{C}^\infty(W)$ of $C^\infty(W)$ is precisely given by

$$\bar{C}^\infty(W) = \{f \in C^\infty_{naive}(W) | (\frac{\partial}{\partial t})^k f_{|\partial W} = 0 \ \forall k > 0\}. \tag{5.9}$$

The space above is now a complete, metrizable and locally convex space, hence a Fréchetspace. Since $C^\infty_{naive}(W)$ is nuclear it follows from the list of statements in section 5.3 that $\bar{C}^\infty(W)$ is also a nuclear space. In case of a stratifold X we get for the completion $\bar{C}^\infty(X)$ of $C^\infty(X)$ the space

$$\bar{C}^\infty(X) = \{f \in C(X) | f \circ \varphi_k \in \bar{C}^\infty(R_k) \ \forall k\}, \tag{5.10}$$

where the maps φ_k denote the charts of X. The algebra $\bar{C}^\infty(X)$ then is a nuclear Fréchetalgebra. Using this and Proposition 5.6.2 implies that $\bar{\Omega}^n_{\bar{C}^\infty(X)}$ can be identified with the completion of $\Omega^n(X)$. It is not hard to show, that the algebraically defined exterior derivation d on $\Omega^n(X)$ (see (4.5)) generalizes to give an exterior derivation d on $\bar{\Omega}^n_{\bar{C}^\infty(X)}$. Hence one also gets de Rham cohomology groups in this case. If we assume, that X has finite dimensional homology groups we have that the de Rham cohomology groups in both cases coincide because of denseness and finite dimensionality.

Now consider the case, where we are given two stratifolds X and Y. We let

$$\bar{C}^\infty_{par}(X \times Y) = \{f \in C(X \times Y) | f \circ i_x \in \bar{C}^\infty(Y), f \circ i_y \in \bar{C}^\infty(X), \forall x \in X, y \in Y\},$$

where

$$i_x : Y \to X \times Y,$$

$$y \mapsto (x, y)$$

and respectively

$$i_y : Y \to X \times Y,$$

$$x \mapsto (x, y)$$

denote the inclusion of the factors into the product. The subindex *par* in $\bar{C}^{\infty}_{par}(X \times Y)$ stands for partial differentiable. The reader should notice that the algebra $\bar{C}^{\infty}_{par}(X \times Y)$ differs from the algebra $\bar{C}^{\infty}(X \times Y)$, where the latter algebra denotes the completion of the algebra of smooth function on the product stratifold $X \times Y$ (see [Kreck00]) Nevertheless in the case when one of the two stratifolds is in fact a smooth manifold (in the naive sense) the two algebras above coincide. The algebra of partial differentiable functions on a product is important because of the following proposition.

Proposition 5.7.1. *Let X and Y be stratifolds, then there is a natural isomorphism*

$$\bar{C}^{\infty}_{par}(X \times Y) \cong \bar{C}^{\infty}(X) \hat{\otimes} \bar{C}^{\infty}(Y)).$$

If either X or Y is a smooth manifold the subscript par can be omitted.

Proof. It is clear how to generalize the concept of smooth complex valued functions on a stratifold X to smooth vector valued functions, at least when the domain is itself a Fréchetspace (see [Treves] page 412). For a Fréchet space E let us denote this algebra with $\bar{C}^{\infty}(X, E)$. There is a canonical isomorphism

$$\bar{C}^{\infty}(X, E) \cong \bar{C}^{\infty}(X) \hat{\otimes}_{\epsilon} E.$$

This fact is proven in [Treves] on page 449 in the case where $\bar{C}^{\infty}(X)$ has been replaced by $C^{\infty}(\Omega)$ where Ω is a domain in $\mathbb{R}^n$. The proof works completely analogous in our case. Now, we can use the identification

$$\bar{C}^{\infty}_{par}(X \times Y) \cong \bar{C}^{\infty}(X, \bar{C}^{\infty}(Y))$$

given by

$$f \mapsto (x \mapsto f(x, -)).$$

From this we get

$$\bar{C}^{\infty}_{par}(X \times Y) \cong \bar{C}^{\infty}(X) \hat{\otimes}_{\epsilon} \bar{C}^{\infty}(Y)$$

and since all the spaces involved are nuclear, this tensorproduct coincides with the projective tensorproduct and we are done with the proof. □

The following corollary of Proposition 5.7.1. will be of major importance when we will study the Hochschild homology of locally coned stratifolds.

Corollary 5.7.1. *Let X be a stratifold and denote with cX the open cone over X. Further let*

$$\bar{C}^{\infty}_{0}(cX) := ker(ev_{pt} : \bar{C}^{\infty}(cX) \to \mathbb{R})$$

be the kernel of the evaluation map at the cone point, which is denoted by pt. Then there is a natural topological isomorphism

$$\bar{C}^{\infty}_{0}(cX) \cong \bar{C}^{\infty}(X) \hat{\otimes} \bar{C}^{\infty}_{0}([0, 1)),$$

where the half open interval $[0, 1)$ is considered as a 1 dimensional c-manifold.

Proof. Consider the following exact sequence

$$0 \to \bar{C}^{\infty}_{0}([0, 1)) \to C^{\infty}((-1, 1)) \to C^{\infty}((-1, 0)),$$

where the right hand map is given by restriction. Since all spaces in the sequence above are nuclear, tensoring this with $\bar{C}^{\infty}(X)$ remains exact (see [Brodzki,Lykova99]). This leads to the following exact sequence

$$0 \to \bar{C}^{\infty}(X) \hat{\otimes} \bar{C}^{\infty}_{0}([0, 1)) \to \bar{C}^{\infty}(X) \hat{\otimes} C^{\infty}(-1, 1) \to \bar{C}^{\infty}(X) \hat{\otimes} C^{\infty}(-1, 0).$$

This sequence embeds in the following commutative diagram, where the lower row is also exact and all vertical maps are given by multiplication in the standard way.

$$\begin{tikzcd}
0 \arrow[r] & \bar{C}^{\infty}(X) \hat{\otimes} \bar{C}^{\infty}_{0}([0, 1)) \arrow[r] \arrow[d] & \bar{C}^{\infty}(X) \hat{\otimes} C^{\infty}(-1, 1) \arrow[r] \arrow[d] & \bar{C}^{\infty}(X) \hat{\otimes} C^{\infty}(-1, 0) \arrow[d] \\
0 \arrow[r] & \bar{C}^{\infty}_{0}(cX) \arrow[r] & \bar{C}^{\infty}(X \times (-1, 1)) \arrow[r] & \bar{C}^{\infty}(X \times (-1, 0))
\end{tikzcd}$$

From Proposition 5.7.1 it follows that both vertical maps on the right side are isomorphisms. A short diagram chase will then show, that the left vertical map is also an isomorphism. □

In general it is unclear, if a short exact sequence of nuclear Fréchetalgebras

$$0 \to A \to B \to C \to 0$$

induces a short exact sequence of the form

$$0 \to \bar{\Omega}^n_A \to \bar{\Omega}^n_B \to \bar{\Omega}^n_C \to 0.$$

In the purely algebraic case there are some theorems about when such a short exact sequence exists (see [Loday91] and [Weibel95]). The situation in our case, concerning the algebra $\bar{C}^\infty(cX)$ is far easier. We have the following proposition where we treat the 1-manifold with boundary $(-1,0]$ in the naive sense.

