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Hirzebruch Homology

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Introduction

For a discrete group π and a rational cohomology class $x \in H^*(K(\pi,1);\mathbb{Q})$, the higher signature determined by x is the characteristic number

$$\mathrm{sig}_x : \Omega_*^{SO}(K(\pi,1)) \longrightarrow \mathbb{Q}$$

$$[M,\alpha] \longmapsto < L(M) \cup \alpha^*(x), [M] >$$

where $L(M)$ is the L-class of M.

By definition, the signature $\mathrm{sig}_1(M,\alpha) =< L(M), [M] >$ only depends on M. Furthermore, according to the Hirzebruch signature theorem, the number $< L(M), [M] >$ is equal to the index of the intersection form of M, and thus it follows that if there exists an orientation-preserving homotopy equivalence $N \sim M$, then for all α, β

$$\mathrm{sig}_1(M,\alpha) = \mathrm{sig}_1(N,\beta).$$

In general however, the higher signatures do also depend on the map α, and consequently one cannot expect to find such an invariance property. For this reason, one says that the higher signature sig_x is homotopy invariant if for every orientation-preserving homotopy equivalence $f : N \to M$ and for every $\alpha : M \to K(\pi,1)$

$$\mathrm{sig}_x(M,\alpha) = \mathrm{sig}_x(N, \alpha \circ f).$$

Novikov discovered that for manifolds with fundamental group $\pi = \mathbb{Z}$ all higher signatures are homotopy invariant, while Rokhlin studied the case of $\pi = \mathbb{Z} \times \mathbb{Z}$. These examples led Novikov to the formulation of the general conjecture:

The Novikov Conjecture. *For any $x \in H^*(K(\pi,1);\mathbb{Q})$, the higher signature determined by x is homotopy invariant.*

It is clear that the Novikov conjecture is a rational statement, and it would be interesting to have an integral formulation of it. A classical approach to this question makes use of L-theory: it has been shown that the Novikov conjecture is equivalent to the assertion that the assembly map is a rational injection, and thus an integral version of the Novikov conjecture can be obtained by requiring the assembly map to be an integral split injection.

What we want to discuss here is a more geometrical and intuitive approach which has been suggested recently by Matthias Kreck. Kreck's idea is to introduce a homology theory $hh_*(-)$, which he calls **Hirzebruch homology**, and which has the following fundamental property:

1. there is a natural transformation $u : \Omega_*^{SO}(-) \to hh_*(-)$
2. there is an isomorphism $\gamma : hh_*(\mathrm{pt}) \xrightarrow{\simeq} \mathbb{Z}[t]$ such that the following diagram commutes:

$$\begin{array}{ccc} \Omega_*^{SO} & \xrightarrow{u_*} & hh_*(\mathrm{pt}) \\ {\scriptstyle\tau}\big\downarrow & \swarrow{\scriptstyle\gamma} & \\ \mathbb{Z}[t] & & \end{array}$$

Here τ is the ring homomorphism

$$\tau : \Omega_*^{SO} \longrightarrow \mathbb{Z}[t]$$

$$[M^n] \longmapsto \mathrm{sig}(M^n) \cdot t^{n/4}$$

Let u_* be the group homomorphism

$$u_* : \Omega_*^{SO}(M) \longrightarrow hh_*(M)$$

induced by the natural transformation u for an n-dimensional oriented manifold M. This homomorphism maps the bordism class $[M, \mathrm{id}] \in \Omega_n^{SO}(M)$ to the element $u_*([M, \mathrm{id}]) \in hh_n(M)$, which we call the Hirzebruch fundamental class of M, and which we denote for simplicity by $[M]$. If $\alpha : M \to X$ is any continuous map, then we indicate by $[M, \alpha]$ the element $\alpha_*([M]) \in hh_n(X)$.

The Hirzebruch fundamental class is said to be homotopy invariant for a discrete group π, if for any orientation-preserving homotopy equivalence $f : N \to M$ and for any map $\alpha : M \to K(\pi, 1)$

$$[M, \alpha] = [N, \alpha \circ f] \in hh_n(K(\pi, 1))$$

Integral Novikov Problem (M. Kreck). *Determine all discrete groups π for which the Hirzebruch fundamental class is homotopy invariant.*

If one takes the tensor product with $\mathbb{Q}$, then there is an isomorphism

$$hh_*(M) \otimes \mathbb{Q} \simeq H_*(M; \mathbb{Q}[t])$$

and it can be proved that the rational Hirzebruch fundamental class $[M] \in hh_n(M) \otimes \mathbb{Q}$ is mapped to the Poincaré dual class of $L(M)$. The Novikov conjecture is therefore equivalent to the homotopy invariance of the rational Hirzebruch fundamental class for every discrete group π, and thus the problem formulated above can be interpreted as an integral generalization of the Novikov conjecture.

Replacing the homotopy invariance of the higher signatures with their topological invariance, one gets a weaker statement, which is nevertheless far from being trivial. In fact, since the discovery of exotic spheres it is known that there are homeomorphisms which are not homotopic to diffeomorphisms and so there is a priori no reason for the Pontrjagin classes to be invariant under homeomorphisms. However it holds the

Theorem (Novikov). *The rational Pontrjagin classes are topological invariant.*

This highly non-trivial result implies the topological invariance of the L-class and consequently of the higher signatures. For the integral Novikov problem the situation looks quite different, and in fact there is no obvious way to reduce this problem to Novikov's theorem. This motivated Kreck to formulate the following

Conjecture. *The Hirzebruch fundamental class is a topological invariant.*

Up to now we have used the properties of Hirzebruch homology without any mention of its construction. Since Hirzebruch homology is after all the subject of this thesis, it is worth to spend some words on it.

According to Milnor, the unitary bordism ring Ω_*^U is isomorphic to the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$. An easy computation shows that one can take a basis sequence $\{x_1, x_2, \ldots\}$ which satisfies the condition

$$\operatorname{sig}(x_n) = \begin{cases} 0 & \text{for } n \neq 2 \\ 1 & \text{for } n = 2 \end{cases}$$

If such a basis sequence $\{x_1, x_2, \ldots\}$ has been fixed, Hirzebruch homology is by definition the homology theory $hh_*(-)$ obtained as the unitary bordism with Baas-Sullivan singularities $\{x_n\}_{n\neq 2}$.

This construction provides only a transformation

$$\Omega_*^U(-) \longrightarrow hh_*(-),$$

while, in order to define Hirzebruch fundamental classes for smooth oriented manifolds, one needs a factorization over $\Omega_*^{SO}(-)$ as illustrated in the following diagram:

$$\begin{array}{ccc} \Omega_*^U(-) & \longrightarrow & hh_*(-) \\ \downarrow & \nearrow & \\ \Omega_*^{SO}(-) & & \end{array}$$

Now, there is no obvious way to find such a dashed transformation, and so it only remains to try to modify the construction of $hh_*(-)$. The general idea is to construct a homology theory $hh'_*(-)$ with $hh'_*(\mathrm{pt}) \simeq \mathbb{Z}[t]$ for which the two diagrams

$$\begin{array}{ccc} \Omega_*^U(-) & \longrightarrow & hh_*(-) \\ \downarrow & & \downarrow \\ \Omega_*^{SO}(-) & \longrightarrow & hh'_*(-) \end{array}$$

and

$$\begin{array}{ccc} hh_*(\mathrm{pt}) & \longrightarrow & hh'_*(\mathrm{pt}) \\ & {\scriptstyle\simeq}\searrow \quad \swarrow{\scriptstyle\simeq} & \\ & \mathbb{Z}[t] & \end{array}$$

commute. By the comparison theorem, the transformation $hh_*(-) \to hh'_*(-)$ is an isomorphism with inverse $hh'_*(-) \to hh_*(-)$, and then the composition

$$\Omega_*^{SO}(-) \longrightarrow hh'_*(-) \longrightarrow hh_*(-)$$

yields the desired factorization.

In order to define the theory $hh'_*(-)$, one could think of mimic the construction of $hh_*(-)$ replacing $\Omega^U_*(-)$ with $\Omega^{SO}_*(-)$. Actually, the situation is now more complicated and in fact, since the coefficients of $\Omega^{SO}_*(-)$ contain elements of 2-torsion, the Baas-Sullivan construction fails (or, to be more precise, there is no more control on the coefficients). Kreck has solved the problem proposing a modified Baas-Sullivan construction in which objects with singularities can be resolved too, and could therewith show the existence of $hh'_*(-)$.

This thesis has the twofold purpose of presenting a natural, geometric construction of Hirzebruch homology, and at the same time of extending the definition of Hirzebruch fundamental class to closed oriented topological manifolds.

The starting point of our construction is a general method to define new homology theories, recently introduced by Kreck, which consists of considering certain classes of topological spaces called stratifolds, and then to pass to their bordism.

Since we want to define fundamental classes for topological manifolds, it is convenient to start with the class of topological stratifolds. The bordism of topological stratifolds without any further assumptions is the zero theory. Thus, to get the right coefficients, we have to restrict our attention to stratifolds admitting some additional structure. The main ingredient is now provided by the theory of self-dual complexes of sheaves: this notion has been introduced and developed by Markus Banagl with the aim of extending the signature to pseudomanifolds with odd codimensional strata, but it is easy to adapt it to topological stratifolds.

Using Banagl's theory we define an H-stratifold as a topological stratifold S together with an H-structure, i.e. a triple $(\mathfrak{o}, \mathbf{A}^\bullet, \nu)$, where $\mathfrak{o}$ is an orientation of the top stratum of S, $\mathbf{A}^\bullet$ is a self-dual complex of sheaves over S, and ν is a normalization of $\mathbf{A}^\bullet$. We denote by $Hh_*(-)$ the bordism theory of H-stratifolds.

Our main result is the

Theorem. *$Hh_*(-)$ is a homology theory for which:*

1. $Hh_*(\mathrm{pt}) \simeq \mathbb{Z}[t]$

2. *there exist two natural transformations*

$$\Omega^{TOP}_*(-) \longrightarrow Hh_*(-) \quad \textit{and} \quad hh_*(-) \longrightarrow Hh_*(-)$$

such that the diagrams

$$\begin{array}{ccc} \Omega_*^U(-) & \longrightarrow & hh_*(-) \\ \downarrow & & \downarrow \\ \Omega_*^{TOP}(-) & \longrightarrow & Hh_*(-) \end{array}$$

and

$$\begin{array}{ccc} hh_*(\mathrm{pt}) & \longrightarrow & Hh_*(\mathrm{pt}) \\ & \searrow \quad \swarrow & \\ & \mathbb{Z}[t] & \end{array}$$

commute.

This theorem allows to achieve the two goals set above. First of all the natural transformation $hh_*(-) \to Hh_*(-)$ is by the comparison theorem an isomorphism of homology theories and therefore it makes sense to consider the bordism theory of H-stratifolds as a new geometric construction of Hirzebruch homology. On the other hand the problem of defining Hirzebruch fundamental classes can be solved using the transformation $\Omega_*^U(-) \to Hh_*(-)$.

The theorem above has another interesting consequence:

Corollary. *The Hirzebruch fundamental class is a topological invariant.*

As observed above, the rational Hirzebruch fundamental class contains the same information as the L-class and therefore as the rational Pontrjagin classes. As integral classes there is however an essential difference: in contrast to the Hirzebruch class the Pontrjagin classes are only defined for smooth manifolds and are not topological invariant. It would be interesting to understand wether the Hirzebruch class really carries more information than the rational Pontrjagin classes do.

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Contents

Chapter 1

Stratifolds

In this chapter we present the definition of topological stratifolds and explain some of their properties. This notion has been firstly introduced by Matthias Kreck in 1998. Through a series of modifications, the term "stratifold" has in the meanwhile come to indicate a different class of spaces, while the original objects are now called p-stratifolds. In this thesis, however, we adopt the original terminology.

1.1 c-Manifolds

This section is devoted to the definition of topological c-manifolds (collared manifolds). In contrast to the case of smooth manifolds, the role played by the choice of a collar for a topological manifold is not so relevant; however, it is convenient to speak of topological c-manifolds in order to underline the analogy between topological and smooth stratifolds.

Except where otherwise indicated, by a manifold we will always mean a *metrizable topological manifold without boundary.*

Let $(W, \partial W)$ be a pair of spaces with ∂W closed in W, and denote by $\mathring{W}$ the open set $W - \partial W$. If $\delta : \partial W \to (0, +\infty)$ is a continuous function, then we introduce the following notation:

$$(\partial W \times [0, +\infty))^{<\delta} := \{(x, t) \in \partial W \times [0, +\infty) | t < \delta(x)\}$$
$$(\partial W \times [0, +\infty))^{\leq\delta} := \{(x, t) \in \partial W \times [0, +\infty) | t \leq \delta(x)\}$$
$$(\partial W \times (0, +\infty))^{<\delta} := \{(x, t) \in \partial W \times (0, +\infty) | t < \delta(x)\}$$

Definition 1.1. *A **collar** is a homeomorphism $c : V \to U$ where V is an open neighborhood of $\partial W \times \{0\}$ of the form $(\partial W \times [0, +\infty))^{<\delta}$ and U is an open neighborhood of ∂W in W, so that for any $x \in \partial W$ it is $c(x, 0) = x$*

Collars are used in differential topology to glue smooth manifolds along their boundary and it is known that the only relevant data of a collar is its behavior in proximity of the boundary. For this reason we say that two collars $c : V \to U$ and $c' : V' \to U'$ are **equivalent** if there is an open neighborhood V'' of $\partial W \times \{0\}$ with $V'' \subset V \cap V'$ and such that $c|_{V''} = c'|_{V''}$. The equivalence class of a collar $c : V \to U$ is called **germ of collars** and will be denoted by $[c : V \to U]$.

If ∂W is compact then every collar is equivalent to one of the form $c : \partial W \times [0, +\varepsilon) \to U$.

Definition 1.2. *An n-dimensional **c-manifold** is a pair $(W, \partial W)$ where*

- *W is a metrizable space;*
- *$\mathring{W}$ and ∂W are manifolds of dimension respectively n and $n-1$*

*together with a germ of collars $[c : V \to U]$. The manifold ∂W is called the **boundary** of W.*

If $c : V \to U$ is a representative of the germ of collars of a c-manifold W, then we denote by π the composition

$$U \xrightarrow{\quad c^{-1} \quad} V \xrightarrow{\quad \pi_1 \quad} \partial W.$$

As one can expect, when speaking of maps of c-manifolds, it is natural to restrict the attention to those maps which are compatible with the collars. This compatibility condition leads to the following

Definition 1.3. *Let W be a c-manifold and M a manifold. A continuous map*

$$f : W \longrightarrow M$$

*is called a **c-map** if there is a representative of the germ of collars $c : V \to U$ such that for all $x \in U$ it holds*

$$f(x) = f(\pi(x)).$$

c-Maps have the following property.

Lemma 1.4. *Let $f : W \to M$ be a continuous map from a c-manifold to a manifold and suppose that there is a closed set A and an open neighborhood U of A so that $f|_U$ is a c-map. Then f is homotopic relative $\partial W \cup A$ to a c-map.*

Another fundamental point which we have to explain is when two c-manifolds have to be considered equivalent.

Definition 1.5. *Let W_1 and W_2 be two c-manifolds. A homeomorphism*

$$f : W_1 \longrightarrow W_2$$

is called an ***isomorphism of c-manifolds*** *if there are representatives of the germs of collars $c_1 : V_1 \to U_1$ and $c_2 : V_2 \to U_2$ such that for all $(x,t) \in V_1$ with $f(c_1(x,t)) \in U_2$ it holds*

$$f(c_1(x,t)) = c_2(f(x),t).$$

If W is a c-manifold and M is any manifold, then it is easily seen that the product $W \times M$ is a c-manifold with boundary $\partial W \times M$.