Proposition 5.7.2. *For $n > 0$ the short exact sequence of nuclear Fréchetalgebras*

$$0 \to \bar{C}^\infty_0(cX) \to \bar{C}^\infty(X \times (-1,1)) \to \bar{C}^\infty(X \times (-1,0]) \to 0$$

induces a short exact sequence of differential forms

$$0 \to \bar{\Omega}^n_{\bar{C}^\infty(cX)} \to \bar{\Omega}^n_{\bar{C}^\infty(X\times(-1,1))} \to \bar{\Omega}^n_{\bar{C}^\infty(X\times(-1,0])} \to 0.$$

Proof. Using the natural topological isomorphisms

$$\bar{C}^\infty(X \times (-1,1)) \cong \bar{C}^\infty(X)\hat{\otimes}C^\infty(-1,1),$$

$$\bar{C}^\infty(X \times (-1,0]) \cong \bar{C}^\infty(X)\hat{\otimes}C^\infty(-1,0]$$

and the result of Proposition 5.6.3 we get the following commutative diagram

$$\begin{array}{ccc} \bar{\Omega}^n_{\bar{C}^\infty(X\times(-1,1))} & \xrightarrow{\cong} & \bar{\Omega}^n_{\bar{C}^\infty(X)}\hat{\otimes}C^\infty(-1,1) \oplus \bar{\Omega}^{n-1}_{\bar{C}^\infty(X)}\hat{\otimes}\bar{\Omega}^1_{C^\infty(-1,1)} \\ \downarrow & & \downarrow \\ \bar{\Omega}^n_{\bar{C}^\infty(X\times(-1,0])} & \xrightarrow{\cong} & \bar{\Omega}^n_{\bar{C}^\infty(X)}\hat{\otimes}C^\infty(-1,0] \oplus \bar{\Omega}^{n-1}_{\bar{C}^\infty(X)}\hat{\otimes}\bar{\Omega}^1_{C^\infty(-1,0]} \end{array} \tag{5.11}$$

where all horizontal maps are isomorphisms and the vertical maps are given by restriction. Since the unitization of $\bar{C}^\infty_0(cX) \cong \bar{C}^\infty(X)\hat{\otimes}\bar{C}^\infty_0([0,1))$ is $\bar{C}^\infty(cX)$ we can use Proposition 5.2.2 to represent any element f in $\bar{C}^\infty(cX)$ as

$$f = \sum_{i=0}^{\infty} \lambda_i g_i \hat{\otimes} h_i + c,$$

such that $g_i \in \bar{C}^\infty(X)$, $h_i \in \bar{C}_0^\infty([0,1))$ and $c \in \mathbb{C}$ is the value of f at the cone point. Since we can neglect constants when calculating differential forms of degree higher than zero it follows again from Proposition 5.6.3 that the canonical map

$$\bar{\Omega}^n_{\bar{C}^\infty(cX)} \to \bar{\Omega}^n_{\bar{C}^\infty(X)} \hat{\otimes} \bar{C}_0^\infty([0,1)) \oplus \bar{\Omega}^{n-1}_{\bar{C}^\infty(X)} \hat{\otimes} \bar{\Omega}^1_{\bar{C}_0^\infty([0,1))} \tag{5.12}$$

$$df = d(\sum_{i=0}^{\infty} \lambda_i g_i \hat{\otimes} h_i) \mapsto \sum_{i=0}^{\infty} \lambda_i dg_i \hat{\otimes} h_i \oplus \sum_{i=0}^{\infty} \lambda_i g_i \hat{\otimes} dh_i$$

is a topological isomorphism. Let us now show, that the right hand side of the expression (5.12) is exactly the kernel of the right hand restriction map in the commutative diagram (5.11). For this reason we tensor the short exact sequence

$$0 \to \bar{C}_0^\infty([0,1)) \to C^\infty((-1,1)) \to C^\infty((-1,0]) \to 0$$

with $\bar{\Omega}^n_{\bar{C}^\infty(X)}$ and the short exact sequence

$$0 \to \bar{\Omega}^1_{\bar{C}_0^\infty([0,1))} \to \bar{\Omega}^1_{C^\infty((-1,1))} \to \bar{\Omega}^1_{C^\infty((-1,0])} \to 0$$

with $\bar{\Omega}^{n-1}_{\bar{C}^\infty(X)}$ and add those two sequences. The resulting sequence is exact again (see [Brodzki,Lykova99]) and that finally proves the proposition.

$\square$

Kapitel 6

Hochschild Homology

In this chapter we give a short introduction to what in general is known as Hochschild homology. There are various versions of Hochschild Homology, depending on how much structure of the underlying algebras is taken into account. The two most important cases are Hochschild homology of general algebras, which we call algebraic Hochschild homology and Hochschild homology of nuclear Fréchetalgebras, which we call continuous Hochschild homology. These two versions will be presented in the following two sections.

6.1 Algebraic Hochschild Homology

Algebraic Hochschild Homology is the most elementary version of Hochschild homology. It is defined for arbitrary not necessarily unital algebras. Throughout this section we assume that A is an associative algebra over a field k of characteristic zero. The field k will also be referred to as the ground field. Algebraic Hochschild Homology has many applications in algebra and algebraic geometry. It was the first version to be defined and resembles the underlying ideas best. Also we think it is helpful to know the algebraic case, before any topological structure is taken into account. This is, why we present this version here, though we actually won't apply it to the algebras we are interested in.

Most of what we present in this section has been taken out of the book "Cyclic Homology" from Loday [Loday91]. Of course we restrict ourselves to the basic definitions and just give some examples for computations in Hochschild homology. The tensor product $\otimes$ always stands for the tensorproduct $\otimes_k$ over the ground field k.

For each $n \in \mathbb{N}$ we associate to A the group

$$C_n(A) := A^{\otimes(n+1)}. \tag{6.1}$$

We define operators b_n and b'_n as follows.

$$b'_n : C_n(A) \to C_{n-1}(A) \tag{6.2}$$

$$b'_n(a_0 \otimes ... \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes ... \otimes a_i a_{i+1} \otimes ... \otimes a_n$$

$$b_n : C_n(A) \to C_{n-1}(A) \tag{6.3}$$

$$b_n(a_0 \otimes ... \otimes a_n) = b'_n(a_0 \otimes ... \otimes a_n) + (-1)^n a_n a_0 \otimes a_1 \otimes ... \otimes a_{n-1}.$$

Since $b_{n-1} \circ b_n = 0 = b'_{n-1} \circ b'_n = 0$ we get two chain complexes $C_*(A) = (C_n(A), b_n)$ and $C_*^{bar}(A) = (C_n(A), b'_n)$. The first complex is called the **Hochschild complex**, the second complex is called the **bar-complex**. Both complexes give rise to homology groups. Let us first consider the case when the algebra A is unital. In this case the maps

$$s_n : C_n(A) \to C_{n+1}(A)$$

$$s_n(a_0 \otimes ... \otimes a_n) = 1_A \otimes a_0 \otimes ... \otimes a_n$$

define a contraction of the complex $C_*^{bar}(A)$. So the bar-complex is not particularly interesting in the unital case. We call the homology groups of the Hochschild complex the Hochschild homology groups.

Definition 6.1.1. *Let A be a unital algebra. We define the n-th Hochschild homology group of A as*

$$HH_n(A) = \frac{ker(b_n : C_n(A) \to C_{n-1}(A))}{im(b_{n+1} : C_{n+1}(A) \to C_n(A))}.$$

Direct calculation yields to the following examples.

Example 6.1.1. *1. If we take $A = k$ to be the ground field, we have*

$$HH_n(k) = \begin{cases} k & \text{if } n = 0, \\ 0 & \text{else} \end{cases} \tag{6.4}$$

2. By definition the first boundary operator b_1 in the Hochschild complex maps $a \otimes b$ to the commutator $[a, b] = a \otimes b - b \otimes a$. Hence we have

$$HH_0(A) = A/[A, A]. \tag{6.5}$$

In the case that A is commutative we have $HH_0(A) = A$.

In general calculations of Hochschild homology groups using the Hochschild complex turn out to be very complicated. As the following proposition shows Hochschild homology groups can also be calculated by using certain projective resolutions of A. Let A^{op} denote the algebra A with the opposite multiplication.

Proposition 6.1.1. *We consider A as a module over $A \otimes A^{op}$ via $(a \otimes b) \cdot c = acb$. Then*

$$HH_*(A) = Tor_*^{A \otimes A^{op}}(A, A).$$

In this way we can use any projective resolution of A over $A \otimes A^{op}$ to calculate the Hochschild homology groups of A.

Proof. see [Loday91] on page 12. □

Any homomorphism $f : A \to B$ of algebras induces a map in the same direction between the Hochschild complexes. So, the association $A \mapsto HH_n(A)$ is a covariant functor. This of course is also clear from the Tor description of the last proposition.

In the following we assume that A is commutative. In this case Hochschild homology can be seen as a refinement of the concept of differential forms for algebras, as constructed in chapter 3. The connection between those two concepts is made by the antisymmetrization which we will consider next.