Now, let W_1 and W_2 be two c-manifolds. As known from differential topology, the product $W_1 \times W_2$ is a manifold with corners and there is a procedure to smoothen the corners and therewith to obtain a c-manifold. The same argument can be used to prove the following lemma.

Lemma 1.6. *The product $W_1 \times W_2$ is a c-manifold with boundary*

$$\partial(W_1 \times W_2) = (\partial W_1 \times W_2) \cup (W_1 \times \partial W_2).$$

The last property which we want to mention is the following

Lemma 1.7. *Let $\mathscr{M}^c$ denote the class of all c-manifolds. There exists a small subclass $\mathscr{M}^c_0 \subset \mathscr{M}^c$ so that every c-manifold is isomorphic to an element of $\mathscr{M}^c_0$.*

1.2 Stratifolds

In this section we give a brief introduction to the theory of the topological stratifolds. The reader is referred to [Kr1] and [Kr2] for proofs and details.

Let X be a topological space. A k-dimensional **strat** of X is a pair (W,f) where W is a k-dimensional c-manifold, and f is a proper continuous map from W to X such that $f|_{\mathring{W}} : \mathring{W} \to f(\mathring{W})$ is a homeomorphism.

The easiest non-trivial example of strat is provided by the cone over a compact manifold. Recall that if X is a topological space, then the (open) **cone** over X is the quotient

$$\mathring{C}X = \frac{X \times [0,1)}{X \times \{0\}}.$$

and that the point corresponding to $X \times \{0\}$ is called the vertex of the cone.

Example. If M is an n-dimensional closed manifold, then the projection

$$M \times [0,1) \longrightarrow \mathring{C}M$$

is an $(n+1)$-dimensional strat of $\mathring{C}M$.

Using the notion of a strat, one can give the following

Definition 1.8. *An n-dimensional **stratifold** is a pair $(X, \mathscr{P})$ where X is a topological space X, and $\mathscr{P}$ is a sequence of strats $\{W_i, f_i\}_{i \leq n}$ which satisfy the following conditions:*

- $\bigsqcup_i f_i(\mathring{W}_i) = X$
- $\dim W_i = i$
- $f_i(\partial W_i) \subset \bigcup_{j \leq i-1} f_j(W_j)$
- *a subset $U \subset X$ is open if and only if for all i the set $f_i^{-1}(U)$ is open in W_i*

*The sequence $\mathscr{P} = \{W_i, f_i\}$ is called the **parametrization** of X, and the restrictions $f_i|_{\partial W_i}$ are called the **attaching maps** of X.*

For simplicity a stratifold $(X, \mathscr{P})$ will be generally denoted just by X.

If X is a stratifold, then the subspaces

$$\Sigma_k(X) := \bigcup_{i=0}^{k} f_i(W_i) \subset X \qquad \text{and} \qquad X_k := \Sigma_k(X) - \Sigma_{k-1}(X)$$

are called respectively the k-th **skeleton** and the k-th **stratum** of X.

A stratifold is said to be **purely n-dimensional** if X_n is dense in X, and it is called **$\mathbb{Z}/2$-orientable** if X_{n-1} is empty.

Remark. If X is a purely n-dimensional stratifold, then the cohomological dimension of X equals n.

The k-th stratum of a stratifold X is by construction homeomorphic to the interior of W_k, and it is therefore an (eventually empty) k-dimensional manifold.

Definition 1.9. *An n-dimensional stratifold X is called **orientable** if it is $\mathbb{Z}/2$-orientable and if the top stratum X_n is orientable. An **orientation** of X is by definition an orientation of X_n.*

A standard argument shows that the collars of the manifolds W_i define, for any k, a canonical germ of retractions

$$\pi_k : V_k \longrightarrow X_k$$

where V_k is an open neighborhood of X_k in X.

We now want to take a look to some constructions which play an important role in the theory of the stratifolds.

Example. Let X and X' be two stratifolds with parametrizations respectively $\mathscr{P} = \{W_i, f_i\}$ and $\mathscr{P}' = \{W_i', f_i'\}$.

1. Any open subset $U \subset X$ is a stratifold with the parametrization $\mathscr{P}|_U := \{f_i^{-1}(U), f_i|_{W_i(U)}\}$.

2. If X is compact, then the cone $\overset{\circ}{C}X$ over X is a stratifold with the parametrization $\mathscr{P}(\overset{\circ}{C}X) := \{W_i(\overset{\circ}{C}X), f_i(\overset{\circ}{C}X)\}$, where

$$W_i(\overset{\circ}{C}X) = \begin{cases} \{\text{pt}\} & \text{for } i = 0 \\ W_{i-1} \times [0, +1) & \text{for } 1 \leq i \leq n+1 \end{cases}$$

   and the maps are given by

$$f_i(\overset{\circ}{C}X) = \begin{cases} \varepsilon & \text{for } i = 0 \\ (f_{i-1} \times \text{i}d) \cup \varepsilon & \text{for } 1 \leq i \leq n+1 \end{cases}$$

   Here ε indicates the constant map.

3. For any manifold M, the product $X \times M$ is a stratifold with the parametrization $\mathscr{P}(X \times M) := \{W_i \times M, f_i \times \text{id}\}$.

4. The product $X \times X'$ is a stratifold with the parametrization

$$\mathscr{P}(X \times X') := \left\{ \bigsqcup_{a+b=i} W_a \times W_b, \bigsqcup_{a+b=i} f_a \times f_b' \right\}.$$

Of course, it is of fundamental importance to explain the notion of isomorphism of stratifolds. Let X and X' be two n-dimensional stratifolds with stratifications respectively $\mathscr{P} = \{W_i, f_i\}$ and $\mathscr{P}' = \{W_i', f_i'\}$.

Definition 1.10. *An* ***isomorphism*** *from X to X' is a homeomorphism*

$$\varphi : X \longrightarrow X'$$

together with a sequence of isomorphisms of c-manifolds $\varphi_i : W_i \to W_i'$ which make the following diagram commutative for every i:

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ \uparrow{\scriptstyle f_i} & & \uparrow{\scriptstyle g_i} \\ W_i(X) & \xrightarrow{\varphi_i} & W_i(Y) \end{array}$$

Now, let X be a topological space and let $\{U_i | i \in J\}$ be a family of open subsets of X with the property that every U_i is a stratifold with parametrization $\mathcal{P}_i$. Furthermore assume that there are isomorphisms of stratifolds

$$\psi_{ji} : (U_i \cap U_j, \mathscr{P}_i|_{U_i \cap U_j}) \xrightarrow{\simeq} (U_i \cap U_j, \mathscr{P}_j|_{U_i \cap U_j}).$$

which satisfy

$$\psi_{ii} = \mathrm{Id}, \qquad \psi_{ij} \circ \psi_{ji} = \mathrm{Id}, \qquad \psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = \mathrm{Id}.$$

Gluing together the strata of the stratifolds U_i, one proves the following

Lemma 1.11. *There is up to isomorphism a unique parametrization $\mathscr{P}$ of X together with a family of isomorphisms*

$$\phi_i : (U_i, \mathscr{P}|_{U_i}) \xrightarrow{\simeq} (U_i, \mathscr{P}_i)$$

so that for any i, j the diagram

$$\begin{array}{ccc} (U_i \cap U_j, \mathscr{P}|_{U_i \cap U_j}) & & \\ \downarrow{\scriptstyle \phi_i|_{U_i \cap U_j}} & \searrow{\scriptstyle \phi_j|_{U_i \cap U_j}} & \\ (U_i \cap U_j, \mathscr{P}_i|_{U_i \cap U_j}) & \xrightarrow{\psi_{ji}} & (U_i \cap U_j, \mathscr{P}_j|_{U_i \cap U_j}). \end{array}$$

commutes

The class $\mathscr{K}$ of the stratifolds can be turned into a category taking the isomorphisms as morphisms. Moreover an easy consequence of Lemma 1.7 is the following

Lemma 1.12. *There is a small subclass $\mathscr{K}_0 \subset \mathscr{K}$ so that every stratifold is isomorphic to an object in $\mathscr{K}_0$.*

Now we want to explain the notion of morphism from a stratifold to a manifold.

Definition 1.13. *Let X be a stratifold and M a manifold. A **morphism** is a map $g : X \to M$ with the property that the composition*

$$g \circ f_i : W^i \longrightarrow M$$

is a c-map for all $i \leq n$.

Applying Lemma 1.4, one can easily show that every continuous map $X \to M$ is homotopic to a morphism. Another interesting property of morphisms is given by the following

Lemma 1.14. *Let A and B be two closed disjoint subspaces of a stratifold X. Then there exists a morphism $\rho : X \to \mathbb{R}$ with $\rho(A) = +1$ and $\rho(B) = -1$.*

Let us fix a stratifold F and denote by $\mathrm{Aut}(F)$ the group of all isomorphisms $F \to F$. A morphism $f : X \to M$ is called a **stratifold bundle** if there is a stratifold F so that (X, f, M) is a locally trivial bundle with fibre F and structure group $\mathrm{Aut}(F)$ (see [St]).

As a consequence of Lemma 1.11 we get the following result.

Corollary 1.15. *Let F be a stratifold. If (X, f, M) is a locally trivial bundle with fibre F and group $\mathrm{Aut}(F)$, then X admits the structure of a stratifold so that $f : X \to M$ is a stratifold bundle.*

The notion of stratifold bundle allows to distinguish an important class of stratifolds.

Definition 1.16. *A stratifold X is called **locally trivial** if, for each k, there is a representative $\pi_k : V_k \to X_k$ of the germ of retractions which is a stratifold bundle.*

In particular:

Definition 1.17. *A stratifold X is said to be **locally conelike** if X is locally trivial and if, for any $x \in X_k$, the fibre of π_k over x is the cone on some compact stratifold.*

Following the approach used in the first section, we introduce now the definition of stratifolds with boundary or, more correctly, of c-stratifolds.

Let $(X, \partial X)$ be a pair of spaces with ∂X closed in X, and suppose that $\mathring{X}$ and ∂X are stratifolds.

Definition 1.18. *A* ***collar*** *is a homeomorphism*

$$c : V \longrightarrow U$$

where V is an open neighborhood of $\partial X \times \{0\}$ in $\partial X \times [0, +\infty)$ of the form $(\partial X \times [0, +\infty))^{<\delta}$ and U is an open neighborhood of ∂X in X, with the following two properties:

- *for any $x \in \partial X$, it is $c(x, 0) = x$*
- *if $\mathring{V} := V \cap (\partial W \times (0, 1)$, and $\mathring{U} := U \cap \mathring{X}$, then the restriction*

$$c|_{\mathring{V}} : \mathring{V} \longrightarrow \mathring{U}$$

  is an isomorphism of stratifolds.

Definition 1.19. *An n-dimensional* ***c-stratifold*** *is a pair $(X, \partial X)$, where $\mathring{X} := X - \partial X$ as well as ∂X are stratifolds of dimension respectively n and $n-1$, together with a germ of collars $[c : V \to U]$.*

As in the case of the c-manifolds, we only consider maps of c-stratifolds which are compatible with the collars. Let X be a c-stratifold and M be any manifold.

Definition 1.20. *A map*

$$f : X \longrightarrow M$$

is called a ***c-morphism*** *if $f|_{\partial X}$ and $f|_{\mathring{X}}$ are morphisms, and furthermore there is a representative of the germ of collars $c : V \to U$ such that*

$$f(c(x, t)) = f(x)$$

for all $(x, t) \in V$.

Next, we want to explain how to glue two stratifolds along the boundary. Let $(X, \partial X)$ and $(X', \partial X')$ be two c-stratifolds and assume that there exists an isomorphism $\varphi : \partial X \xrightarrow{\simeq} \partial X'$. If δ and δ' are two distance functions on ∂X and $\partial X'$ respectively, then we set

$$(\partial X \times \mathbb{R})^{-\delta<\delta'} := \{(x,t) \in \partial X \times \mathbb{R} | -\delta(x) < t < \delta'(\varphi(x))\}.$$

Now, let $c : (\partial X \times [0,\infty))^{\delta} \to U$ and $c' : (\partial X' \times [0,\infty))^{\delta'}$ be two representatives of the germ of collars of X and X' respectively. If $\sim$ is the equivalence relation on

$$X \cup (\partial X \times \mathbb{R})^{-\delta<+\delta'} \cup X'$$

obtained identifying the points

$$(x,t) \sim \begin{cases} c(x,t) & t < 0 \\ c(x,-t) & t < 0 \end{cases}$$

then the quotient space $X \cup_{\partial} X'$ is naturally decomposed in the union of the two open sets A and B which correspond to $X \sqcup X'$ and to $(\partial X \times \mathbb{R})^{-\delta<\delta'}$. Let $\mathscr{P}_A$ and $\mathscr{P}_B$ denote the parametrizations of A and B. The collars induce an isomorphism of stratifolds

$$(A \cap B, \mathscr{P}_A|_{A\cap B}) \xrightarrow{\simeq} (A \cap B, \mathscr{P}_B|_{A\cap B})$$

and, applying Lemma 1.11, we get a canonical parametrization $\mathscr{P}$ on $X \cup_{\partial} X'$.

We conclude this section comparing the definition of stratifold with that of topological pseudomanifold. The notion of pseudomanifold has proved to be quite useful as a model of a manifold with singularities, but it does not seem to be suitable to the construction of new homology theories as bordism theories, and for this reason we will instead work in the category of stratifolds.

Definition 1.21. *A **pseudomanifold** of dimension n is a paracompact Hausdorff space X which admits a filtration by closed subsets*

$$X = \Sigma_n \supset \Sigma_{n-1} \supset \Sigma_{n-2} \supset \ldots \supset \Sigma_1 \supset \Sigma_0$$

such that for each point $p \in X_i = \Sigma_i - \Sigma_{i-1}$ there exists a distinguished neighborhood N of p in X, a compact space L with a $n-i-1$ dimensional stratification,

$$L = L_{n-i-1} \supset \ldots \supset L_1 \supset L_0 \supset L_{-1} = \emptyset$$

and a homeomorphism

$$\mathbb{R}^i \times \overset{\circ}{C}L \longrightarrow N$$

which takes each $\mathbb{R}^i \times \overset{\circ}{C}L_j$ homeomorphically to $N \cap \Sigma_{i+j+1}$.

The following lemma is an immediate consequence of the definitions.

Lemma 1.22. *Every locally conelike stratifold is a pseudomanifold.*

1.3 Transversality

In this section we show how to extend the transversality theorem to the class of locally conelike topological stratifolds.

Transversality for manifolds

We begin by recalling the notion of a bicollar. Let (M, N) be a pair of spaces with N closed in M. If $\delta_1 : N \to (-\infty, 0)$ and $\delta_2 : N \to (0, +\infty)$ are two continuous functions, then we set

$$^{\delta_1 <}(N \times \mathbb{R})^{<\delta_2} := \{(x, s) \in N \times \mathbb{R} \,|\, \delta_1(x) < s < \delta_2(x)\}.$$

Definition 1.23. *A* ***bicollar*** *of N is a homeomorphism $\varphi : V \to U$ where V is an open neighborhood of $N \times \{0\}$ of the form $^{\delta_1 <}(N \times \mathbb{R})^{<\delta_2}$ and U is an open neighborhood of N in M, so that for any $x \in N$ it is $\varphi(x, 0) = x$.*

Two bicollars $\varphi : V \to U$ and $\varphi' : V' :\to U'$ are called **equivalent** if there is an open neighborhood V'' of $N \times \{0\}$ with $V'' \subset V \cap V'$ and so that $\varphi|_{V''} = \varphi'|_{V''}$. The equivalence class of a bicollar is called **germ of bicollars**. As usual, if N is compact, then every bicollar is equivalent to one of the form $\varphi : N \times (-\varepsilon, +\varepsilon) \to U$.

Now, let M be a topological manifold and consider a continuous function

$$\rho : M \longrightarrow \mathbb{R}.$$

Definition 1.24. *A real number $t \in \mathbb{R}$ is called a* ***regular value*** *of ρ, if*

- *$N := \rho^{-1}(t)$ is an $(n-1)$-dimensional manifold together with a germ of bicollars $[\varphi]$;*
- *there exists a representant $\varphi : V \to U$ of the germ of bicollars so that it results*
$$\rho(\varphi(x, s)) = s + t$$
for all $(x, t) \in V$.

Equivalently, we will also say that ρ is ***transversal*** *at t.*

Using a very indirect and sophisticated argument based on surgery theory (see [KiSi], [Ma] and [Quin]), one can prove the following

Theorem 1.25 (The transversality theorem). *Let A be a closed subset of M and suppose that there is an open neighborhood O of A such that 0 is a regular value of $\rho|_O$. Then there exists a homotopy*

$$H : M \times [0,1] \longrightarrow \mathbb{R}$$

of $H(-,0) = \rho$ so that

- *$H(x,s) = \rho(x)$ for $x \in A$ and for all $s \in [0,1]$;*

- *0 is a regular value of $H(-,1)$.*

Next, we want to extend the transversality theorem to the class of c-manifolds.

Let

$$\rho : W \longrightarrow \mathbb{R}.$$

be a continuous c-function from an n-dimensional c-manifold to $\mathbb{R}$.

Definition 1.26. *A real number $t \in \mathbb{R}$ is a **regular value** of ρ, if t is a regular value of $\rho|_{\partial W}$ and of $\rho|_{\mathring{W}}$.*

If t is a regular value of ρ, then $Z := \rho^{-1}(t)$ is a bicollared c-submanifold of W with boundary $\partial Z := \partial W \cap Z$.

Remark. Observe that, even though ρ has been assumed to be a c-function, the transversality of $\rho|_{\mathring{W}}$ at a point t does not imply automatically that of $\rho|_{\partial W}$. A counterexample is provided by the map

$$\mathbb{R}^4 \times [0,1) \longrightarrow Y \times \mathbb{R} \times [0,1) \xrightarrow{\pi_2} \mathbb{R}$$

where Y is the non-manifold constructed in [Bi].

Next, we need the following

Lemma 1.27. *Let $K \subset W$ be a closed set and let $L \subset \partial W$ be another closed set so that $K \cap \partial W \subset \mathring{L}$ (where $\mathring{L}$ denotes the interior of L). Then there is a representative of the germ of collars $c : V \to U$ so that it holds*

$$x \in K \cap U \quad \Longrightarrow \quad \pi(x) \in L.$$

Proof. If $c' : V' \to U'$ is any representative of the germ of collars, then there is by definition a continuous function $\delta' : \partial W \to (0, +\infty)$ such that $V' = (\partial W \times [0, +\infty))^{<\delta'}$.

Since the projection π is a closed map and A is closed, the number

$$m(x) := \min\{t | (x, t) \in A\}$$

is defined and for $x \in (\partial W - \mathring{L}) \cap A$ it holds $m(x) > 0$.

For this reason, we can define a function δ over the closed set $K \cup (\partial W - \mathring{L})$ setting

$$\delta(x) = \begin{cases} \delta'(x) & \text{if } x \in K \\ \min\{\frac{m(x)}{2}, \delta'(x)\} & \text{if } x \in \partial W - \mathring{L} \end{cases}$$

By Tietze's extension theorem, δ can be extended to a continuous function $\partial W \to (0, +\infty)$, which we denote again by δ. Using this new function, we set

$$\begin{aligned} V &:= (\partial W \times [0, +\infty))^{<\delta} \\ c &:= c'|_V \\ U &:= c(V) \end{aligned}$$

The new collar $c : V \to U$ is by construction equivalent to c'.

Finally observe that, if $\pi(x) \in \partial W - L$ for some $x \in U$, then it must be $\delta(\pi(x)) < \min\{t | (x, t) \in A\}$ and so $x \notin A$. ■

The transversality theorem has now the following consequence.

Corollary 1.28 (Transversality for c-manifolds). *Let*

$$\rho : W \longrightarrow \mathbb{R}$$

be a c-function. Furthermore assume that there is a closed c-set $A \subset W$ and an open neighborhood O of A such that 0 is a regular value of $\rho|_O$. Then there exists a homotopy

$$H : W \times [0, 1] \longrightarrow \mathbb{R}$$

of $H(-, 0) = \rho$ so that:

- *$H(x, s) = \rho(x)$ for all $x \in A$ and all $s \in [0, 1]$;*
- *the map $H(-, s)$ is a c-function for all $s \in [0, 1]$;*
- *0 is a regular value for $H(-, 1)$.*

Proof. Let us choose a representative of the germ of collars $c : V \to U$ such that $\rho(x) = \rho(\pi(x))$ for all $x \in U$. Since W and ∂W are normal spaces, there exist two closed sets $K \subset W$ and $L \subset \partial W$ so that

- $A \subset \mathring{K} \subset K \subset O$
- $K \cap \partial W \subset \mathring{L} \subset L \subset O \cap \partial W$

By assumption, 0 is a regular value of $\rho|_{O \cap \partial W}$, and so, by Theorem 1.25, there is a homotopy

$$h : \partial W \times [0,1] \longrightarrow \mathbb{R}$$

of $h(-,0) = \rho|_{\partial W}$ such that

- $h(x,s) = \rho(x)$ for all $x \in L$;
- 0 is a regular value of $h(-,1)$.

Now, using Lemma 1.27, we can find a representative of the germ of collars so that

$$x \in K \cap U \quad \Longrightarrow \quad \pi(x) \in L$$

Composing with the projection $\pi : U \to \partial W$, we get a homotopy h'

$$U \times [0,1] \xrightarrow{\pi \times \mathrm{id}} \partial W \times [0,1] \xrightarrow{h} \mathbb{R}$$

with the property that

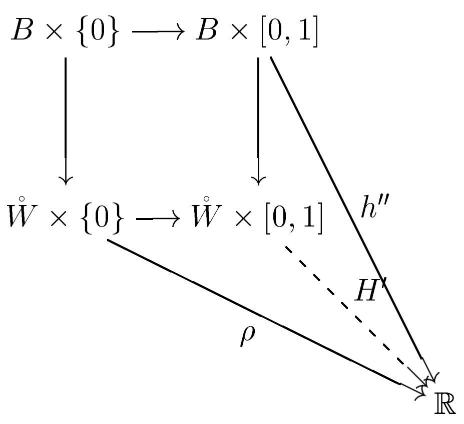
$$h'(x,s) = h(\pi(x)) = \rho(\pi(x)) = \rho(x).$$

for each $x \in K \cap U$.

On the closed subspace $B := K \cup c((\partial W \times [0,+\infty))^{\leq \delta/2}) \subset W$ we define a homotopy h'' setting:

$$h''(x,s) = \begin{cases} \rho(x) & \text{if } x \in K \\ h'(x,s) & \text{else} \end{cases}$$

The c-manifold W is a metrizable space and so it is in particular binormal (recall that a space X is called binormal if the product $X \times [0,1]$ is a normal space). According to Borsuk's homotopy extension theorem (see [Sp]), there is an extension



and the function $\rho' := H'(-,1)$ has the following properties:

- ρ' is a c-function;
- $\rho'(x) = \rho(x)$ for all $x \in B$;
- 0 is a regular value of $\rho'|_{\mathring{B}}$.

By construction $\mathring{B} \cap \mathring{W}$ is an open neighborhood of $c((\partial W \times (0,+\infty))^{\leq \delta/3}) \cup (A \cap \mathring{W})$ and so, applying the transversality theorem 1.25 to $H'(-,1)$, we get a homotopy

$$H'' : \mathring{W} \times [0,1] \longrightarrow \mathbb{R}$$

of $\rho'|_{\mathring{W}}$ which fixes $(A \cup (\partial W \times [0,+\infty))^{\leq \delta/3}) \cap \mathring{W}$ and such that 0 is a regular value of $H''(-,1)$. The map H'' extends uniquely to a homotopy $H''' : W \times [0,1] \to \mathbb{R}$ with the property that $H'''(-,s)$ is a c-function for all s, and finally we define H as the composition of H' with H'''. ∎

Transversality for stratifolds

Now let us pass to the category of stratifolds.

Definition 1.29. *Let $\rho : X \to \mathbb{R}$ be a morphism from an n-dimensional stratifold X to $\mathbb{R}$. A number $t \in \mathbb{R}$ is called a* ***regular value*** *of ρ, if*

- *$Y := \rho^{-1}(t)$ is an $(n-1)$-dimensional stratifold together with a germ of bicollars $[\varphi]$;*
- *there is a representative of the germ of bicollars $\varphi : V \to U$ with*

$$\rho(\varphi(x,s)) = s + t$$

  for all $(x,s) \in V$.

Again, if the conditions above are satisfied, we will say that ρ is ***transversal*** *at t.*

Observe that if X is a purely n-dimensional oriented stratifold, then the preimage of a regular value is automatically purely $(n-1)$-dimensional and oriented.

The proof of a general transversality theorem for stratifolds seems to be extremely complicated, and for this reason we will restrict our attention to the subclass of locally trivial stratifolds. A fundamental role is therefore played by the

Lemma 1.30. *Let $\pi : X \to M$ be a stratifold bundle and consider a continuous function $\rho : M \to \mathbb{R}$. If ρ is transversal at zero then the map defined by the composition $\rho \circ \pi$ is also transversal at zero.*

Proof. By definition, the set $N := \rho^{-1}(0)$ is a bicollared submanifold of M, i.e. there is a homeomorphism $\varphi : V \to U$ where V is an open neighborhood of $N \times \{0\}$ in $N \times \mathbb{R}$ and U is an open neighborhood of N in M. Now, the set $E := \pi^{-1}(N)$ is the total space of the bundle

$$\pi|_E : E \longrightarrow N$$

and therefore, by Corollary 1.15, E is a stratifold. Since U is homeomorphic to a space of the form ${}^{\delta_1<}(N \times \mathbb{R})^{<\delta_2}$, we get a bundle isomorphism

$$\begin{array}{ccc} {}^{\delta_1<}(E \times \mathbb{R})^{<\delta_2} & \longrightarrow & \pi^{-1}(U) \\ \Big\downarrow {\scriptstyle \pi|_E \times \mathrm{Id}} & & \Big\downarrow {\scriptstyle \pi|_{\pi^{-1}(U)}} \\ U & = & U \end{array}$$

which defines a bicollar of E in X. ■

The transversality theorem for stratifolds can be now stated as follows.

Proposition 1.31. *Let X be a locally trivial stratifold and let*

$$\rho : X \longrightarrow \mathbb{R}$$

be a morphism. Moreover assume that there is a closed set A and an open neighborhood O of A so that 0 is a regular value of $\rho|_O$. Then there exists a homotopy

$$H : X \times [0,1] \longrightarrow \mathbb{R}$$

of $H(-,0) = \rho$ with the following properties:

- *$H(x,s) = \rho(x)$ for all $x \in A$;*
- *$H(-,1)$ is a morphism;*
- *0 is a regular value of $H(-,1)$.*

Proof. For simplicity we suppose $A = \emptyset$. As explained in the previous section, the assumption that X is locally trivial means that, for any i, there is a representative of the germ of projections $\pi_i : V_i \to X_i$ which is a stratifold-bundle. The homotopy H will be constructed inductively as the composition of a sequence of homotopies

$$H_k : X \times [0,1] \longrightarrow \mathbb{R},$$

for which it holds:

- $H_0(x,0) = \rho(x)$ and $H_{k+1}(x,0) = H_k(x,1)$ for every $x \in X$;
- $H_k(-,1)$ is a morphism of stratifolds for any k;
- there is an open neighborhood U_k of Σ_k in X, such that 0 is a regular value of $H_k(-,1)|_{U_k}$.

The stratifold Σ_0 is just a discrete set and thus there is a homotopy

$$h : X_0 \times [0,1] \longrightarrow \mathbb{R}$$

so that 0 is not hit by $h(-,1)$. Let Z_0 and U_0 be two open neighborhoods of X_0 with $\overline{U_0} \subset Z_0 \subset \overline{Z_0} \subset V_0$. The composition

$$V_0 \times [0,1] \xrightarrow{\pi_0 \times \mathrm{Id}} X_0 \times [0,1] \xrightarrow{h} \mathbb{R}$$

defines a homotopy whose restriction on Z_0 can be extended to a homotopy of ρ

$$k : X \times [0,1] \longrightarrow \mathbb{R}.$$

The map $k(-,1)$ is homotopic relative U_0 to a morphism, and if l is such a homotopy, then we define H_0 as the composition of k and l.

Now, suppose to have already defined H_k and let us construct H_{k+1}. If (W,f) is the $(k+1)$-dimensional strat of X and if we set $\xi := H_k(-,1)$, then it follows from the inductive assumptions that the function $\xi \circ f$ is transversal at 0 on $f^{-1}(U_k) \cap \mathring{W}$. Applying the transversality theorem for manifolds we get a homotopy

$$h : W \times [0,1] \longrightarrow \mathbb{R}$$

of $\xi \circ f$ which fixes a smaller open neighborhood of the boundary and so that $h(-,1)|_{\mathring{W}}$ is transversal at 0. Let Z_{k+1} be an open neighborhood of X_{k+1} with $\overline{Z_{k+1}} \subset V_{k+1}$ and let B_k be an open neighborhood of Σ_k with $\overline{B_k} \subset U_k$. The composition

$$V_{k+1} \times [0,1] \xrightarrow{\pi_{k+1} \times \mathrm{Id}} X_{k+1} \times [0,1] \xrightarrow{f^{-1}| \times \mathrm{Id}} \mathring{W} \times [0,1] \xrightarrow{h} \mathbb{R}$$

defines a homotopy which we denote by h'.

By Borsuk's homotopy extension theorem, the homotopy

$$(\overline{Z_{k+1}} \cup \overline{B}_k) \times [0,1] \longrightarrow \mathbb{R}$$

$$(x,s) \longmapsto \begin{cases} h'(x,s) & \text{if } x \in \overline{Z_{k+1}} \\ \xi(x) & \text{if } x \in \overline{B_k} \end{cases}$$

can be extended to a homotopy of ξ

$$k : X \times [0,1] \longrightarrow \mathbb{R}.$$

The map $k(-,1)$ is homotopic relative an open neighborhood U_{k+1} of Σ_{k+1} to a morphism and we define H_{k+1} as the composition of k with the latter homotopy. Finally, the transversality at zero of $H_{k+1}(-,1)|_{U_{k+1}}$ is a consequence of Lemma 1.30. ∎

A procedure similar to the one used at the beginning of this section to extend the transversality theorem from manifolds to c-manifolds can be used to obtain the transversality theorem in its most general form.