Let us denote with Σ_n the group of permutations of the set $\{1, 2, ...n\}$. There is an operation of Σ_n on $C_n(A)$ given by

$$\sigma \cdot (a_0 \otimes ... \otimes a_n) = a_0 \otimes a_{\sigma^{-1}(1)} \otimes ... \otimes a_{\sigma^{-1}(n)}.$$

k-linear extension induces an operation of the group algebra $k[\Sigma_n]$ of Σ_n on $C_n(A)$. We let $\epsilon_n \in k[\Sigma_n]$ be the element

$$\epsilon_n = \sum_{\sigma \in \Sigma_n} sign(\sigma)\sigma.$$

ϵ_n induces a map which we denote by ϵ_n again

$$\epsilon_n : A \otimes \Lambda^n A \to C_n(A)$$

$$a_0 \otimes a_1 \wedge ... \wedge a_n \mapsto \epsilon_n \cdot (a_0 \otimes ... \otimes a_n).$$

It is not hard to show (see [Loday91],page 27) that this map factors to a well defined A-linear map

$$a_0 da_1 ... da_n \to \epsilon_n \cdot (a_0 \otimes ... \otimes a_n).$$

We call this map the **antisymmetrization map** and denote it

$$\epsilon_n : \Omega^n_A \to HH_n(A) \tag{6.6}$$

We also have a natural map $\pi_n : C_n(A) \to \Omega^n_A$ in the other direction. π_n is given by

$$\pi_n(a_0 \otimes ... \otimes a_n) = a_0 da_1 ... da_n.$$

One readily verifies, that $\pi_n \circ b = 0$. So π_n induces a map

$$\pi_n : HH_n(A) \to \Omega^n_A \tag{6.7}$$

The maps π_n and ϵ_n are related in the following way.

Proposition 6.1.2. *Let A be a unital, commutative algebra. Then the composition $\pi_n \circ \epsilon_n$ is multiplication with $n!$ on Ω^n_A. Since $char(k) = 0$, this is an isomorphism. In particular ϵ_n is injective and Ω^n_A is a direct summand of $HH_n(A)$.*

Proof. This follows from

$$a_0 da_{\sigma^{-1}(1)} \wedge ... \wedge da_{\sigma^{-1}(n)} = sign(\sigma) a_0 da_1 \wedge ... \wedge da_n$$

for all $\sigma \in \Sigma_n$ and $|\Sigma_n| = n!$. $\square$

In general the question remains whether the map ϵ_n is an isomorphism or not. In the algebraic case, there is the Hochschild-Kostant-Rosenberg theorem, which states that ϵ_n is an isomorphism, whenever the algebra A is smooth (see [Loday91], page 102).

Let us now turn to the case where the algebra A is not necessarily unital. We already mentioned that nonunital algebras play a role in our considerations. The Hochschild complex of A is still defined and a natural thing would be, as in the unital case to define the Hochschild homology groups of A as the homology groups of the Hochschild complex. It turns out, that in general this is not the right definition. From the topological point of view, the situation should be compared to the case, where a homology theory on the category of pointed topological spaces is transferred to a homology theory on the category of topological space by simply adding a base point and then take the cokernel of the map, which is induced by the inclusion of this base point. Homomorphism of unital algebras take the unit element into the unit element, hence can be compared to morphisms in the category of pointed spaces. Adding a base point can be compared to adding a unit element. In this sense the following definition seems to be natural, at least from the topological point of view.

Definition 6.1.2. *Let A denote a not necessarily unital algebra and A_+ its unitization. The n-th Hochschild homology group of A is defined as*

$$HH_n(A) = coker(i_* : HH_n(k) \to HH_n(A_+)),$$

where $i : k \to A_+$ denotes the inclusion.

In the case that A is unital, this definition coincides with Definition 6.1.1. In the nonunital case we have that in general the Hochschild homology groups as defined in Definition 6.1.2 doesn't coincide with the homology groups of the Hochschild complex. The latter groups are called the **naive Hochschild homology** groups and will be denoted as $HH_n^{naive}(A)$. The importance of these groups will show up in the following. Let us denote the homology groups of the bar-complex of A as $H_n^{bar}(A)$. The following proposition relates the three homology groups defined above.

Proposition 6.1.3. *Let A be a not necessarily unital algebra. Then there is a long exact sequence*

$$... \to HH_n^{naive}(A) \to HH_n(A) \to H_n^{bar}(A) \to HH_{n-1}^{naive}(A) \to ...$$

Proof. see [Loday91], page 30 □

From the long exact sequence above one can deduce, that when the bar-complex is acyclic, naive Hochschild homology and Hochschild homology coincide. We mentioned earlier that in the unital case the bar-complex is contractible, hence acyclic. So the following definition is a generalization of being unital.

Definition 6.1.3. *Let A be a not necessarily unital algebra. We call A* **H-unital** *if the bar-complex of A is acyclic, in equal*

$$H_n^{bar}(A) = 0 \ \forall n \in \mathbb{N}.$$

To decide, whether a nonunital algebra is H-unital or not in general can be quite difficult. One can show ([Loday91],page 32) that an algebra with local units is H-unital. So for example the algebra $C_0^\infty(cX)$ of smooth functions on the cone over a stratifold X which vanish at the cone point is H-unital.

The following proposition is of major importance. It tells us in which cases Hochschild homology behaves like a homology theory for algebras, in equal carries short exact sequences of algebras into long exact sequences of Hochschild homology groups.

Proposition 6.1.4. *Let A be a unital algebra and let $I \subset A$ be an ideal which is H-unital. Then there is a long exact sequence of Hochschild homology groups*

$$... \to HH_n(I) \to HH_n(A) \to HH_n(A/I) \to HH_{n-1}(I) \to ...$$

6.2 Continuous Hochschild Homology

The version of Hochschild homology we will use in chapter 7, to determine the Hochschild homology of a locally coned stratifold is not the standard one, as we discussed in the previous section, but a topological version. Most of what is presented in this section can be seen as a suitable completion of the algebraic case. A reference for this section is the original work of Connes [Connes87]. We also refer to the articles of Wodzicki [Wodzicki89] and Brodzki/Lykova [Brodzki,Lykova99] about excision in continuous Hochschild homology. From now on we assume that A is a nuclear Fréchetalgebra. For any natural number $n \in \mathbb{N}$ let

$$\bar{C}_n(A) = A^{\hat{\otimes}(n+1)}$$

be the $(n+1)$-fold completed tensor product. Since A is nuclear by our assumption, it doesn't matter which of the two tensorproducts we use at this point. Clearly, when considered on the Cartesian product the operators b_n and b'_n in (6.2) respectively. (6.3) are multi linear and continuous. By the universal property of $\hat{\otimes}$ they induce operators $\bar{C}_n(A) \to \bar{C}_{n-1}(A)$ again denoted by b_n and b'_n. We call

$$\bar{C}_*(A) = (\bar{C}_n(A), b_n) \tag{6.8}$$

the **continuous Hochschild complex** and

$$\bar{C}_*^{bar}(A) = (\bar{C}_n(A), b'_n) \tag{6.9}$$

the **continuous bar-complex**. The complexes $C_*(A)$ and $C_*^{bar}(A)$ can be considered as dense subcomplexes of the corresponding complexes.

Definition 6.2.1. *Under the assumptions above the n-th continuous Hochschild homology group of A is defined as the A module*

$$HH_n(A) = \frac{ker(b_n : \bar{C}_n(A) \to \bar{C}_{n-1}(A))}{im(b_{n+1} : \bar{C}_{n+1}(A) \to \bar{C}_n(A))}.$$

In the following we will sometimes omit the word continuous in front of Hochschild homology. Whether we mean algebraic or continuous Hochschild homology should then be clear from the context. As in the algebraic case, these groups are modules over A. As quotient spaces of topological vectorspaces, the Hochschild homology groups are also topological vectorspaces. In fact they are topological modules over A. In general though, these vectorspace are non-Hausdorff. This often makes things difficult. For example a Kuenneth like theorem for the Hochschild homology of $A\hat{\otimes}B$ doesn't seem to appear in the literature. On the other side, if the continuous Hochschild homology groups are Hausdorff , then they are automatically nuclear Fréchet (see our list on section 5.3) and most constructions work. In our case, that is $A = \bar{C}^\infty(X)$ for a stratifold X the Hochschild homology groups will turn out to be Hausdorff and we are on the safe side.

Using the definition one can compute the continuous Hochschild homology of $\mathbb{C}$ similar as in (6.4). For an arbitrary unital nuclear Fréchetalgebra the same calculation as in (6.5) shows that $HH_1(A) = A/[A, A]$.

We will now show, that Hochschild homology can also be described as a topological version of a particular torsionproduct. This will enable us to calculate the Hochschild homology groups in certain cases by using projective resolutions. Of course, we have to define these terms first.

Definition 6.2.2. *A locally convex topological vectorspace M is called a topological module over A if M is a module over A and scalar multiplication as well as addition is continuous. M is called topological projective if it is a topological direct summand of a module of the form $N = A\hat{\otimes}E$, where E is a locally convex vectorspace.*

Projectivity can also be characterized by a universal property which is similar to the algebraic case, where homomorphisms are replaced by admissible homomorphisms. For a general treatment of the category of nuclear Fréchet algebras and admissible maps, the reader should consult the book of Helemskii [Helemskii]. We come to what is called a projective resolution.

Definition 6.2.3. *Let M be a topological module over A. A topological projective resolution of M is an exact sequence of topological projective A modules and A-linear maps*

$$...M_2 \xrightarrow{b_2} M_1 \xrightarrow{b_1} M_0 \xrightarrow{b_0} M \ ,$$

which admits an $\mathbb{C}$-linear continuous contraction

$$s_i : M_i \to M_{i+1},$$

$$b_{i+1}s_i + s_{i-1}b_i = id \ \forall i.$$

Now let A^{op} denote the algebra A with the opposite multiplication and $B = A\hat{\otimes}A^{op}$. The algebra A itself becomes a topological B-module by setting

$$(a\hat{\otimes}b) \cdot c = acb.$$

The following proposition gives an answer to how to compute Hochschild homology groups using projective resolutions.

Proposition 6.2.1. *Let (M_n, b_n) be a topological projective resolution of A over B. Then the Hochschild homology groups of A coincide with the homology groups of the complex*

$$...M_3\hat{\otimes}_B A \xrightarrow{b_3} M_2\hat{\otimes}_B A \xrightarrow{b_2} M_1\hat{\otimes}_B A \xrightarrow{b_1} M_0\hat{\otimes}_B A \ .$$

There is a standard projective resolution of A over $A\hat{\otimes}A^{op}$ called the **bar-resolution**. This resolution is constructed similar to the bar-resolution defined in (6.1),(6.2) though shouldn't be confused with the latter, since it is by construction a resolution over $A\hat{\otimes}A^{op}$ rather than k as in (6.1),(6.2). It can be obtained as follows. For $n \in \mathbb{N}$ we take $M_n = A^{\hat{\otimes}(n+2)}$. We consider this as a module over $A\hat{\otimes}A^{op}$ via

$$(a\hat{\otimes}b) \cdot (a_0\hat{\otimes}...\hat{\otimes}a_{n+1}) = aa_0\hat{\otimes}...\hat{\otimes}a_{n+1}b.$$

From the isomorphism

$$M_n \cong (A\hat{\otimes}A^{op})\hat{\otimes}A^{\hat{\otimes}n}$$

it follows that M_n is projective in the sense of Definition 6.2.2. We define a differential

$$b' : M_n \to M_{n-1}$$

$$b'(a_0\hat{\otimes}...\hat{\otimes}a_{n+1}) = \sum_{i=0}^{n}(-1)^i a_0\hat{\otimes}..\hat{\otimes}a_i a_{i+1}\hat{\otimes}...\hat{\otimes}a_{n+1}.$$

It is not hard to verify that this complex is continuous and k-linear contractible via

$$s_n : M_n \to M_{n+1}$$

$$s_n(a_0\hat{\otimes}...\hat{\otimes}a_{n+1}) = 1_A\hat{\otimes}a_0\hat{\otimes}...\hat{\otimes}a_{n+1}.$$

Hence M_* is a projective resolution of A over $A\hat{\otimes}A^{op}$. We call this resolution the bar-resolution. To compute the Hochschild homology of A , we have to tensor the bar-resolution with A over $A\hat{\otimes}A^{op}$. Some easy calculation then shows that the resulting complex is precisely $(\bar{C}_*(A), b)$.

We do now use this proposition to calculate the Hochschild homology in the case where $A = C^\infty(B)$ consists of smooth complex valued functions on the open unit disc B in $\mathbb{R}^n$. We will construct an explicit projective resolution and show that

$$HH_k(C^\infty(B)) \cong \Omega^k(B), \forall k \in \mathbb{N}.$$

Here the right hand side denotes complex differential forms on B. We will later use this result to proof a similar result for locally coned stratifolds. For each $k \in \mathbb{N}$ we define modules over $C^\infty(B \times B)$

$$M_k := C^\infty(B \times B, \Lambda^k(\mathbb{C}^{n*})).$$

Here $\mathbb{C}^{n*}$ denotes the space of linear forms on $\mathbb{C}^n$. Clearly

$$M_k \cong C^\infty(B \times B)\hat{\otimes}\Lambda^k(\mathbb{C}^{n*}),$$

where completion is actually unnecessary, since the vectorspace $\Lambda^k(\mathbb{C}^{n*})$ is finite dimensional. Nevertheless, it follows that each M_k is free, hence projective. Further let γ denote the difference function

$$\gamma : B \times B \to \mathbb{R}^n \subset \mathbb{C}^n,$$

$$\gamma(a, b) = b - a.$$

This map induces maps we denote with i_γ

$$i_\gamma : M_{k+1} \to M_k,$$

$$i_\gamma\omega(a, b)(v_1, ..., v_k) = \omega(a, b)(\gamma(a, b), v_1, ...v_k) = \omega(a, b)(b - a, v_1, ...v_k).$$

Here $\omega \in M_{k+1}$ denotes a form, a, b are points in $B \subset \mathbb{R}^n$ and $v_1, ...v_k$ are elements of $\mathbb{C}^n$ In other words i_γ is contraction with the vectorfield γ. Let us now consider the following sequence

$$0 \longleftarrow C^\infty(B) \xleftarrow{\Delta^*} C^\infty(B \times B) = M_0 \xleftarrow{i_\gamma} M_1 \xleftarrow{i_\gamma} M_2 \xleftarrow{i_\gamma} \cdots,$$

where $\Delta : B \to B \times B$ denotes the diagonal map. To show that this sequence defines a topological projective resolution of $C^\infty(B)$ over $C^\infty(B \times B)$ we have to give a continuous $\mathbb{C}$-linear contraction. For this let $s_k : M_k \to M_{k+1}$ be defined as follows. Let $e_1^*, ..., e_n^*$ denote the dual basis of the standard canonical basis of $\mathbb{C}^n$, and let $\omega \in M_k$ be given as

$$\omega(a, b) = f(a, b)e_{i_1}^* \wedge ... \wedge e_{i_k}^*,$$

where $f \in C^\infty(B \times B)$ is a smooth function on $B \times B$ and $i_1, ..., i_k \in \{1, ...n\}$. In this case we define

$$s_k\omega(a, b) := \sum_{j=1}^{n} \int_0^1 \frac{\partial f}{\partial y_j}(a, a + t(b - a))t^k e_j^* \wedge e_{i_1}^* \wedge ... \wedge e_{i_k}^* dt.$$

In the following we suppress the subscript k and simply write $s\omega$. We have

$$(i_\gamma s\omega)(a, b) = \sum_{j=1}^{n}(\int_0^1 \frac{\partial f}{\partial y_j}(a, a + t(b - a))t^k$$

$$\cdot\{\sum_{l=1}^{k}(-1)^{l+1}(b-a)_{i_l}e_j^* \wedge e_{i_1}^* \wedge ... \wedge \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^*\} + (b-a)_j e_{i_1}^* \wedge ... \wedge e_{i_k}^*).$$

From this we get the expression

$$(i_\gamma s\omega)(a,b) = \{\sum_{j=1}^{n}\int_0^1 \frac{\partial f}{\partial y_j}(a, a+t(b-a))t^k(b-a)_j e_{i_1}^* \wedge ... \wedge e_{i_k}^* dt\}$$

$$+\{\sum_{l=1}^{k}\sum_{j=1}^{n}\int_0^1 (-1)^{l+1}\frac{\partial f}{\partial y_j}(a, a+t(b-a))t^k(b-a)_{i_l}e_j \wedge e_{i_1}^* \wedge ... \wedge \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^* dt\}.$$