Proposition 1.32 (Transversality for c-stratifolds). *Let*

$$\rho : X \longrightarrow \mathbb{R}$$

be a c-morphism and consider a closed set $A \subset X$. Suppose that there is an open neighborhood O of A so that 0 is a regular value of $\rho|_O$. Then there exists a homotopy

$$H : X \times [0,1] \longrightarrow \mathbb{R}$$

of $H(-,0) = \rho$ so that

- *the function $H(-,s)$ is a c-map for all $s \in [0,1]$;*
- *$H(x,s) = \rho(x)$, for all $x \in A$;*
- *0 is a regular value of $H(-,1)$.*

Chapter 2

Sheaf theory

The aim of this chapter is to recall some elements of sheaf theory and to introduce some notation. The reader is referred to any standard book on sheaf theory for proofs and details (see for example [KaSc]).

2.1 Sheaves and presheaves

Let X be a locally compact space.

Definition 2.1. *A* ***sheaf*** *over X is a pair $(\mathbf{A}, \pi)$ where*

1. *$\mathbf{A}$ is a topological space*
2. *$\pi : \mathbf{A} \to X$ is a local homeomorphism*
3. *the stalk of $\mathbf{A}$ at x, (i.e. the set $\mathbf{A}_x := \pi^{-1}(x)$) is a real vector space for any $x \in X$*
4. *if $\mathbf{A} \oplus \mathbf{A}$ is the subset $\{(a, b) \in \mathbf{A} \times \mathbf{A} | \pi(a) = \pi(b)\}$ with the induced topology, then the maps*

$$\mathbf{A} \oplus \mathbf{A} \longrightarrow \mathbf{A}$$

$$(a, b) \longmapsto (a - b)$$

and, for $k \in \mathbb{R}$,

$$\mathbf{A} \longrightarrow \mathbf{A}$$

$$a \longmapsto k \cdot a$$

are continuous.

For simplicity we denote a sheaf $(\mathbf{A}, \pi)$ *by* $\mathbf{A}$.

For any subset U of X, we indicate by $\Gamma(U, \mathbf{A})$ the vector space of the continuous sections of $(\mathbf{A}, \pi)$.

Given two sheaves $\mathbf{A}$ and $\mathbf{B}$ over X, a **morphism** $\mathbf{A} \to \mathbf{B}$ is a continuous map which commutes with the projections and which is $\mathbb{R}$-linear on the stalks. We denote by $Sh(X)$ the category of the sheaves over X. The set $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ of all morphisms from $\mathbf{A}$ to $\mathbf{B}$ is in natural way a vector space.

Definition 2.2. *A **presheaf** over* X *is a contravariant functor*

$$\mathbf{A} : \mathcal{U}(X) \longrightarrow \{\mathbb{R} - \text{Vector Spaces}\}$$

where $\mathcal{U}(X)$ *is the category whose objects are the open subsets of* X *and with the inclusions as morphisms.*

The class of all presheaves over X form a category which we denote by $Pr(X)$. The functor

$$Sh(X) \longrightarrow Pr(X)$$

$$\mathbf{A} \longmapsto \{U \mapsto \Gamma(U, \mathbf{A})\}$$

has a left adjoint functor $Pr(X) \to Sh(X)$, which is called **sheafification**.

Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism of sheaves. The sheafifications of the presheaves

$$U \longmapsto \mathrm{Ker}(\Gamma(U, \mathbf{A}) \xrightarrow{\alpha_U} \Gamma(U, \mathbf{B}))$$

and

$$U \longmapsto \mathrm{Im}(\Gamma(U, \mathbf{A}) \xrightarrow{\alpha_U} \Gamma(U, \mathbf{B}))$$

are called respectively **kernel** and **image** of α.

If Y is another space and f is a continuous map

$$f : X \longrightarrow Y$$

then the **direct-image** of $\mathbf{A} \in Sh(X)$ under f is by definition the sheaf $f_*\mathbf{A} \in Sh(Y)$ associated to the presheaf

$$U \longmapsto \Gamma(f^{-1}(U), \mathbf{A}).$$

The **direct-image with proper support** of A under f is the sheaf $f_!\mathbf{A}$ obtained sheafifying the presheaf

$$U \longmapsto \{s \in \Gamma(f^{-1}(U), \mathbf{A}) | f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to U \text{ is proper}\}.$$

The category of sheaves admits also a contravariant construction. Let $\mathbf{B}$ be a sheaf over Y; the sheaf $f^*\mathbf{B} \in Sh(X)$ defined by the pull-back diagram

$$\begin{array}{ccc} f^*\mathbf{B} & \longrightarrow & \mathbf{B} \\ \big\downarrow & & \big\downarrow \pi \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

is called **inverse-image** of $\mathbf{B}$ under f. In particular, if f is the inclusion of a subset $U \subset X$, then the sheaf $f^*\mathbf{B}$ is also called the restriction of $\mathbf{A}$ and is denoted by $\mathbf{B}|_U$.

A sheaf $\mathbf{A}$ is called **locally constant** if every point $x \in X$ has a neighborhood U such that $\mathbf{A}|_U$ is isomorphic to the constant sheaf with stalk $\mathbf{A}_x$.

All constructions presented here are functorial, and therefore one can associate to a map f three functors

$$f_* : Sh(X) \longrightarrow Sh(Y),$$

$$f_! : Sh(X) \longrightarrow Sh(Y),$$

$$f^* : Sh(Y) \longrightarrow Sh(X).$$

Two other fundamental constructions in sheaf theory are the tensor product and the Hom functor. If $\mathbf{A}$ and $\mathbf{B}$ are two sheaves over X, then $\mathbf{A} \otimes \mathbf{B}$ and $\mathbf{Hom}(\mathbf{A}, \mathbf{B})$ are the sheaves obtained sheafifying

$$U \longmapsto \Gamma(U, \mathbf{A}) \otimes \Gamma(U, \mathbf{B})$$

and

$$U \longmapsto \mathrm{Hom}(\mathbf{A}|_U, \mathbf{B}|_U)$$

respectively. In particular, for the tensor product of sheaves it holds

$$(\mathbf{A} \otimes \mathbf{B})_x \simeq \mathbf{A}_x \otimes \mathbf{B}_x$$

for all $x \in X$.

2.2 Complexes of sheaves

Definition 2.3. *A* ***complex of sheaves*** *over X is a sequence of sheaves and morphisms of sheaves*

$$\dots \longrightarrow \mathbf{A}^n \xrightarrow{d^n} \mathbf{A}^{n+1} \xrightarrow{d^{n+1}} \dots$$

with the property that $d^{n+1} \circ d^n = 0$ for each $n \in \mathbb{Z}$. A complex $(\mathbf{A}^n, d^n)$ will be normally indicated by $\mathbf{A}^\bullet$.

A complex $\mathbf{A}^\bullet$ is called **bounded**, if there exist two integers p and q such that $\mathbf{A}^n = 0$, for $n < p$ or $n > q$.

We denote by $C^b(X)$ the category of bounded complexes of sheaves over X; as usual in homological algebra, a morphism between two complexes is a sequence of morphisms of sheaves which commutes with the coboundary operators.

A sheaf $\mathbf{A}$ will be generally identified with the complex concentrated in degree 0.

Definition 2.4. *The n-th* ***cohomology sheaf*** *of a complex $\mathbf{A}^\bullet$ is the sheaf*

$$\mathbf{H}^n(\mathbf{A}^\bullet) = \frac{\mathbf{Ker}\ d^n}{\mathbf{Im}\ d^{n-1}}$$

associated to the presheaf

$$U \longmapsto H^n(\Gamma(U, \mathbf{A}^\bullet)).$$

A complex of sheaves is said to be **cohomologically locally constant** if the associated cohomology sheaves are locally constant.

By construction, any morphism of complexes induces a sheaf-morphism in cohomology, and in particular a morphism of complexes is called a **quasi-isomorphism** if it induces isomorphisms in cohomology.

The functors f_*, $f_!$ and f^* associated to a continuous map $f : X \to Y$ can be extended to functors of complexes of sheaves. For simplicity we use the same notation to indicate a functor on the level of sheaves and its extension to the complexes of sheaves. This construction produces thus three functors.

$$f_* : C^b(X) \longrightarrow C^b(Y),$$

$$f_! : C^b(X) \longrightarrow C^b(Y),$$

$$f^* : C^b(Y) \longrightarrow C^b(X).$$

The next step is to extend the functors $\otimes$ and **Hom** to the category of complexes of sheaves. If $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are two complexes of sheaves, then taking the tensor product one gets at first a double complex $\mathbf{A}^p \otimes \mathbf{B}^q$, and one can define an ordinary complex setting $(\mathbf{A}^\bullet \otimes \mathbf{B}^\bullet)^n := \bigoplus_{p+q=n} \mathbf{A}^p \otimes \mathbf{B}^q$. The same can be done for **Hom**, and so we get two bifunctors

$$\otimes : C^b(X) \times C^b(X) \longrightarrow C^b(X),$$

$$\mathbf{Hom}^\bullet : C^b(X) \times C^b(X) \longrightarrow C^b(X).$$

Finally we want to explain how to shift and to truncate a complex of sheaves. Let $\mathbf{A}^\bullet$ be a complex of sheaves over X and fix an integer $m \in \mathbb{Z}$. The shifted complex $\mathbf{A}^\bullet[m]$ is defined by

$$(\mathbf{A}^\bullet[m])^n := \mathbf{A}^{n+m}$$

$$(d[m])^n := (-1)^m d^{n+m}$$

The truncated complex $\tau_{\leq m}\mathbf{A}^\bullet$ is by definition given by the sequence

$$(\tau_{\leq m}\mathbf{A}^\bullet)^n := \begin{cases} 0 & \text{for } n > m \\ \text{Ker } d^n & \text{for } n = m \\ \mathbf{A}^n & \text{for } n > m \end{cases}$$

with coboundary operator

$$(\tau_{\leq m} d)^n = \begin{cases} 0 & \text{for } n \geq m \\ d^n & \text{for } n < m \end{cases}$$

A sheaf $\mathbf{I} \in Sh(X)$ is called **injective** if the functor

$$\text{Hom}(-, \mathbf{I}) : Sh(X) \longrightarrow \{\mathbb{R} - \text{Vector Spaces}\}$$

is exact. A complex $\mathbf{I}^\bullet \in C^b(X)$ is said to be injective, if it consists of injective sheaves.

An **injective resolution** of a sheaf $\mathbf{A}$ is a quasi-isomorphism

$$\mathbf{A} \longrightarrow \mathbf{I}^\bullet$$

where $\mathbf{I}^\bullet$ is a bounded injective complex of sheaves which is concentrated in positive degree.

If X has finite cohomological dimension, then every sheaf $\mathbf{A} \in Sh(X)$ admits an injective resolution: such a resolution can be obtained applying a constructive method (see for example [GM2], Section 1.5), and for this reason we will call it the canonical injective resolution of $\mathbf{A}$.

Now, we want to explain the analogous notions for a complex of sheaves. Ba definition, an injective resolution of the complex $\mathbf{A}^\bullet \in C^b(X)$ is a quasi-isomorphism $\mathbf{A}^\bullet \to \mathbf{I}^\bullet$, where $\mathbf{I}^\bullet$ is a bounded injective complex.

Let $\mathbf{A}^\bullet$ be a complex of sheaves. If $\mathbf{I}^\bullet$ denotes the complex of sheaves associated to the double complex obtained putting together the canonical injective resolutions of the sheaves $\mathbf{A}^n$, then the induced quasi-isomorphism $\mathbf{A}^\bullet \to \mathbf{I}^\bullet$ is called the canonical injective resolution of $\mathbf{A}^\bullet$.

Applying the functor $\Gamma(X, -)$ to the canonical injective resolution of $\mathbf{A}^\bullet$, one gets a cochain-complex

$$\Gamma(X, \mathbf{I}^0) \longrightarrow \Gamma(X, \mathbf{I}^1) \longrightarrow \Gamma(X, \mathbf{I}^2) \longrightarrow \ldots$$

whose n-th cohomology space $\mathcal{H}^n(X, \mathbf{A}^\bullet)$ is called the **n-th hypercohomology space of X with coefficients in $\mathbf{A}^\bullet$.**

It can be shown that the quasi-isomorphisms induce isomorphisms in hypercohomology.

2.3 The derived category

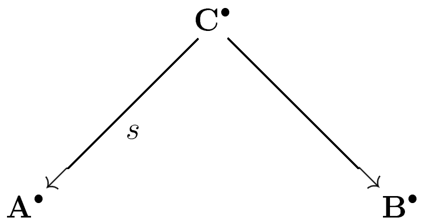
Many results in sheaf theory require to identify complexes of sheaves which are quasi-isomorphic. Such an identification can be realized constructing a new category of bounded complexes of sheaves which is called the derived category and which has the property that quasi-isomorphisms can be inverted. The construction of the derived category consists essentially of two steps.

From $C^b(X)$ to $K^b(X)$

Using the ordinary definition of homotopy of cochain-morphism, one introduces an equivalence relation on the set of all morphisms between two objects. The homotopy category of bounded complexes $K^b(X)$ is then defined as the category whose objects are the bounded complexes of sheaves and with the homotopy classes of morphisms as arrows. Since homotopic morphisms induce the same morphism in cohomology, it makes still sense to speak of quasi-isomorphisms.

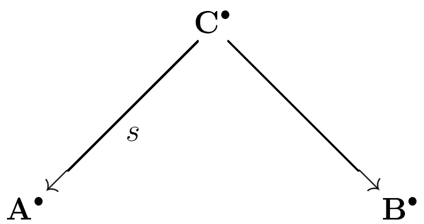
From $K^b(X)$ to $D^b(X)$

The starting point of the second step is the fact that the class of all quasi-isomorphisms build a multiplicative system in $K^b(X)$. Localizing $K^b(X)$ with respect to the quasi-isomorphisms, one gets a category $D^b(X)$ of complexes of sheaves called the **derived category**. In order to describe the morphisms in $D^b(X)$ let us consider two complexes $\mathbf{A}^\bullet, \mathbf{B}^\bullet \in D^b(X)$ and all diagrams (in $K^b(S)$) of the form

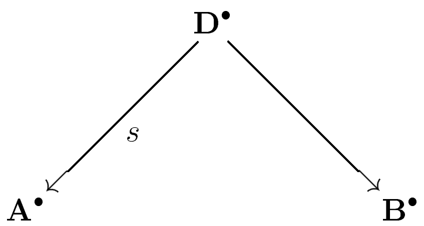


where the arrow decorated by s represents the homotopy class of a quasi-isomorphism.

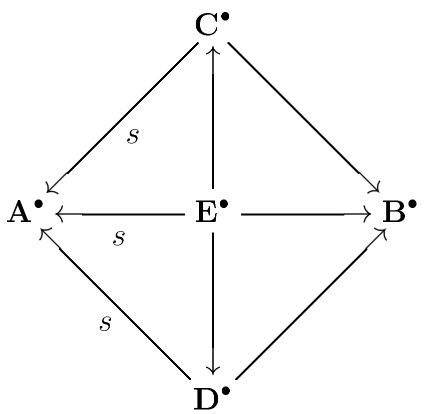
Two diagrams



and

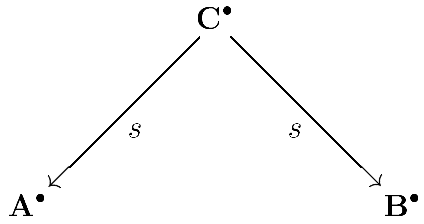


are equivalent if there is a commutative diagram (in $K^b(X)$)



A morphism from $\mathbf{A}^\bullet$ to $\mathbf{B}^\bullet$ is defined as the equivalence class of a diagram $\mathbf{A}^\bullet \overset{s}{\leftarrow} \mathbf{C}^\bullet \rightarrow \mathbf{B}^\bullet$.