The chain-rule of differentiation applied to the first sum gives

$$i_\gamma s\omega(a,b) = \int_0^1 \frac{d}{dt}f(a, a+t(b-a))t^k e_{i_1}^* \wedge ... \wedge e_{i_k}^* dt$$

$$+\{\sum_{l=1}^{k}\sum_{j=1}^{n}\int_0^1 (-1)^{l+1}\frac{\partial f}{\partial y_j}(a, a+t(b-a))t^k(b-a)_{i_l}e_j \wedge e_{i_1}^* \wedge ... \wedge \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^* dt\}.$$

Now we perform partial integration with the first integral on the right side. This yields us to the following expression

$$i_\gamma s\omega(a,b) = f(a, a+t(b-a))t^k|_0^1 e_{i_1}^* \wedge ... \wedge e_{i_k}^*$$

$$-\int_0^1 f(a, a+t(b-a))kt^{k-1}e_{i_1}^* \wedge ... \wedge e_{i_k}^* dt$$

$$+\{\sum_{l=1}^{k}\sum_{j=1}^{n}\int_0^1 (-1)^{l+1}\frac{\partial f}{\partial y_j}(a, a+t(b-a))t^k(b-a)_{i_l}e_j \wedge e_{i_1}^* \wedge ... \wedge \widehat{e_{i_l}^*} \wedge ... \wedge e_{i_k}^* dt\}.$$

Calculating the first term on the right side is easy and gives

$$i_\gamma s\omega(a,b) = \omega(a,b) + R(a,b),$$

where $R(a,b)$ denotes the rest, i.e. the integral and the double sum on the right side.

Now we calculate the expression $si_\gamma\omega(a,b)$. The definition give us

$$si_\gamma\omega(a,b) = \sum_{l=1}^{k}(-1)^l\{\sum_{j=1}^{n}\int_0^1 \frac{\partial f}{\partial y_j}(a, a+t(b-a))(a+t(b-a)-a)_{i_l}$$

$$-f(a,a+t(b-a))\delta_{j,i_l}t^{k-1}dt\ e_j^*\wedge e_{i_1}^*\wedge ...\widehat{e_{i_l}^*}\wedge ...\wedge e_{i_k}^*\}.$$

Reordering terms and evaluation of the Kronecker symbol δ_{j,i_l} yields to

$$si_\gamma\omega(a,b) = \sum_{l=1}^{k}(-1)^l\{\sum_{j=1}^{n}\int_0^1 \frac{\partial f}{\partial y_j}(a, a+t(b-a))(b-a)_{i_l}t^k dt\ e_j^*\wedge e_{i_1}^*\wedge ...\widehat{e_{i_l}^*}\wedge ...\wedge e_{i_k}^*\}$$

$$+(-1)^{l+1}\int_0^1 f(a,a+t(b-a))t^{k-1}dt\ e_{i_l}^*\wedge e_{i_1}^*\wedge ...\wedge \widehat{e_{i_l}^*}\wedge ...\wedge e_{i_k}^*.$$

Shuffling $e_{i_l}^*$ from the first to the i_l-th position in $e_{i_l}^*\wedge e_{i_1}^*\wedge ...\wedge \widehat{e_{i_l}^*}\wedge ...\wedge e_{i_k}^*$ changes the sign by the factor $(-1)^{l-1}$. This cancels with the factor $(-1)^{l+1}$ in front of the second term on the right side, and we see, that this term is actually independent of the summation index l. Hence for this term summation over l is just multiplication with k. Taking a close look on the summands we can recognize, that we end up with $-R(a,b)$, where $R(a,b)$ was defined on the previous page. So we get

$$si_\gamma\omega(a,b) + i_\gamma s\omega(a,b) = -R(a,b) + \omega(a,b) + R(a,b) = \omega(a,b).$$

This proves

$$si_\gamma + i_\gamma s = id.$$

It is not hard to see, that s is continuous and $\mathbb{C}$-linear. So far we have constructed a topological projective Resolution of $C^\infty(B)$ over $C^\infty(B\times B)$. We are now able to prove the following proposition.

Proposition 6.2.2. *For any $k\in\mathbb{N}$ we have*

$$HH_k(C^\infty(B)) \cong \Omega^k(B)$$

Proof. We calculate the Hochschild homology of $C^\infty(B)$ by tensoring the topological projective resolution from above over $C^\infty(B\times B)$ with $C^\infty(B)$. For any $k\in\mathbb{N}$ we have

$$M_k\hat{\otimes}_{C^\infty(B\times B)}C^\infty(B) = (C^\infty(B\times B)\hat{\otimes}\Lambda^k(\mathbb{C}^{n*}))\hat{\otimes}_{C^\infty(B\times B)}C^\infty(B)$$

$$= \Lambda^k(\mathbb{C}^{n*})\hat{\otimes}C^\infty(B) = \Omega^k(B).$$

Since γ as defined in the construction of our resolution is zero on the diagonal, we have

$$i_\gamma\hat{\otimes}_{C^\infty(B\times B)}id_{C^\infty(B)} = 0.$$

Hence the tensored complex has zero differentials and we get $\Omega^k(B)$ for the k-th homology group of this complex. □

So far, the isomorphism above is more or less abstract. From the universal properties of the various constructions involved, it follows that the maps ϵ_n and π_n as defined in the algebraic case in (6.6) and (6.7) induce corresponding maps

$$\epsilon_n : \bar{\Omega}^n_A \to HH_n(A) \tag{6.10}$$

$$\pi_n : HH_n(A) \to \bar{\Omega}^n_A \tag{6.11}$$

for any unital,nuclear and commutative Fréchetalgebra and $n \in \mathbb{N}$. Here $HH_n(A)$ stands of course for the continuous Hochschild homology of A. It is not hard to see, that the isomorphism of Proposition 6.2.2 is given by these maps. In general we have the following proposition, which is the continuous counterpart to Proposition 6.1.2.

Proposition 6.2.3. *Let A be a unital commutative nuclear Fréchetalgebra. Then the composition $\pi_n \circ \epsilon_n$ is multiplication with $n!$ on $\bar{\Omega}^n_A$. Hence $\bar{\Omega}^n_A$ is a topological direct summand of $HH_n(A)$ and ϵ_n is an embedding.*

Proof. This is completely analogous as in Proposition 6.1.2 □

Let us briefly say something about the functorial properties of continuous Hochschild homology. Clearly a continuous homomorphism between two nuclear Fréchet algebras

$$f : A \to B$$

induces a chain map between the Hochschild complexes and hence maps

$$f_* : HH_n(A) \to HH_n(B),\ \forall n \in \mathbb{N}.$$

The following result is stated in [Karoubi] and can be seen as a Kuenneth like theorem for chain complexes in the world of nuclear Fréchetspaces.

Lemma 6.2.1. *Assume we have two chain complexes*

$$0 \longleftarrow M_0 \xleftarrow{d} M_1 \xleftarrow{d} M_2 ...$$

$$0 \longleftarrow N_0 \xleftarrow{d'} N_1 \xleftarrow{d'} N_2 ...$$

consisting of nuclear Fréchetspaces. Let us further assume, that all homology groups of these two complexes are Hausdorff, in equal the boundary maps have closed images. Then the completed tensorproduct $(M_* \hat{\otimes} N_*, d\hat{\otimes}1 + (-1)^* 1\hat{\otimes}d')$ of both complexes is again a chain complex of nuclear Fréchetspaces and there is a natural isomorphism

$$H_n(M_* \hat{\otimes} N_*) \cong \sum_{p+q=n} H_p(M_*) \hat{\otimes} H_q(N_*).$$

As an application of Lemma 6.2.1 we have the following proposition. It will help us, to prove our main theorem about Hochschild homology of stratifolds in chapter 7.

Proposition 6.2.4. *Let X and Y be stratifolds and assume that $\forall n \in \mathbb{N}$ the antisymmetrization maps*

$$\epsilon_n^X : \bar{\Omega}^n_{\bar{C}^\infty(X)} \to HH_n(\bar{C}^\infty(X))$$

$$\epsilon_n^Y : \bar{\Omega}^n_{\bar{C}^\infty(Y)} \to HH_n(\bar{C}^\infty(Y))$$

are topological isomorphisms. Then the same is true for the antisymmetrization maps

$$\epsilon_n^{X\times Y} : \bar{\Omega}^n_{\bar{C}^\infty_{par}(X\times Y)} \to HH_n(\bar{C}^\infty_{par}(X \times Y)).$$

Proof. Since we know from the assumption that the Hochschild homology groups of $\bar{C}^\infty(X)$ and $\bar{C}^\infty(Y)$ are Hausdorff and furthermore from Proposition 5.7.1 we have $\bar{C}^\infty_{par}(X \times Y) \cong \bar{C}^\infty(X)\hat{\otimes}\bar{C}^\infty(Y)$, we can apply Lemma 6.2.1 as well as Proposition 5.6.3 to get the following commutative diagram where the horizontal maps are isomorphisms

$$\begin{tikzcd}
\bar{\Omega}^n_{\bar{C}^\infty_{par}(X\times Y)} \arrow[r, "\cong"] \arrow[d, "\epsilon_n"] & \sum_{p+q=n} \bar{\Omega}^p_{\bar{C}^\infty(X)} \hat{\otimes} \bar{\Omega}^q_{\bar{C}^\infty(Y)} \arrow[d, "\sum_{p+q=n} \epsilon_p \hat{\otimes} \epsilon_q"] \\
HH_n(\bar{C}^\infty_{par}(X \times Y)) \arrow[r, "\cong"] & \sum_{p+q=n} HH_p(\bar{C}^\infty(X)) \hat{\otimes} HH_q(\bar{C}^\infty(Y))
\end{tikzcd}.$$

That this diagram is indeed commutative follows from compatibility of the antisymmetrization map with products (see [Weibel95], page 322).

∎

We should now consider the nonunital case. The unitization A_+ of a possibly nonunital nuclear Fréchetalgebra A is as a vectorspace isomorphic to $A \oplus \mathbb{C}$ and hence has a natural nuclear Fréchet structure. As in the algebraic case we define the continuous Hochschild homology of A as follows.

Definition 6.2.4. *Let A be a not necessarily unital nuclear Fréchetalgebra. We define its Hochschild homology by*

$$HH_n(A) := coker(i_* : HH_n(\mathbb{C}) \to HH_n(A_+)),$$

where A_+ denotes the unitization of A and i_* denotes the map which is induced by the natural inclusion of $\mathbb{C}$ into A_+.

Clearly, this definition coincides with the older one, in the case A already was unital. Furthermore, we have

$$HH_0 = A_+/k = A$$

$$HH_n(A) = HH_n(A_+), \forall n > 0.$$

Nonunital nuclear Fréchetalgebras often occur as closed ideals in unital nuclear Fréchetalgebras. The nonunital nuclear Fréchetalgebra we are mainly interested in is given by the kernel of the evaluation map

$$ev_x : \bar{C}^\infty(X) \to \mathbb{R}.$$

Analogous to the algebraic case, we have continuous versions of naive Hochschild homology and bar homology which we again denote with $HH_n^{naive}(A)$ and $H_n^{bar}(A)$. The following definition is the continuous counterpart of Definition 6.1.3.

Definition 6.2.5. *Let A be a possibly nonunital nuclear Fréchetalgebra. We call A* **H-unital** *if the continuous bar-complex of A is acyclic, in equal*

$$H_n^{bar}(A) = 0 \ \forall n \in \mathbb{N}.$$

A continuous version of Proposition 6.1.3 can be found in [Brodzki,Lykova99]. In this work one can also find the following excision theorem which is the continuous counterpart of Proposition 6.1.4.

Proposition 6.2.5. *Let $0 \to I \to A \to A/I \to 0$ be an exact sequence of nuclear Fréchetalgebras such that A is unital and I is H-unital. Then there is a long exact sequence of continuous Hochschild homology groups*

$$\cdots \longrightarrow HH_n(I) \longrightarrow HH_n(A) \longrightarrow HH_n(A/I) \xrightarrow{\delta} HH_{n-1}(I) \longrightarrow \cdots .$$

As for continuous Hochschild homology there is also a description of continuous bar-homology as particular torsion product. To be more precise there is a topological isomorphism

$$H_*^{bar}(A) = \overline{Tor}_*^{A_+}(\mathbb{C}, \mathbb{C}), \tag{6.12}$$

where $\overline{Tor}$ denotes the reduced tor groups (see [Wodzicki89]).

As one can possibly imagine, in general it turns out to be very difficult to determine whether a closed ideal I in a unital nuclear Fréchetalgebra is H-unital or not. In our case, we can use a technique introduced by Wodzicki (see [Wodzicki89]) and a result by Voigt (see [Voigt]) to prove the following proposition.

Proposition 6.2.6. *Let B be a unital nuclear Fréchet algebra and*

$$\bar{C}_0^\infty([0,1)) = ker(res : C^\infty((-1,1)) \to C^\infty(-1,0))$$

the completed algebra of smooth function on the c-manifold $[0,1)$ vanishing at zero. Then the nonunital nuclear Fréchet algebra $\bar{C}_0^\infty([0,1))\hat{\otimes}B$ is H-unital.

Proof. Let

$$\alpha = \sum_{i=0}^{\infty} \lambda_i (f_0^i \hat{\otimes} b_0^i) \hat{\otimes} \ldots \hat{\otimes} (f_n^i \hat{\otimes} b_n^i) \in \bar{C}_n(\bar{C}_0^\infty([0,1))\hat{\otimes}B)$$

be an element in the continuous bar complex. Here λ_i is a sequence of complex numbers such that $\sum_{i=0}^{\infty} |\lambda_i| < 1$ and f_j^i respectively b_j^i converge to zero as i goes to infinity (see proposition 5.2.2). The factorization theorem of Voigt (see [Voigt], Thm. 3.4) applied to $\bar{C}_0^\infty([0,1))$ and the sequence (f_0^i) gives us functions $g^i \in \bar{C}_0^\infty([0,1))$ for all $i \in \mathbb{N}$ and $h \in \bar{C}_0^\infty([0,1))$ with the following properties.

1. $f_0^i = h \cdot g^i \; \forall i \in \mathbb{N}$
2. $g^i \in \overline{\bar{C}_0^\infty([0,1)) \cdot (f_0^i | i \in \mathbb{N})}$

The expression in condition 2 denotes the closure of the ideal in $\bar{C}_0^\infty([0,1))$ which is generated by the functions f_0^i. Let us define $\beta \in \bar{C}_n(\bar{C}_0^\infty([0,1))\hat{\otimes}B)$ as

$$\beta = \sum_{i=0}^{\infty} \lambda_i (g^i \hat{\otimes} b_0^i) \hat{\otimes} ... \hat{\otimes} (f_n^i \hat{\otimes} b_n^i).$$

From condition 2 on the previous page, it follows that

$$\beta \in \overline{\bar{C}_0^\infty([0,1))\hat{\otimes}B) \cdot \alpha} \subset \bar{C}_n(\bar{C}_0^\infty([0,1))\hat{\otimes}B). \tag{6.13}$$

Here the term in the middle denotes the closure of the ideal generated by α. A simple calculation shows, that

$$\alpha = b'((h\hat{\otimes}1_B)\hat{\otimes}\beta) + (h\hat{\otimes}1_B)\hat{\otimes}b'(\beta). \tag{6.14}$$

Let us now assume that α is a cycle in the continuous bar complex. Then $b'(\alpha) = 0$. Hence by continuity and $\bar{C}_0^\infty([0,1))\hat{\otimes}B$ linearity of b' it follows from (6.13) that $b'(\beta) = 0$. Hence by (6.14) we have that

$$\alpha = b'((h\hat{\otimes}1_B)\hat{\otimes}\beta)$$

is a boundary in the continuous bar complex and the bar complex is acyclic. □

We will soon use the following corollary.