In particular, two complexes $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are isomorphic in $D^b(X)$ if there is a third complex $\mathbf{C}^\bullet$ and two quasi-isomorphisms



Derived Functors

If $\mathcal{A}$ is any Abelian category and F is an exact functor

$$F: \quad Sh(X) \longrightarrow \mathcal{A},$$

then there is an induced functor

$$D^b(X) \longrightarrow D^b(\mathcal{A})$$

$$(\mathbf{A}^n, d^n) \longmapsto (F(\mathbf{A}^n), F(d^n))$$

where $D^b(\mathcal{A})$ denotes the derived category of bounded cochain-complexes in $\mathcal{A}$. For example, the functor f^* associated to a map $f: X \to Y$ is exact and extends thus to a functor

$$D^b(Y) \longrightarrow D^b(X).$$

which we again denote by f^*.

The situation is more complicated if one tries to extend functors which are not exact (for example f_* or $f_!$). However, at least in some case, there is still the possibility to find a sort of best approximation. If F is *left-exact*, then this approximation is given by the **right-derived functor** RF which is defined setting

$$RF(\mathbf{A}^\bullet) := F(\mathbf{I}^\bullet),$$

where $\mathbf{A}^\bullet \to \mathbf{I}^\bullet$ is the canonical injective resolution of $\mathbf{A}^\bullet$. If $\mathbf{A}^\bullet \to \mathbf{J}^\bullet$ is another injective resolution of $\mathbf{A}^\bullet$, then the identity of $\mathbf{A}^\bullet$ lifts to an isomorphism $F(\mathbf{I}^\bullet) \simeq F(\mathbf{J}^\bullet)$ and so RF is up to canonical isomorphism independent from the choice of the resolution.

The higher derived functors R^iF are defined by

$$R^iF : D^b(X) \longrightarrow \mathcal{A}$$

$$\mathbf{A}^\bullet \longmapsto H^i(F(\mathbf{I}^\bullet))$$

Since F was assumed to be left-exact, it follows that $R^0F = F$.

According to Borel (see [Bor] V,5.18), the right-derived of a functor F can be computed using any resolution by F-acyclic sheaves. Recall that a sheaf $\mathbf{B}$ is called F-acyclic if, for $i > 0$, it holds

$$R^iF(\mathbf{B}) = 0.$$

Dually, in order to define the derived functor associated to a right-exact functor, one has to consider projective (left)-resolutions. We do not want to carry out this construction in details. Instead, we prefer to take a look at the case of the tensor product, which is after all the only right-exact functor occurring in this thesis. For a sheaf $\mathbf{A} \in Sh(X)$, let $\mathbf{A} \otimes (-)$ denote the right-exact functor

$$Sh(X) \longrightarrow Sh(X)$$

$$\mathbf{B} \longmapsto \mathbf{A} \otimes \mathbf{B}.$$

As in the case of a left-exact functor, it is enough to consider resolutions by $\mathbf{A} \otimes (-)$-acyclic sheaves, and we claim that every sheaf $\mathbf{B} \in Sh(X)$ is $\mathbf{A} \otimes (-)$-acyclic. In fact, if

$$\mathbf{P}^\bullet \longrightarrow \mathbf{B}$$

is any projective resolution of a sheaf $\mathbf{B}$, then the sequence

$$\mathbf{A} \otimes \mathbf{P}^\bullet \longrightarrow \mathbf{A} \otimes \mathbf{B}$$

is exact too: the algebraic Künneth formula provides an identification

$$\mathbf{H}^n(\mathbf{A} \otimes \mathbf{P}^\bullet)_x \simeq \mathbf{H}^n(\mathbf{A}_x \otimes \mathbf{P}^\bullet_x) \simeq \bigoplus_{i+j=n} \mathbf{H}^i(\mathbf{A})_x \otimes \mathbf{H}^j(\mathbf{P}^\bullet)_x = 0.$$

On the other hand, the complex $\mathbf{P}^\bullet$ is quasi-isomorphic to $\mathbf{B}$, and thus the cohomology sheaf $\mathbf{H}^j(\mathbf{P}^\bullet)_x$ is zero for all $j < 0$. Consequently, for $i < 0$, it results $L^i(\mathbf{A} \otimes \mathbf{B}) = 0$ and this means that $\mathbf{B}$ is $\mathbf{A} \otimes (-)$-acyclic. In particular the trivial resolution

$$\mathbf{B}^\bullet \longrightarrow \mathbf{B}^\bullet$$

can be used to compute $\mathbf{A}^\bullet \overset{L}{\otimes} (-)$, and so there is a canonical isomorphism

$$\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet \simeq \mathbf{A}^\bullet \otimes \mathbf{B}^\bullet.$$

Summarizing, if $f : X \to Y$ is any continuous map and $\mathbf{A}^\bullet$ is a complex, then we have the following functors:

$$Rf_*(-) : D^b(X) \longrightarrow D^b(Y),$$

$$f^*(-) : D^b(Y) \longrightarrow D^b(X),$$

$$Rf_!(-) : D^b(X) \longrightarrow D^b(Y),$$

$$R\mathbf{Hom}^\bullet(\mathbf{A}, -) : D^b(X) \longrightarrow D^b(X),$$

$$\mathbf{A}^\bullet \otimes (-) : D^b(X) \longrightarrow D^b(X).$$

2.4 Verdier duality

The **dualizing complex** of a topological space X is the complex of sheaves $\mathbb{D}_X^\bullet$ associated to the complex of presheaves $\mathbf{C}^\bullet$

$$U \longmapsto \mathbf{C}^{-p}(U) := C_p(X, X - U; \mathbb{R}),$$

where $C_p(X, X - U; \mathbb{R})$ denotes the space of the singular p-chains on $(X, X - U)$. By construction, the complex $\mathbb{D}_X^\bullet$ vanishes in degree higher than 0 and, if we assume the space X to be locally contractible, then the cohomology of the stalk of $\mathbb{D}_X^{-p}$ at the point x can be identified with the local homology vector space $H_p(X, X - x; \mathbb{R})$. Moreover, if X has finite homological dimension n, then the cohomology sheaf $\mathbf{H}^p(\mathbb{D}_X^\bullet)$ vanishes for $p < -n$, and so there is a quasi-isomorphism

$$\tau_{\leq -n}\mathbb{D}_X^\bullet \xrightarrow{\simeq} \mathbb{D}_X^\bullet.$$

The complex $\tau_{\leq -n}\mathbb{D}_X^\bullet$ is clearly bounded, and consequently the isomorphism above allows to identify the dualizing complex with a bounded complex.

Example. If M is an n-dimensional topological manifold without boundary, then the cohomology sheaf of $\mathbb{D}_M^\bullet$ is locally constant with stalk

$$H^i(\mathbb{D}_M^\bullet)_x \simeq \begin{cases} \mathbb{R} & \text{for } i = -n \\ 0 & \text{otherwise} \end{cases}$$

In particular the diagram

$$\begin{array}{ccc} & \tau_{\leq -n}\mathbb{D}^\bullet_M & \\ \swarrow & & \searrow \\ \mathbb{D}^\bullet_M & & \mathbf{H}^{-n}(\mathbb{D}^\bullet_M)[n] \end{array}$$

provides an isomorphism

$$\mathbb{D}^\bullet_M \xrightarrow{\ \simeq\ } \mathbf{H}^{-n}(\mathbb{D}^\bullet_M)[n]$$

and, using the fact that there is a one-to-one correspondence between orientations of M and trivializations of $\mathbf{H}^{-n}(\mathbb{D}^\bullet_M)$, one sees that an orientation of M is the same as an isomorphism

$$\mathbb{D}^\bullet_M \xrightarrow{\ \ \simeq\ \ } \mathbb{R}[n].$$

For a locally contractible finite-dimensional space X, the dualizing complex $\mathbb{D}^\bullet_X \in D^b(X)$ defines a functor

$$\mathcal{D}_X : \ D^b(X) \longrightarrow D^b(X)$$

$$\mathbf{A}^\bullet \longmapsto \mathcal{D}_X(\mathbf{A}^\bullet) := R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, \mathbb{D}^\bullet_X)$$

which is called the **duality functor**.

Furthermore, if X is compact, then by standard homological algebra there is an isomorphism

$$\mathcal{H}^{-n}(X, \mathcal{D}\mathbf{A}^\bullet) \xrightarrow{\simeq} \mathrm{Hom}(\mathcal{H}^n(X, \mathbf{A}^\bullet), \mathbb{R}).$$

Theorem 2.5 (Verdier duality). *For any map $f : X \to Y$, the functor $Rf_!$ has a right adjoint*

$$f^! : \quad D^b(Y) \longrightarrow D^b(X)$$

such that, given two complexes $\mathbf{A}^\bullet \in D^b(X)$ and $\mathbf{B}^\bullet \in D^b(Y)$, there is a canonical isomorphism in $D^b(Y)$

$$Rf_*R\mathbf{Hom}^\bullet(\mathbf{A}^\bullet, f^!\mathbf{B}^\bullet) \simeq R\mathbf{Hom}^\bullet(Rf_!\mathbf{A}^\bullet, \mathbf{B}^\bullet).$$

The functor $f^!$ can be used to characterize the dualizing complex: in fact, if f is the constant map $X \to \mathrm{pt}$, then there is a canonical isomorphism

$$f^!\mathbb{R}_{\{\mathrm{pt}\}} \simeq \mathbb{D}^\bullet_X,$$

where $\mathbb{R}_{\mathrm{pt}}$ is the sheaf over $\{\mathrm{pt}\}$ with stalk $\mathbb{R}$.

2.5 Constructibility

Let X and Y be two pseudomanifolds. According to ([GM2], Section 1.2), a map

$$f : X \longrightarrow Y$$

is called **stratified**, if it has the following two properties:

- for any connected component S of a stratum Y_k, the set $f^{-1}(S)$ is a union of connected components of strata of X;

- for each point $p \in Y_i$ there exists a neighborhood N of p in $\Sigma_i(Y)$, a stratifold F and an isomorphism

  $$F \times N \longrightarrow f^{-1}N$$

  such that the diagram

  $$\begin{array}{ccc} F \times N & \longrightarrow & f^{-1}N \\ & \searrow \quad \swarrow & \\ & N & \end{array}$$

  commutes.

Open inclusions, immersions of the form $X \to X \times \mathbb{R}$ and projections $\pi_j : X_1 \times X_2 \to X_j$ are the most important examples of stratified maps.

A complex of sheaves $\mathbf{A}^\bullet$ on a stratifold X is called **constructible** if, for each j, $\mathbf{A}^\bullet|_{\Sigma_j(X)}$ is cohomologically locally constant with finitely generated stalk.

It can be shown that the dualizing complex of a stratifold is constructible. Moreover the functors $\otimes$, $\mathbf{Hom}^\bullet$ and, if f is stratified, f^*, $f^!$, Rf_* and $Rf_!$ preserve constructibility.

We conclude this chapter reporting a list of standard identities taken from ([GM2], Section 1.13).

Theorem 2.6. *Suppose that X, Y and Z are topological pseudomanifolds and $f : X \to Y$ is a stratified map, and moreover let $\mathbf{A} \in D^b(X)$ and $\mathbf{B}^\bullet, \mathbf{C}^\bullet, \mathbf{D}^\bullet \in D^b(Y)$ be constructible. Then there are natural isomorphisms in $D^b(X)$ and $D^b(Y)$:*

1. $\mathbb{D}_X^\bullet \simeq \mathcal{D}(\mathbb{R}_X)$

2. $\mathbf{A}^\bullet \simeq \mathcal{D}(\mathcal{D}(\mathbf{A}^\bullet))$

3. $R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet \otimes \mathbf{C}^\bullet, \mathbf{E}^\bullet) \simeq R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, R\mathbf{Hom}^\bullet(\mathbf{C}^\bullet, \mathbf{E}^\bullet))$

4. $\mathcal{D}_X f^! \mathbf{B}^\bullet \simeq f^* \mathcal{D}_Y \mathbf{B}^\bullet$

5. $\mathcal{D}_Y R f_! \mathbf{A}^\bullet \simeq R f_* \mathcal{D}_X \mathbf{A}^\bullet$

6. $f^*(\mathbf{B}^\bullet \otimes \mathbf{C}^\bullet) \simeq f^* \mathbf{B}^\bullet \otimes f^* \mathbf{C}^\bullet$

7. $f^! R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{C}^\bullet) \simeq R\mathbf{Hom}^\bullet(f^* \mathbf{B}^\bullet, f^! \mathbf{C}^\bullet)$

8. *If $g : Y \times Z \to Y$ is the projection to the first factor, then*

$$g^* R\mathbf{Hom}^\bullet(\mathbf{B}^\bullet, \mathbf{C}^\bullet) \simeq R\mathbf{Hom}^\bullet(g^* \mathbf{B}^\bullet, g^* \mathbf{C}^\bullet)$$

9. *If $g : X \to Y$ is the inclusion of a subset, then*

$$X \text{ open in } Y \Rightarrow g^! \mathbf{B}^\bullet \simeq g^* \mathbf{B}^\bullet$$

$$X \text{ closed in } Y \Rightarrow R g_! \mathbf{A}^\bullet \simeq R g_* \mathbf{B}^\bullet.$$

Chapter 3

H-Stratifolds

Using the notion of a self-dual complex of sheaves recently introduced by Markus Banagl, we present in this chapter the definition of H-stratifold and discuss some related constructions. The notion of H-stratifolds will play a fundamental role in the construction of Hirzebruch homology theory $Hh_*(-)$.

3.1 Self-dual complexes of sheaves

In this first section we recall some definitions and facts from the theory of the self-dual complexes of sheaves (see [Ba1]).

First of all we make the following two assumptions:

- **all stratifolds from now on are assumed to be locally conelike, $\mathbb{Z}/2$-orientable, and purely n-dimensional.**
- **all complexes of sheaves from now on are assumed to be constructible.**

Let X be an n-dimensional stratifold with k-th skeleton Σ_k. For any integer $0 \leq k \leq n$, we indicate by U_k the open subset $X - \Sigma_{n-k}$ and by i_k the inclusion $U_k \hookrightarrow U_{k+1}$.

As we have seen in section 2.4, if M is a manifold, then an orientation of M is the same as a trivialization of the dualizing complex $\mathbb{D}^\bullet_M$, and we can use this fact to reformulate the orientability of a stratifold in the language of sheaf theory.

Definition 3.1. *A stratifold X is* ***orientable*** *if there exists an isomorphism (called* ***orientation****)*

$$\mathfrak{o} : \mathbb{D}^\bullet_{U_2} \xrightarrow{\simeq} \mathbb{R}_{U_2}[n].$$

A pair of the form $(X, \mathfrak{o})$ is called an ***oriented stratifold****.*

Let $(X, \mathfrak{o}_X)$ and $(Y, \mathfrak{o}_Y)$ be two oriented stratifolds. By definition an orientation-preserving isomorphism is an isomorphism of stratifolds which respects the orientations on the top strata. Again it is convenient to translate this condition in the language of sheaf theory.

Definition 3.2. *An isomorphism $f : X \to Y$ is said to be* ***orientation-preserving*** *if the diagram*

$$\begin{array}{ccc} \mathbb{D}^\bullet_{U_2(X)} & \xrightarrow{\;\simeq\;} & (f|_{U_2(X)})^* \mathbb{D}^\bullet_{U_2(Y)} \\ & {\scriptstyle \mathfrak{o}_X} \searrow \quad \swarrow {\scriptstyle (f|_{U_2(X)})^* \mathfrak{o}_Y} & \\ & \mathbb{R}_X[n] & \end{array}$$

commutes. The top arrow of the diagram is the canonical identification

$$\mathbb{D}^\bullet_{U_2(X)} \simeq (f|_{U_2(X)})^! \mathbb{D}^\bullet_{U_2(Y)} \simeq (f|_{U_2(X)})^* \mathbb{D}^\bullet_{U_2(Y)}.$$

Remark. Any open subset $U \subset X$ of an oriented stratifold $(X, \mathfrak{o})$ is naturally oriented. In fact, if we denote by i_2 the open inclusion $U \cap U_2 \hookrightarrow U_2$, there is a canonical isomorphism $\mathbb{D}^\bullet_{U \cap U_2} \simeq i_2^!(\mathbb{D}^\bullet_{U_2})$ and thus a trivialization of $\mathbb{D}^\bullet_{U \cap U_2}$ is given by $i_2^!(\mathfrak{o})$.

For simplicity, we will omit the orientation of a stratifold except where necessary and, consequently, we will denote an oriented stratifold $(X, \mathfrak{o})$ by X.

Using the terminology explained above, we can now introduce the definition of a self-dual complex of sheaves.

Definition 3.3. *Let $(X, \mathfrak{o})$ be an oriented stratifold. A constructible complex of sheaves $\mathbf{A}^\bullet \in D^b(X)$ is said to be* **self-dual** *if it satisfies*

- **(SD1):** Normalization: *there is an isomorphism:*

  $$\nu : \mathbf{A}^\bullet|_{U_2} \xrightarrow{\;\simeq\;} \mathbb{R}_{U_2}[n].$$

  The isomorphism ν is called the ***normalization of*** $\mathbf{A}^\bullet$*.*

- **(SD2):** Lower bound: $\mathbf{H}^i(\mathbf{A}^\bullet) = 0$*, for $i < -n$.*

- **(SD3):** Stalk condition for the upper middle perversity $\overline{n}$:

$$\mathbf{H}^i(\mathbf{A}^\bullet|_{U_{k+1}}) = 0, \quad for\ i > \overline{n}(k) - n,\ k \geq 2.$$

  Recall that the upper middle perversity $\overline{n}$ *is defined setting* $\overline{n}(k) = [\frac{k-1}{2}]$, *for* $k \geq 2$.

- **(SD4):** Self-Duality: *there is an isomorphism* $d : \mathcal{D}\mathbf{A}^\bullet[n] \xrightarrow{\simeq} \mathbf{A}^\bullet$ *such that* $\mathcal{D}d[n] = (-1)^n d$ *and the diagram*

$$\begin{array}{ccc}
\mathbb{R}_{U_2}[n] & \xleftarrow[\simeq]{\nu} & \mathbf{A}^\bullet|_{U_2} \\
\mathfrak{o} \uparrow \simeq & & \simeq \uparrow d|_{U_2} \\
\mathbb{D}^\bullet_{U_2} & \xrightarrow[\mathcal{D}\nu[n]]{\simeq} & \mathcal{D}\mathbf{A}^\bullet|_{U_2}[n]
\end{array}$$

  commutes.

The class of all self-dual complexes of sheaves on a stratifold X is actually independent from the orientation of X. In fact, if $\mathfrak{o}$ and $\mathfrak{o}'$ are two orientations of X, and $\mathbf{A}^\bullet$ is a self-dual complex of sheaves over $(X, \mathfrak{o})$, then it is enough to modify the normalization of $\mathbf{A}^\bullet$ to see that $\mathbf{A}^\bullet$ is also a self-dual complex of sheaves over $(X, \mathfrak{o}')$. For this reason, if X is an orientable stratifold, we denote by $\mathrm{SD}(X)$ the full subcategory of $D^b(X)$ whose objects are the self-dual complexes of sheaves over X for some orientation.

One of the first properties of the self-dual complexes of sheaves is given by the following

Lemma 3.4. *Any open inclusion* $U \hookrightarrow X$ *induces a functor*

$$i^* : \mathrm{SD}(X) \longrightarrow \mathrm{SD}(U).$$

Proof. What we have to show is that, if $\mathbf{A}^\bullet$ is a self-dual complex of sheaves over X (with normalization ν and self-duality isomorphism d), then $\mathbf{A}^\bullet|_U$ is a self-dual complex of sheaves over U (with normalization $\nu|_U$ and self-duality isomorphism $d|_U$).

Axiom **(SD1)** is satisfied using the isomorphism $\nu|_U$. Axioms **(SD2)** and **(SD3)** are also satisfied, since i^* is an exact functor and commutes therefore with cohomology.

Now, since i is an open inclusion, there is a natural equivalence of functors

$$i^!(-) \simeq i^*(-)$$

and therefore it holds

$$\mathcal{D}(i^!(-)) \simeq i^*\mathcal{D}(-) \simeq i^!\mathcal{D}(-).$$

This canonical identification can be used to define $d|_U$ as the isomorphism:

$$\mathcal{D}(i^!\mathbf{A}^\bullet)[n] \simeq i^!\mathcal{D}\mathbf{A}^\bullet[n] \xrightarrow{i^!(d)} i^!\mathbf{A}^\bullet$$

Finally, in order to see that $d|_U$ is a self-duality isomorphism, we observe that it holds

$$\mathcal{D}(i^!d)[n] = i^!(\mathcal{D}d[n]) = (-1)^n i^!(d).$$

and that, for this reason, the diagrams

$$\begin{array}{ccc}
i_2^!(\mathbb{R}_{U_2}[n]) & \xleftarrow{\quad i_2^!(\nu) \quad} & i_2^!(\mathbf{A}^\bullet_{U_2}) \\
{\scriptstyle i_2^!(\mathfrak{o})}\uparrow & & \uparrow{\scriptstyle i_2^!(d)} \\
i_2^!(\mathbb{D}^\bullet_{U_2}) & \xrightarrow[\quad i_2^!(\mathcal{D}\nu[n]) \quad]{} & i_2^!(\mathcal{D}\mathbf{A}^\bullet|_{U_2}[n])
\end{array}$$

and

$$\begin{array}{ccc}
\mathbb{R}_{U_2\cap U}[n] & \xleftarrow{\quad \nu|_U \quad} & \mathbf{A}^\bullet|_{U\cap U_2} \\
{\scriptstyle \mathfrak{o}|_U}\uparrow & & \uparrow{\scriptstyle d|_U} \\
\mathbb{D}^\bullet_{U\cap U_2} & \xrightarrow[\quad \mathcal{D}\nu|_U[n] \quad]{} & \mathcal{D}\mathbf{A}^\bullet|_{U\cap U_2}[n]
\end{array}$$

are commutative. ■

Let $\mathbf{E}^\bullet$ and $\mathbf{F}^\bullet$ be two complexes of sheaves over X, and denote by $\mathrm{Hom}_{D^b(X)}(\mathbf{E}^\bullet, \mathbf{F}^\bullet)$ the set of all morphisms from $\mathbf{E}^\bullet$ to $\mathbf{F}^\bullet$ in $D^b(X)$ (or, equivalently, in $\mathrm{SD}(X)$).

Lemma 3.5. *The restriction on the top stratum induces a monomorphism*

$$\mathrm{Hom}_{D^b(X)}(\mathbf{E}^\bullet, \mathbf{F}^\bullet) \to \mathrm{Hom}_{D^b(U_2)}(\mathbf{E}^\bullet|_{U_2}, \mathbf{F}^\bullet|_{U_2}).$$

Proof. The statement follows by an iterated application of ([Ba1], Lemma 2.2). ∎

This lemma has the interesting consequence that the self-duality isomorphism is, if it exists, completely determined by the orientation and by the normalization. In fact it holds the

Lemma 3.6. *Let* $\mathbf{A}^\bullet$ *be a self-dual complex of sheaves over* $(X, \mathfrak{o})$*, with normalization* $\nu : \mathbf{A}|^\bullet_{U_2} \xrightarrow{\simeq} \mathbb{R}_{U_2}[n]$*. Then there is a unique self-duality isomorphism* d *over* $\mathbf{A}^\bullet$ *which is compatible with* ν*.*

Proof. If $d, d' : \mathcal{D}\mathbf{A}^\bullet[n] \xrightarrow{\simeq} \mathbf{A}^\bullet$ are two self-duality isomorphisms, then, by Lemma 3.5, one has

$$d = d' \quad \Longleftrightarrow \quad d|_{U_2} = d'|_{U_2},$$

but, on the other hand, it holds

$$d|_{U_2} = \nu^{-1} \circ \mathfrak{o} \circ (\mathcal{D}\nu[n])^{-1} = d'|_{U_2}.$$

∎

The last point which we want to mention, is the problem of determining the structure of the category $\mathrm{SD}(X)$. This problem can be reduced to the determination of the relation between the categories $\mathrm{SD}(U_k)$ and $\mathrm{SD}(U_{k+1})$.

A partial answer is given by the following

Theorem 3.7 (Goreski-MacPherson). *If* k *is even, then the restriction functor*

$$i_k^* : \mathrm{SD}(U_{k+1}) \longrightarrow \mathrm{SD}(U_k)$$

is an equivalence of categories whose inverse is given by the functor

$$\tau_{\leq \overline{m}(k)-n} Ri_*(-) : \mathrm{SD}(U_k) \longrightarrow \mathrm{SD}(U_{k+1}).$$

In order to reconstruct all information contained in $\mathrm{SD}(X)$, we also need to understand the case of odd k, which has been investigated by Banagl in the above cited work. Before we begin to present Banagl's results we have to explain some terminology.

Let $\mathbf{A}^\bullet$ be a self-dual complex of sheaves over the open set U_k, with k odd and set $s := \overline{n}(k) - n$.

Definition 3.8. *The* **lifting obstruction** $\mathcal{O}(\mathbf{A}^\bullet)$ *is the complex*

$$\mathcal{O}(\mathbf{A}^\bullet) := \mathbf{H}^s(Ri_{k*}\mathbf{A}^\bullet)[-s] \in D^b(U_{k+1}).$$

Using the lifting obstruction, one can introduce the notion of a Lagrangian structure on a self-dual complex of sheaves.

Definition 3.9. *A* **Lagrangian structure on** $\mathbf{A}^\bullet$ *is a morphism*

$$\phi : \mathcal{L} \longrightarrow \mathcal{O}(\mathbf{A}^\bullet)$$

which induces injections $\mathbf{H}^*(\phi) : \mathbf{H}^*(\mathcal{L}) \to \mathbf{H}^*(\mathcal{O}(\mathbf{A}^\bullet))$ *and such that some distinguished triangle on* ϕ *is a null-bordism for the self-dual lifting obstruction* (See [Ba1], Def. 2.3).

This definition is unfortunately too long and technical to be presented completely in this thesis and, for this reason, the reader is referred to Banagl's work.

An alternative approach is suggested in ([Ba1], Remark 2.3). Let $\mathbf{A}^\bullet \in \mathrm{SD}(U_k)$, and denote by i and j respectively the inclusions $U_k \hookrightarrow U_{k+1}$ and $\Sigma := U_{k+1} - U_k \hookrightarrow U_{k+1}$. Moreover set $\mathbf{H} := \mathbf{H}^s(j^*Ri_*\mathbf{A}^\bullet) \simeq \mathcal{O}(\mathbf{A}^\bullet)|_\Sigma$.

By ([Ba1], Lemma 2.3), the self-duality isomorphism d induces an isomorphism $\delta : \mathcal{D}\mathcal{O}(\mathbf{A}^\bullet)[n+1] \xrightarrow{\simeq} \mathcal{O}(\mathbf{A}^\bullet)$ and therefore a non-singular pairing

$$\mathbf{H} \otimes \mathbf{H} \longrightarrow \mathbb{R}_\Sigma.$$

A subsheaf $\mathbf{E} \subset \mathbf{H}$ is called **Lagrangian** if, for every $x \in \Sigma$, the stalk $\mathbf{E}_x$ is a Lagrangian subspace of $\mathbf{H}_x$.

Lemma 3.10. *The map*

$$\left\{ \begin{array}{c} \text{Lagrangian structures} \\ \text{of } \mathbf{A}^\bullet \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Lagrangian subsheaves} \\ \text{of } \mathbf{H} \end{array} \right\}$$

$$(\mathcal{L}, \phi) \longmapsto (\mathbf{H}^s(\phi)(\mathcal{L}))|_\Sigma \subset \mathcal{O}(\mathbf{A})|_\Sigma$$

is a bijection.

A **morphism of Lagrangian structures** is a commutative diagram in $D^b(U_{k+1})$

$$\begin{array}{ccc} \mathcal{L}_A & \xrightarrow{\phi_A} & \mathcal{O}(\mathbf{A}^\bullet) \\ \downarrow & & \downarrow \mathcal{O}(f) \\ \mathcal{L}_B & \xrightarrow{\phi_B} & \mathcal{O}(\mathbf{B}^\bullet) \end{array}$$

for some $f : \mathbf{A}^\bullet \to \mathbf{B}^\bullet$.

It follows from the functoriality of the lifting obstruction that the composition of morphisms of Lagrangian structures is well defined and thus that the Lagrangian structures form a category $\mathrm{Lag}(U_{k+1} - U_k)$.

The categories $\mathrm{SD}(U_k)$ and $\mathrm{Lag}(U_{k+1} - U_k)$ can be used to construct a new category which is called the twisted product category and which is denoted by $\mathrm{SD}(U_k) \rtimes \mathrm{Lag}(U_{k+1} - U_k)$. By definition this is the category whose objects are the pairs

$$(\mathbf{A}^\bullet, \phi : \mathcal{L} \to \mathcal{O}(\mathbf{A}^\bullet)) \in \mathrm{SD}(U_k) \rtimes \mathrm{Lag}(U_{k+1} - U_k),$$

and whose morphisms are the pairs (f, g) with first component a morphism $f \in \mathrm{Hom}_{D^b(U_k)}(\mathbf{A}^\bullet, \mathbf{B}^\bullet)$ and second component a commutative square

$$\begin{array}{ccc} \mathcal{L}_A & \xrightarrow{\phi_A} & \mathcal{O}(\mathbf{A}^\bullet) \\ g \big\downarrow & & \big\downarrow \mathcal{O}(f) \\ \mathcal{L}_B & \xrightarrow{\phi_B} & \mathcal{O}(\mathbf{B}^\bullet) \end{array}$$

If $\mathbf{A}^\bullet$ is a self-dual complex of sheaves on U_{k+1}, then there is a constructive way to extract from $\mathbf{A}^\bullet$ a lagrangian structure on $i_k^* \mathbf{A}^\bullet$. This procedure allows to define a functor

$$\Lambda : \mathrm{SD}(U_{k+1}) \longrightarrow \mathrm{Lag}(U_{k+1} - U_k)$$

and Banagl's main result can be thus formulated as follows.

Theorem 3.11 (Banagl). *The functor*

$$(i_k^*, \Lambda) : \mathrm{SD}(U_{k+1}) \longrightarrow \mathrm{SD}(U_k) \rtimes \mathrm{Lag}(U_{k+1} - U_k)$$

is an equivalence of categories. The inverse of (i_k^*, Λ) is denoted by $\boxplus$.

Finally, putting this result together with Goreski-MacPherson's theorem, one gets the following structure theorem for the category of self-dual complexes of sheaves (see [Ba1], Theorem 2.10).

Theorem 3.12. *Let X be an n-dimensional stratifold. Then there is an equivalence of categories*

$$\mathrm{SD}(X) \simeq \mathrm{Const}(U_2) \rtimes \mathrm{Lag}(U_4 - U_3) \rtimes \ldots \rtimes \mathrm{Lag}(U_{n-2} - U_{n-3}) \rtimes \mathrm{Lag}(U_n - U_{n-1}).$$

3.2 H-Stratifolds

In this section we present the definition of H-stratifold and consider some constructions in the category of H-stratifolds.