Corollary 6.2.1. *Let X be a stratifold and let cX denote the cone over X. Then the nuclear Fréchetalgebra $\bar{C}_0^\infty(cX)$ which consists of the smooth maps on cX which vanish at the cone point is H-unital.*

Proof. From Proposition 5.7.1. we have

$$\bar{C}_0^\infty(cX) \cong \bar{C}^\infty(X)\hat{\otimes}\bar{C}_0^\infty([0,1)).$$

The corollary now follows from Proposition 6.2.6 by setting $B = \bar{C}^\infty(X)$. □

The next proposition shows, that when we know the antisymmetrization map is an isomorphism for a stratifold X, it also is for the coned stratifold cX. Besides the localization result in chapter 7, this is the main step towards proving our general result about the Hochschild homology of locally coned stratifolds in section 7.3.

Proposition 6.2.7. *Let X be a stratifold such that $\forall n \in \mathbb{N}$ the antisymmetrization maps for X*

$$\epsilon_n^X : \bar{\Omega}^n_{\bar{C}^\infty(X)} \to HH_n(\bar{C}^\infty(X))$$

are topological isomorphisms, then the same is true for the antisymmetrization maps for cX

$$\epsilon_n^{cX} : \bar{\Omega}^n_{\bar{C}^\infty(cX)} \to HH_n(\bar{C}^\infty(cX)).$$

Proof. Since for $n = 0$ there is nothing to show we can assume $n \geq 1$. By naturality of the antisymmetrization map and Proposition 5.7.2, H-unitality of $\bar{C}_0^\infty(cX)$ induces the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & \bar{\Omega}^n_{\bar{C}^\infty(cX)} \arrow[r] \arrow[d] & \bar{\Omega}^n_{\bar{C}^\infty(X\times(-1,1))} \arrow[r] \arrow[d] & \bar{\Omega}^n_{\bar{C}^\infty(X\times(-1,0])} \arrow[r] \arrow[d] & 0 \\
\cdot\cdot \arrow[r, "\delta"] & HH_n(\bar{C}^\infty(cX)) \arrow[r] & HH_n(\bar{C}^\infty(X \times (-1,1))) \arrow[r] & HH_n(\bar{C}^\infty(X \times (-1,0])) \arrow[r, "\delta"] & \cdot\cdot
\end{tikzcd}$$

Here the half open interval $(1, 0]$ has been treated in the naive sense. The vertical maps in this diagram are given by the various antisymmetrization maps. The ones at the right side are isomorphisms by Proposition 6.2.6.Since we have the diagram available $\forall n \geq 1$ it follows that the connecting homomorphism δ is zero. Hence we can replace "..." in the diagram by 0 and the proposition follows from the five lemma. $\square$

Kapitel 7

Hochschild Homology of Stratifolds

In the case that M is a closed manifold, Alain Connes proved in [Connes87] that the continuous Hochschild homology of the algebra $C^\infty(M)$ is isomorphic to the module of differential forms on M, where both are considered as modules over $C^\infty(M)$.Using methods of Teleman it can be shown, that the latter is true also for non compact manifolds with boundary. In this chapter we will generalize this result to the case where X is a locally coned stratifold. Not much is known about the algebraic Hochschild homology of $C^\infty(M)$, so we won't say anything about the algebraic Hochschild homology of $C^\infty(X)$ for a stratifold X.

7.1 The Hochschild Complex of a Stratifold

In this section, we will rewrite the Hochschild complex of a stratifold X, which by definition is the continuous Hochschild complex of the algebra $\bar{C}^\infty(X)$ in form of smooth functions on Cartesian products of X. This makes the Hochschild complex more favourable to topological constructions such as partitions of unity etc. Since continuous Hochschild homology is only defined on nuclear Fréchetalgebras, it is necessary to work with the completed version $\bar{C}^\infty(X)$ of $C^\infty(X)$. To shorten the notation we write $C_*(X)$ for the Hochschild complex of X.

In the following sections we use the natural isomorphism of Proposition 5.7.1 to identify the Hochschild complex with the following complex

$$C_n(X) = \bar{C}^\infty_{par}(X^{n+1}) \tag{7.1}$$

$$(bF)(x_0, ..., x_{n-1}) = \sum_{i=0}^{n-1} (-1)^i F(x_0, ..., x_i, x_i, ..., x_{n-1}) + (-1)^n F(x_0, ..., x_{n-1}, x_0), \tag{7.2}$$

where F denotes an n chain interpreted as a function on the $(n+1)$-fold Cartesian product of X. The subscript *par* is explained in Proposition 5.7.1. This form of the Hochschild complex of X will be of particular importance in the following section.

7.2 Localization of the Hochschild Complex

In this section we show, that the Hochschild complex of a stratifold X contains a large acyclic subcomplex. This subcomplex consists of the Hochschild chains

$$F : X^{n+1} \to \mathbb{R},$$

which vanish in a neighbourhood of the diagonal $\Delta_{n+1} \subset X^{n+1}$. The methods applied by Teleman in [Teleman98] to show this for the case of a smooth manifold, also work in the case of a stratifold, once we have proven the following lemma. For a matter of completeness we also illustrate Teleman's ideas.

Lemma 7.2.1. *Let X be a stratifold, then there exists a metric d on X which generates the topology and satisfies*

$$d^2 \in \bar{C}^{\infty}_{par}(X \times X).$$

Proof. To show the existence of such a metric d on X, we will modify the proof of the Urysohn metrization theorem, which states that every regular T_1 space with countable base of topology is metrizable. During the discussion of the basic properties of a stratifold in chapter 1, we mentioned that, for any two disjoint and closed subsets A and B of X there is a function $f_{A,B} \in C^{\infty}(X)$ such that $A \subset f_{A,B}^{-1}(0)$ and $B \subset f_{A,B}^{-1}(1)$. This function also belongs to $\bar{C}^{\infty}(X)$. Let us now consider a complete family F of such function, that is for any two disjoint and closed subsets A and B of X there is $f_{A,B} \in F$ as above. We can assume that F is countable. Let $[0,1]^F$ denote the space $map(F, [0,1])$ where $[0,1]$ denotes the unit interval and the topology is given by the product topology. Let us assume that F is given by the family $\{f_n | n \in \mathbb{N}\}$. Then we can identify $[0,1]^F$ with the space of infinite sequences $(x_n)_{n\in\mathbb{N}}$ with entries in the interval $[0,1]$. It is a standard exercise in analysis that

$$d_\infty((x_n),(y_n)) := \sqrt{\sum_{i=1}^{\infty} \frac{1}{2^n}(x_n - y_n)^2}$$

is a metric on $[0,1]^F$ which generates the topology. Obviously this metric has the property, that when fixing all but one coordinate, it's square depends smoothly on that free coordinate. Now, as one can see in the book [Kelley] on page 125 for example, the map

$$\psi : X \to [0,1]^F$$

$$x \mapsto (f(x))_{f\in F}$$

is a topological embedding. Since all component functions are elements of $\bar{C}^\infty(X)$ it is clear that ψ is also smooth. Here we consider a map on the infinite dimensional space $[0,1]^F$ as smooth, if and only if it is partially smooth. Since composition of smooth maps is smooth we find that

$$\psi^* d_\infty : X \times X \to \mathbb{R}$$

$$(x,y) \mapsto d_\infty(\psi(x),\psi(y))$$

has the property $(\psi^* d_\infty)^2 \in \bar{C}^\infty_{par}(X \times X)$. Setting $d := \psi^* d_\infty$ will finish the proof. $\square$

We can now proceed with the Teleman method. Let $\lambda : [0,\infty) \to [0,1]$ be a smooth function, such that $supp(\lambda) \subset [0,1]$ and $\lambda_{|[0,1/2]} \equiv 1$. For $t > 0$ we define

$$\lambda_t : [0,\infty) \to [0,1],$$

$$\lambda_t(s) := \lambda(s/t).$$

These functions have the following properties :

1. $supp(\lambda_t) \subset [0,t]$
2. $\lambda_{t|[0,t/2]} \equiv 1$

Now for any $k \in \mathbb{N}$ let us define functions ρ_k as

$$\rho_k : X^{k+1} \to [0, \infty)$$

$$\rho_k(x_0, x_1, ..., x_k) = d(x_0, x_1)^2 + d(x_1, x_2)^2 + ... + d(x_k, x_0)^2. \tag{7.3}$$

Here d denotes a function on $X \times X$ such as in Lemma 7.2.1 . In words, ρ_k measures the distance of a point in X^{k+1} from the diagonal. Clearly $\rho_k \in \bar{C}^{\infty}_{par}(X^{k+1})$. Let

$$U_{t,k} := \{(x_0, x_1, ..., x_k) | \rho_k(x_0, ..., x_k) < t\}$$

be the t-neighbourhood of the diagonal $\Delta_{k+1} \subset X^{k+1}$. Let $C^t_*(X)$ be the subcomplex of the Hochschild complex $C_*(X)$ where $C^t_k(X)$ contains the elements of $C_k(X)$ vanishing on $U_{t,k}$. Let

$$C^0_*(X) = \varinjlim C^t_*(X)$$

where the limit goes as t goes to zero. The complex $C^0_*(X)$ consists of the chains vanishing in an arbitrary neighbourhood of the diagonal.