Let X be an n-dimensional oriented stratifold.

Definition 3.13. *An* **H-structure** $\mathcal{S}$ *over* X *is a pair* $(\mathbf{A}^\bullet, \nu)$ *where* $\mathbf{A}^\bullet$ *is a self-dual complex of sheaves over* X *and* ν *is a normalization of* $\mathbf{A}^\bullet$.

If $\mathcal{S} = (\mathbf{A}^\bullet, \nu)$ is an H-structure over X, then, by Lemma 3.6, there is unique self-duality isomorphism $d : \mathcal{D}\mathbf{A}^\bullet[n] \xrightarrow{\simeq} \mathbf{A}^\bullet$ which is compatible with the orientation of X and with ν. Consequently, we do not need to specify d as a part of $\mathcal{S}$.

Definition 3.14. *Let* $\mathcal{S}_1 = (\mathbf{A}_1^\bullet, \nu_1)$ *and* $\mathcal{S}_2 = (\mathbf{A}_2^\bullet, \nu_2)$ *be two* H*-structures over a stratifold* X*. An* ***isomorphism of*** H***-structures***

$$\varphi : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$$

is an isomorphism of complexes of sheaves $\varphi : \mathbf{A}_1^\bullet \to \mathbf{A}_2^\bullet$*, for which the diagram*

$$\begin{array}{ccc} \mathbf{A}_1^\bullet|_{U_2} & \xrightarrow{\varphi|_{U_2}} & \mathbf{A}_2^\bullet|_{U_2} \\ & {\scriptstyle \nu_1} \searrow \quad \swarrow {\scriptstyle \nu_2} & \\ & \mathbb{R}_{U_2} & \end{array}$$

commutes.

Definition 3.15. *An* **H-stratifold** *is a pair* $(X, \mathcal{S})$*, where* X *is an oriented topological stratifold and* $\mathcal{S}$ *is an* H*-structure over* X.

If $(X, \mathcal{S})$ is an H-stratifold, then we denote by $-(X, \mathcal{S})$ the H-stratifold $(-X, \mathcal{S})$ obtained reversing the orientation of X and considering $\mathcal{S}$ as an H-structure over $-X$.

Now, let $\varphi : X \to Y$ be an orientation-preserving isomorphism of stratifolds and let $\mathcal{S} = (\mathbf{A}^\bullet, \nu)$ be an H-structure on Y.

Lemma/Definition 3.16. *The pair* $\varphi^*\mathcal{S} = (\varphi^*\mathbf{A}^\bullet, \varphi^*\nu)$ *is an* H*-structure on* X *which is called the* ***pull-back*** H***-structure***.

Proof. We have to check that $\varphi_* \mathbf{A}^\bullet$ is a self-dual complex of sheaves over X. Axiom **(SD1)** is satisfied with the normalization $\varphi^* \nu$. Axioms **(SD2)** and **(SD3)** are satisfied since it holds respectively

$$\mathbf{H}^i(\varphi^* \mathbf{A}^\bullet) \simeq \varphi^* \mathbf{H}^i(\mathbf{A}^\bullet)$$

and

$$\mathbf{H}^i((\varphi^* \mathbf{A}^\bullet)|_{U_{k+1}(X)}) \simeq \varphi^* \mathbf{H}^i(\mathbf{A}^\bullet|_{U_{k+1}(Y)}).$$

Finally, axiom **(SD4)** is satisfied using the self-duality isomorphism

$$\mathcal{D}(\varphi^* \mathbf{A}^\bullet)[n] \simeq \varphi^!(\mathcal{D}\mathbf{A}^\bullet[n]) \xrightarrow{\simeq} \varphi^! \mathbf{A}^\bullet \simeq \varphi^* \mathbf{A}^\bullet.$$

The reader is referred for the details to the proof of Lemma 3.4. ■

The previous definition allows us to explain the notion of isomorphism of H-stratifolds.

Definition 3.17. *Let $(X_1, \mathcal{S}_1)$ and $(X_2, \mathcal{S}_2)$ be two H-stratifolds with H-structure $\mathcal{S}_1 = (\mathbf{A}_1^\bullet, \nu_1)$ and $\mathcal{S}_2 = (\mathbf{A}_2^\bullet, \nu_2)$. An **isomorphism of H-stratifolds** (or H-isomorphism)*

$$\varphi : (X_1, \mathcal{S}_1) \xrightarrow{\simeq} (X_2, \mathcal{S}_2)$$

is a pair (φ_1, φ_2), where $\varphi_1 : X_1 \xrightarrow{\simeq} X_2$ is an orientation-preserving isomorphism of stratifolds and $\varphi_2 : \mathcal{S}_1 \xrightarrow{\simeq} \varphi_1^* \mathcal{S}_2$ is an isomorphism of H-structures.

An isomorphism of H-stratifolds is automatically compatible with the self-duality isomorphisms. In fact it holds the

Lemma 3.18. *For any isomorphism of H-stratifold*

$$\varphi = (\varphi_1, \varphi_2) : (X_1, (\mathbf{A}_1^\bullet, \nu_1)) \xrightarrow{\simeq} (X_2, (\mathbf{A}_2^\bullet, \nu_2)),$$

the diagram

$$\begin{array}{ccc} \mathcal{D}\mathbf{A}_1^\bullet[n] & \xleftarrow[\simeq]{\mathcal{D}\varphi_2[n]} & \mathcal{D}\varphi_1^* \mathbf{A}_2^\bullet[n] \simeq \varphi_1^* \mathcal{D}\mathbf{A}_2^\bullet[n] \\ {\scriptstyle d_1} \downarrow {\scriptstyle \simeq} & & {\scriptstyle \simeq} \downarrow {\scriptstyle \varphi_1^* d_2} \\ \mathbf{A}_1^\bullet & \xrightarrow[\varphi_2]{\simeq} & \varphi_1^* \mathbf{A}_2^\bullet \end{array}$$

commutes.

Proof. According to Lemma 3.5 it holds:

$$d_1 = \varphi_2^{-1} \circ \varphi_1^* d_2 \circ (\mathcal{D}\varphi_2[n])^{-1} \quad \Longleftrightarrow \quad d_1|_{U_2} = \varphi_2^{-1}|_{U_2} \circ \varphi_1^* d_2|_{U_2} \circ (\mathcal{D}\varphi_2[n]|_{U_2})^{-1}.$$

On the other hand, one has by definition $\nu_1 = \varphi_1^* \nu_2 \circ \varphi_2|_{U_2}$, and thus it holds

$$\begin{aligned} d_1|_{U_2} &= \nu_1^{-1} \circ \mathfrak{o}_1 \circ \mathcal{D}(\nu_1^{-1})[n] \\ &= \varphi_2^{-1}|_{U_2} \circ \varphi_1^* \nu_2^{-1} \circ \mathfrak{o}_1 \circ (\varphi_1^* \mathcal{D}\nu_2^{-1} \circ \mathcal{D}\varphi_2^{-1}|_{U_2}[n] \\ &= \varphi_2^{-1}|_{U_2} \circ \varphi_1^* (\nu_2^{-1} \circ \mathfrak{o}_2 \circ \mathcal{D}\nu_2^{-1}[n]) \circ \mathcal{D}\varphi_2^{-1}|_{U_2}[n] \\ &= \varphi_2^{-1}|U_2 \circ \varphi_1^* d_2|_{U_2} \circ (\mathcal{D}\varphi_2[n]_{U_2})^{-1}. \end{aligned}$$

■

The class $\mathscr{C}$ of all H-stratifolds can be turned into a category with the H-isomorphisms as morphisms.

- For any object $(X, \mathcal{S})$ there is an H-isomorphism $1_{(X,\mathcal{S})} := (1_X, 1_{\mathbf{A}^\bullet})$.
- Given $(\varphi_1, \varphi_2) : (X_1, \mathcal{S}_1) \to (X_2, \mathcal{S}_2)$ and $(\varphi_1', \varphi_2') : (X_2, \mathcal{S}_2) \to (X_3, \mathcal{S}_3)$, the composition is defined setting $(\varphi_1', \varphi_2') \circ (\varphi_1, \varphi_2) := (\varphi_1' \circ \varphi_1, \varphi_1^*(\varphi_2') \circ \varphi_2)$

Finally, we want to explain how to realize an oriented manifold as an H-stratifold.

Let M be an oriented n-dimensional topological manifold. The trivial sheaf $\mathbb{R}_M[n]$ is a self-dual complex of sheaves with the identity as normalization and with the self-duality isomorphism

$$\mathcal{D}(\mathbb{R}_M[n])[n] \simeq \mathcal{D}\mathbb{R}_M \simeq \mathbb{D}_M^\bullet \xrightarrow{\simeq} \mathbb{R}_M[n].$$

In other words, if $\mathscr{M}$ denotes the category of oriented topological manifolds with the oriented isomorphisms as morphisms, there is a natural inclusion of categories

$$\mathscr{M} \hookrightarrow \mathscr{C}$$

$$M \longmapsto (M, (\mathbb{R}_M[n], \mathrm{Id})).$$

3.3 Some properties of H-stratifolds

In this section we present some constructions which will play a fundamental role in the construction of Hirzebruch homology.

3.3.1 Restriction of H-structures

The first construction which we want to consider is the restriction of an H-structure to an open subset.

Let $\mathcal{S} = (\mathbf{A}^\bullet, \nu)$ be an H-structure over a stratifold X and let U be an open subset of X. As we have seen in Lemma 3.4, the complex of sheaves $\mathbf{A}^\bullet|_U \in D^b(U)$ is a self-dual complex of sheaves with normalization $\nu|_U$. For this reason we can formulate the following

Definition 3.19. *The H-structure $\mathcal{S}|_U = (\mathbf{A}^\bullet|_U, \nu|_U)$ is called the* **restriction** *of $\mathcal{S}$ to U.*

Open subsets of an H-stratifold will be always considered as H-stratifolds with the induced H-structure.

3.3.2 Disjoint union of H-stratifolds

Now we want to consider the disjoint union (or topological sum) of two H-stratifolds.

Definition 3.20. *The **disjoint union** of two H-stratifolds $(X_1, \mathcal{S}_1)$ and $(X_2, \mathcal{S}_2)$ is the H-stratifold*

$$(X_1 + X_2, \mathcal{S}_1 + \mathcal{S}_2)$$

where

$$\mathcal{S}_1 + \mathcal{S}_2 := (\mathbf{A}_1^\bullet + \mathbf{A}_2^\bullet, \nu_1 + \nu_2).$$

By $\mathbf{A}_1^\bullet + \mathbf{A}_2^\bullet$ we mean here the complex whose n-th sheaf is the disjoint union $\mathbf{A}_1^n + \mathbf{A}_2^n$. Recall that if, for $i = 1, 2$, X_i is a topological space and $\mathbf{A}_i$ is a sheaf over X_i with projection $\pi_i : \mathbf{A}_i \to X_i$, then the topological sum $\mathbf{A}_1 + \mathbf{A}_2$ is a sheaf with the projection $\pi_1 + \pi_2 : \mathbf{A}_1 + \mathbf{A}_2 \to X_1 + X_2$.

3.3.3 Product structures

This section is devoted to the definition of the product of an H-stratifold with an oriented manifold. In the second part we will consider the relates problem of determining all H-structures on a bicollared substratifold of an H-stratifold.

Let X be an n-dimensional stratifold, M an m-dimensional oriented manifold and denote by π_i and p_i the projections:

$$\pi_1 : X \times M \longrightarrow X, \quad \pi_2 : X \times M \longrightarrow M$$

and

$$p_1 : U_2 \times M \longrightarrow U_2, \quad p_2 : U_2 \times M \longrightarrow M.$$

Lemma/Definition 3.21. *If $\mathcal{S} = (\mathbf{A}^\bullet, \nu))$ is an H-structure over X, then the pair*

$$\pi_1^! \mathcal{S} = (\pi_1^! \mathbf{A}^\bullet, p_1^! \nu)$$

is an H-structure over $X \times M$. The H-stratifold $(X \times M, \pi_1^! \mathcal{S})$ will be called the ***product*** *of $(X, \mathcal{S})$ with M.*

Proof. According to ([KaSc], Prop. 3.3.2), there are natural isomorphisms of functors

$$\pi_1^!(-) \simeq \pi_1^*(-) \otimes \pi_2^* \mathbb{D}_M^\bullet$$

and

$$p_1^!(-) \simeq p_1^*(-) \otimes p_2^* \mathbb{D}_M^\bullet.$$

The orientation of M induces thus a natural equivalence

$$\pi_1^!(-) \simeq \pi_1^*(-) \otimes \pi_2^* \mathbb{D}_M^\bullet \xrightarrow{\simeq} \pi_1^*(-) \otimes \mathbb{R}_{X \times M}[m] \simeq \pi_1^*[m](-)$$

and a natural isomorphism

$$p_1^!(-) \xrightarrow{\simeq} p_1^*[m](-).$$

In particular, the dualizing complex of $U_2 \times M$ is isomorphic to $p_1^* \mathbb{D}_{U_2}^\bullet[m]$ and the orientation of X induces an orientation of $X \times M$

$$\mathfrak{o}_{X \times M} : \mathbb{D}_{U_2 \times M}^\bullet \xrightarrow{\simeq} p_1^* \mathbb{D}_{U_2}^\bullet[m] \xrightarrow{\simeq} \mathbb{R}_{U_2 \times M}[m+n].$$

Now, we have to show that $\pi_1^! \mathbf{A}^\bullet$ is a self-dual complex over $X \times M$.

- **(SD1)** is of course satisfied with normalization $p_1^! \nu$.
- In order to show that **(SD2)** is satisfied, it is enough to observe that it holds

  $$\mathbf{H}^i(\pi_1^! \mathbf{A}^\bullet) \simeq \mathbf{H}^i(\pi_1^* \mathbf{A}^\bullet[m]) \simeq \pi_1^* \mathbf{H}^i(\mathbf{A}^\bullet[m]) = \pi_1^* \mathbf{H}^{i+m}(\mathbf{A}^\bullet),$$

  and that, if $i < -n - m$, then it results $\mathbf{H}^i(\pi^! \mathbf{A}^\bullet) = 0$ since $\mathbf{A}^\bullet$ satisfies **(SD2)**.
- Axiom **(SD3)** can be proved using the isomorphism

  $$\mathbf{H}^i((\pi_1^! \mathbf{A}^\bullet)|_{U_{k+1}(X \times M)}) \simeq \mathbf{H}^i((\pi_1^* \mathbf{A}^\bullet[m])|_{U_{k+1} \times M}) \simeq \pi_1^* \mathbf{H}^{i+m}(\mathbf{A}^\bullet|_{U_{k+1}}).$$

  In fact, if $i > \overline{n}(k) - (n+m)$, then $i + m > \overline{n}(k) - n$ and the last term is zero since $\mathbf{A}^\bullet$ satisfies **(SD3)**.