Proposition 7.2.1. *Let X be a stratifold.The complex $C^0_*(X)$ is acyclic.*

Proof. We define an operator

$$E_t : C_k(X) \to C_{k+1}(X),$$

$$E_t(F)(x_0, ..., x_{k+1}) = \lambda_t(d(x_0, x_1)^2) \cdot F(x_1, ..., x_{k+1}), \forall F \in C_k(X)$$

This operator maps C^s_k into $C^{s/4}_{k+1}$ which can easily be verified. A calculation also shows that

$$b \circ E_t + E_t \circ b = 1 - N_t,$$

where N_t is defined as

$$N_t(F)(x_0, ..., x_k) = (-1)^k \lambda_t(d(x_0, x_1)^2) \cdot \{F(x_1, x_2, ...x_k, x_0) - F(x_1, x_2, ...x_k, x_1)\}$$

$$\forall F \in C_k(X).$$

Direct calculation also shows that $b \circ N_t = N_t \circ b$. Let's consider the k-th power of N_t that is $(N_t)^k$. For $F \in C_k(X)$ we get

$$(N_t)^k F(x_0, ..., x_k) = \prod_{i=0}^{k-1} \lambda_t(d(x_i, x_{i+1})^2) \cdot G(x_0, ..., x_k),$$

where $G(x_0, ..., x_k)$ is a linear combination of functions built out of F by restricting to certain diagonals and permutation of some arguments. For the product in front of G to be not zero, we must have $d(x_i, x_{i+1})^2 < t$ for each $0 \leq i \leq k-1$. The triangle equation shows that in this case we also have $d(x_0, x_k) < kt^{1/2}$. Hence we have

$$\rho_k(x_0, ..., x_k) = \sum_{i=0}^{k-1} d(x_i, x_{i+1})^2 + d(x_k, x_0)^2 < kt + k^2 t.$$

Hence for $F \in C_k^{(k+k^2)t}(X)$ we have that $(N_t)^k(F) = 0$. Let's define another operator

$$K_t : E_t \cdot \sum_{r=0}^{k-1} (N_t)^r : C_k^{(k+k^2)t}(X) \to C_{k+1}^{(k+k^2)4^{-(k+1)}t}(X).$$

By construction this operator satisfies

$$b \circ K_t + K_t \circ b = 1,$$

which proves the theorem by taking the direct limit where t goes to zero. $\square$

From the previous proposition we know, that any Hochschild class in $HH_n(\bar{C}^\infty(X))$ can now be represented by a cycle F which has support arbitrary close to the diagonal. One can now use a partition of unity and the $\bar{C}^\infty(X)$ module structure on $HH_n(\bar{C}^\infty(X))$ to see that the following corollary is true.

Corollary 7.2.1. *Let X be a stratifold and $(U_i|i \in I)$ be a locally finite open covering of X, where I is some index set. Let further $F \in C_n(X)$ be a Hochschild cycle. Then there are Hochschild cycles $F_i \in C_n(X)$ such that $supp(F_i) \in (U_i)^{n+1}$ $\forall i \in I$ and*

$$F \sim \sum_{i \in I} F_i$$

are homologous.

7.3 Hochschild homology of locally coned stratifolds

In this section we will finally show that the Hochschild homology of the algebra $\bar{C}^\infty(X)$ of a locally coned stratifold X is isomorphic to the module $\bar{\Omega}^n_{\bar{C}^\infty(X)}$ of differential forms. Besides the result on de Rham cohomology of stratifolds this can be seen as the main result of this work. After all the work we did in chapters 5 and 6 and in the beginning of chapter 7, the proof seems to be quite easy.

Theorem 7.3.1. *Let X be a locally coned stratifold. Then $\forall n \in \mathbb{N}$ the antisymmetrization maps*

$$\epsilon_n : \bar{\Omega}^n_{\bar{C}^\infty(X)} \to HH_n(\bar{C}^\infty(X))$$

are topological isomorphisms.

Proof. Let $n \in \mathbb{N}$. We have to show that the antisymmetrization map ϵ_n is surjective in equal any Hochschild cycle in $HH_n(\bar{C}^\infty(X))$ is antisymmetric. From Corollary 7.2.1. it suffices to show, that this is locally the case. Hence we can assume that our stratifold is of the kind $B^k \times cL$ where B^k denotes the open unit ball of dimension k and cL denotes the open cone over a stratifold of dimension less than the dimension of X. Using induction on the dimension, we can assume that the antisymmetrization maps ϵ_n^L for L are isomorphisms $\forall n \in \mathbb{N}$. From Proposition 6.2.7 it then follows that the antisymmetrization maps for cL are also isomorphisms. From Proposition 6.2.2. it follows, that the antisymmetrization maps for B^k are isomorphisms. Hence the theorem follows from Proposition 6.2.6. □

As we mentioned earlier, the same proof goes through for locally product coned stratifolds.

7.4 Some Remarks on Cyclic Homology of Stratifolds

This section is only informal, so we don't give any proofs and don't bother to define things exactly.

If we divide out a cyclic action from the Hochschild complex (6.8), in equal identifying cycles, which arise from another by cyclic permutation, we get another complex, which is sometimes called Connes' complex. The homology groups of this complex are called **cyclic homology groups** and will be denoted by

$$HC_n(A).$$

These groups are related to the Hochschild homology groups by the so called Connes' exact sequence

$$...HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \xrightarrow{I} \cdots .$$

The operator S is the so called Connes periodicity operator and corresponds via some identifications to the Bott periodicity operator in K-theory. In the commutative case it is not hard to show, that via the antisymmetrization map, up to a factor the operator

$$B \circ I : HH_n(A) \to HH_{n+1}(A)$$

exactly corresponds to the operator

$$d : \bar{\Omega}^n_A \to \bar{\Omega}^{n+1}_A.$$

Using this and Connes' exact sequence one can proceed exactly as in [Connes87] to prove the following.

Proposition 7.4.1. *Let X be a locally coned stratifold with finite dimensional homology groups. Then $\forall n \in \mathbb{N}$ there is a natural topological isomorphism*

$$HC_n(\bar{C}^\infty(X)) \cong \bar{\Omega}^n_{\bar{C}^\infty(X)}/d\bar{\Omega}^{n-1}_{\bar{C}^\infty(X)} \oplus H^{n-2}_{dR}(X) \oplus H^{n-4}_{dR}(X)....$$

7.5 Closing Remarks

In the end, the reader has the right to ask, why it might be important to know something about the Hochschild homology of locally coned stratifolds. In the framework of index theory on manifolds as well as in the framework of noncommutative geometry, Hochschild homology and in particular cyclic homology have been proven successful. One could say that this door has been opened by Connes' work about the cyclic homology of the algebra $C^\infty(M)$ for a smooth manifold M. For example, people studied Hochschild and cyclic homology of algebras consisting of pseudo differential operators on manifolds (see [Schulze]). Motivated by questions from theoretical physics, people began studying the analysis of singular spaces. In their considerations, some kind of differential operators on singular spaces play a role. One might now hope to learn something about these, by studying their Hochschild homology for example. Though we must clearly say, that the approach on singular spaces, which we have taken in this work is probably to naive and not suitable for more complicated analytic constructions (like for example differential operators, connections etc.) it is to my knowledge the first complete result about the Hochschild homology of some version of singular spaces. We hope the reader thinks this is justification enough to have spent some of his time reading this work.

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