- Finally a self-duality isomorphism $d_{X\times M}$ is given by the composition:

$$\mathcal{D}\pi_1^!\mathbf{A}^\bullet[m+n] \simeq \pi_1^*\mathcal{D}\mathbf{A}^\bullet[n][m] \xrightarrow{\simeq} \pi_1^!(\mathcal{D}\mathbf{A}^\bullet[n]) \xrightarrow{\pi_1^! d} \pi_1^!\mathbf{A}^\bullet$$

where the second isomorphism is induced by $\mathfrak{o}_M = d_M$. The equalities $\mathcal{D}d_X[n] = (-1)^n d_X$ and $\mathcal{D}d_M[m] = (-1)^m d_M$, imply

$$\mathcal{D}d_{X\times M}[m+n] = (-1)^{m+n} d_{X\times M}.$$

Finally, the commutativity of

$$\begin{array}{ccc} \mathbb{R}_{U_2\times M}[m+n] & \longleftarrow & (\pi_1^!\mathbf{A}^\bullet)|_{U_2\times M} \\ \uparrow & & \uparrow \\ \mathbb{D}^\bullet_{U_2\times M} & \longrightarrow & (\mathcal{D}\pi_1^!\mathbf{A}^\bullet[n+m])|_{U_2\times M} \end{array}$$

follows from that of

$$\begin{array}{ccc} p_1^!(\mathbb{R}_{U_2}[n]) & \longleftarrow & p_1^!(\mathbf{A}^\bullet|_{U_2}) \\ \uparrow & & \uparrow \\ p_1^!(\mathbb{D}^\bullet_{U_2}) & \longrightarrow & p_1^!(\mathcal{D}\mathbf{A}^\bullet|_{U_2}[n]) \end{array}$$

■

The product of an H-stratifold with a manifold allows to define a map

$$\mathscr{M} \times \mathscr{C} \to \mathscr{C}.$$

Actually, it is possible to generalize this construction to a product

$$\mathscr{C} \times \mathscr{C} \to \mathscr{C}.$$

Such a map is however not essential in the definition of Hirzebruch homology (as additive homology theory), and for this reason we prefer to postpone the treatment of this point to Section 3.6.

In the second part of this section we want instead to restrict our attention to the case of $M = \mathbb{R}$. This case is particularly interesting, since, in order to define the boundary operator for the Mayer-Vietoris sequence, we have to understand all possible H-structures over the products of the form $X \times \mathbb{R}$.

Let us assume $\mathbb{R}$ to be endowed with a fixed orientation, and denote by j and j_2 the inclusions $X = X \times \{0\} \hookrightarrow X \times \mathbb{R}$ and $U_2 \hookrightarrow X \times \mathbb{R}$ respectively.

Lemma 3.22. *Let $\mathcal{S} = (\mathbf{A}^\bullet, \nu)$ be an H-structure over $X \times \mathbb{R}$. Then there is a unique (up to isomorphism) H-structure $j^! \mathcal{S}$ over X, such that*

$$\mathcal{S} \simeq \pi^! j^! \mathcal{S}.$$

Proof. As we have seen in the proof of Lemma/Definition 3.21, the orientation of $\mathbb{R}$ induces an isomorphism $\pi^!(-) \simeq \pi^*[1](-)$. Moreover, according to ([Ba1], Lemma 5.2), there is a natural identification $\pi^* j^*(-) \simeq \mathrm{Id}$ and consequently it results

$$j^![1](-) \simeq j^! \pi^* j^*[1](-) \simeq j^! \pi^! j^* \simeq j^*(-).$$

The H-structure $j^! \mathcal{S}$ is defined setting:

$$j^! \mathcal{S} := (j^! \mathbf{A}^\bullet, j_2^! \nu).$$

The orientation of X is here given by the isomorphism $j_2^!(\mathfrak{o})$. As usual, we only have to show that $j^! \mathbf{A}^\bullet$ is a self-dual complex of sheaves.

- **(SD1)** is clear since a normalization is given by

$$j_2^!(\nu) : j_2^!(\mathbf{A}^\bullet|_{U_2}) \xrightarrow{\simeq} j_2^!(\mathbb{R}_{U_2})[n] \simeq \mathbb{R}_{U_2 \cap X}[n-1].$$

- **(SD2)** and **(SD3)** are consequences of the fact that $j^! \mathbf{A}^\bullet \simeq j^* \mathbf{A}^\bullet[-1]$.

- The self-duality isomorphism is given by the composition

$$\mathcal{D}(j^! \mathbf{A}^\bullet)[n-1] \simeq j^* \mathcal{D}(\mathbf{A}^\bullet)[n-1] \simeq j^! \mathcal{D}(\mathbf{A}^\bullet)[n] \xrightarrow{\simeq} j^! \mathbf{A}^\bullet.$$

The only thing left to show is the isomorphism

$$\pi^! j^! \mathcal{S} \simeq \mathcal{S}$$

but this is a just consequence of the functorial identification

$$\pi^! j^!(-) \simeq \pi^* j^*(-) \simeq \mathrm{Id}.$$

■

3.4 Existence of a small subclass

This section is devoted to the construction of a small subcategory $\mathscr{C}_0$ of $\mathscr{C}$ which has the property that every H-stratifold is isomorphic to an H-stratifold in $\mathscr{C}_0$. This construction is based on the following

Lemma 3.23. *For a fixed n-dimensional stratifold X, there is a small subclass $\mathrm{SD}_0(X) \subset \mathrm{SD}(X)$ such that any self-dual complex of sheaves over X is isomorphic to a complex of sheaves in $\mathrm{SD}_0(X)$.*

Proof. We proceed by induction on the codimension of the strata of X.

By definition, every self-dual complex of sheaves on U_2 is isomorphic to the trivial complex $\mathbb{R}_{U_2}[n](\mathfrak{o}, \mathbb{R}_{U_2}[n], \nu)$ and so we can set $\mathrm{SD}_0(U_2) = \mathrm{Const}(U_2)$.

Now, we suppose to have already defined $\mathrm{SD}_0(U_k)$ and we show how to construct $\mathrm{SD}(U_{k+1})$.

- Let k be even. According to Goreski-MacPherson (see Theorem 3.7 of this thesis), the restriction functor i_k^* induces an equivalence of categories $\mathrm{SD}(U_{k+1}) \simeq \mathrm{SD}(U_k)$ and the subclass $\mathrm{SD}_0(U_{k+1})$ can be defined as the preimage under i_k^* of $\mathrm{SD}_0(U_k)$.

- Now let k be odd. According to Banagl (see Theorem 3.11 of this thesis), there is an equivalence of categories.

$$\mathrm{SD}(U_{k+1}) \simeq \mathrm{SD}(U_k) \rtimes \mathrm{Lag}(U_{k+1} - U_k).$$

  By inductive assumption, there is a small class $\mathrm{SD}_0(U_k) \subset \mathrm{SD}(U_k)$ so that every complex $\mathbf{A}^\bullet \in \mathrm{SD}(U_k)$ is isomorphic to a complex in $\mathrm{SD}_0(U_k)$. Let $\mathrm{Lag}_0(U_{k+1}-U_k)$ be the subclass of $\mathrm{Lag}(U_{k+1}-U_k)$ defined setting:

$$\mathrm{Lag}_0(U_{k+1} - U_k) := \{\mathcal{L} \subset \mathcal{O}(\mathbf{A}^\bullet) | \mathbf{A}^\bullet \in \mathrm{SD}_0(U_k), \mathcal{L} \in \mathrm{Lag}(U_{k+1} - U_k)\}.$$

  In other words an element of $\mathrm{Lag}_0(U_{k+1}-U_k)$ is a Lagrangian subsheaf of $\mathcal{O}(\mathbf{A}^\bullet)$, for some $\mathbf{A}^\bullet \in \mathrm{SD}_0(U_k)$. The class $\mathrm{Lag}_0(U_{k+1} - U_k)$ is by construction a set and, moreover, every lagrangian structure is isomorphic to an element of $\mathrm{Lag}_0(U_{k+1} - U_k)$. In fact, if $(\mathcal{L}, \phi)$ is a lagrangian structure over $\mathbf{A}^\bullet \in \mathrm{SD}(U_k)$, then there exist a complex $\mathbf{B}^\bullet \in \mathrm{SD}_0(U_k)$ and an isomorphism $\alpha : \mathbf{A}^\bullet \xrightarrow{\simeq} \mathbf{B}^\bullet$. The Lagrangian structure $(\mathcal{L}, \phi)$ is now isomorphic to $(\mathcal{L}, \mathcal{O}(\alpha) \circ \phi)$ and consequently, since the map $\mathcal{O}(\alpha) \circ \phi : \mathcal{L} \to \mathcal{O}(\mathbf{B}^\bullet)$ induces an injection in cohomology, $\mathcal{L}$ can be identified up to isomorphism with its image under $\mathcal{O}(\alpha) \circ \phi$ (here we are using the fact that the cohomology of $\mathcal{L}$ is concentrated in degree s, and that for this reason $\mathcal{L}$ is canonically isomorphic to $\mathbf{H}^s(\mathcal{L})[-s]$). Finally, $\mathrm{SD}_0(U_{k+1})$ is defined as the set of all complexes of sheaves of the form $\mathbf{A}^\bullet \boxplus \mathcal{L}$ for $\mathbf{A}^\bullet \in \mathrm{SD}_0(U_k)$ and $\mathcal{L} \in \mathrm{Lag}_0(U_{k+1} - U_k)$.

■

A consequence of this result and of Lemma 1.12 is the

Proposition 3.24. *There is a small subcategory $\mathscr{C}_0 \subset \mathscr{C}$ of the category of H-stratifolds so that every H-stratifold is isomorphic to an element of $\mathscr{C}_0$.*

Proof. Let $\mathscr{C}_0$ be the class defined by

$$\mathscr{C}_0 := \{(X, \mathcal{S}) | X \in \mathscr{K}_0, \mathcal{S} = (\mathbf{A}^\bullet, \nu) \text{ with } \mathbf{A}^\bullet \in SD_0(X)\}.$$

If $(X, \mathcal{S})$ is an H-stratifold, then we know by Lemma 1.12 that there is a stratifold $Y \in \mathscr{K}_0$ and an isomorphism $\varphi : Y \xrightarrow{\simeq} X$. The pull-back construction provides an H-structure $\varphi^*\mathcal{S}$ over Y with the property that $(Y, \varphi^*\mathcal{S})$ is isomorphic to $(X, \mathcal{S})$ as an H-stratifold. Now, according to Lemma 3.23, there is an isomorphism $\alpha : \mathbf{B}^\bullet \xrightarrow{\simeq} \varphi^*\mathbf{A}^\bullet$ with $\mathbf{B}^\bullet \in \mathrm{SD}_0(Y)$ and in particular it results

$$\varphi^*\mathcal{S} \simeq \mathcal{T} := (\mathbf{B}^\bullet, (\varphi|_{U_2})^*(\nu) \circ (\alpha|_{U_2})).$$

Collecting these facts together, we get an isomorphism

$$(X, \mathcal{S}) \simeq (Y, \varphi^*\mathcal{S}) \simeq (Y, \mathcal{T}) \in \mathscr{C}_0$$

and so the only thing left to show is that the class $\mathscr{C}_0$ is small, but this is clear since the orientations of a stratifold are a set, and the same holds for the class of all normalizations of a fixed complex.

■

3.5 Collared H-Stratifolds

In this section we introduce the notion of H-stratifold with boundary or, more precisely, of collared H-stratifold.

Let $(X, \partial X)$ be a pair of spaces with ∂X closed in X, and suppose that $(\mathring{X}, \mathcal{S})$ and $(\partial X, \partial\mathcal{S})$ are two H-stratifolds of dimension n and $n-1$ respectively. Moreover, denote by i and j the inclusions in X of $\mathring{X}$ and ∂X, and by π the projection $\partial X \times (0, +\infty) \to X$.

Definition 3.25. *A **collar** is a pair (c, φ) where*

- *$c : V \to U$ is a collar of X as a topological stratifold in the sense of Definition 1.18;*

- *φ is an isomorphism of H-stratifolds*

$$\varphi : (V - \partial X \times \{0\}, (\pi^! \partial \mathcal{S})|_{V-\partial X \times \{0\}}) \xrightarrow{\simeq} (U - \partial X, \mathcal{S}|_{U-\partial X})$$

whose first component φ_1 is equal to $c|_{V-\partial X\times\{0\}}$. Here $\pi^! \partial \mathcal{S}$ is the product structure on $\partial X \times (0, +\infty)$.

Two collars $(c, \varphi) : V \to U$ and $(c', \varphi') : V' \to U'$ are said to be **equivalent** if there is an open subset $V'' \subset V \cap V'$, such that $(c, \varphi)|_{V'} = (c, \varphi')|_{V''}$. An equivalence class of collars is called a **germ of collars**. If ∂X is compact, then it is possible to assume the collar to be of constant length, that is to say of the form

$$(c, \varphi) : \partial X \times [0, +\varepsilon) \to U.$$

Definition 3.26. *A **collared H-stratifold** is a pair of spaces $(X, \partial X)$, where $(\mathring{X}, \mathcal{S})$ is an n-dimensional H-stratifold, ∂X is a closed subspace and an $(n-1)$-dimensional H-stratifold with H-structure $\partial\mathcal{S}$, together with a germ of collars $[(c, \varphi)]$. The H-stratifold $(\partial X, \partial \mathcal{S})$ is called the **boundary** of $(X, \partial X)$.*

Example. Let $(X, \mathcal{S})$ be an H-stratifold with $\mathcal{S} = (\mathbf{A}^\bullet, \nu)$.

1. The product $X \times [0, 1]$ is a collared H-stratifold with boundary $(X, \mathcal{S}) + (-X, \mathcal{S})$, where $-X$ denotes the stratifold obtained reversing the orientation of X and the H-structure on $X \times (0, 1)$ is given by the product structure $\pi^! \mathcal{S}$ described in Section 3.3.3.

2. Let i denote the inclusion of $X \times (0, 1)$ in CX. <u>**If**</u> there exists an H-structure $\mathcal{T}$ on CX so that $i^* \mathcal{T}$ is isomorphic to $\pi^! \mathcal{S}$, then $(CX, \mathcal{T})$ is a collared H-stratifold whose boundary is isomorphic to $(X, \mathcal{S})$. We will see in the next chapter under which conditions such an H-structure $\mathcal{T}$ exists.

The H-structure on the boundary of a collared H-stratifold X can be deduced directly from the H-structure on the interior of X, as we are going to show. In order to simplify the notation, we assume the collar to be of constant length, but the same argument applies in the general case.

Using the collar we can restrict our attention to a space of the form $X \times [0, +\varepsilon)$ and we denote by i, j, π the maps indicated in the following diagram.

$$\begin{array}{ccc} X \times (0,+\varepsilon) & \overset{i}{\hookrightarrow} & X \times [0,+\varepsilon) \\ & \pi \searrow \quad \nearrow j = j_0 & \\ & X & \end{array}$$

Lemma 3.27. *For any sheaf* $\mathbf{A} \in Sh(X)$ *there is a natural isomorphism*

$$j^* i_* \pi^* \mathbf{A} \simeq \mathbf{A}.$$

Proof. If p denotes the projection $X \times [0,+\varepsilon) \to X$, then it is enough to show

$$p^* \mathbf{A} \simeq i_* \pi^* \mathbf{A}.$$

In fact it follows from the last isomorphism that there is an isomorphism

$$\mathbf{A} \simeq j^* p^* \mathbf{A} \simeq j^* i_* \pi^* \mathbf{A}.$$

The identity of $i^* p^* \mathbf{A}^\bullet$ induces by adjunction a morphism

$$\alpha : p^* \mathbf{A} \longrightarrow i_* i^* p^* \mathbf{A} = i_* \pi^* \mathbf{A}$$

which we want to prove to be an isomorphism. For any point $x \in X$ and any $t < \varepsilon$, the stalk of $p^* \mathbf{A}$ at (x,t) is isomorphic to $\mathbf{A}_x$. If we take $t \neq 0$, then it results

$$(i_* \pi^* \mathbf{A})_{(x,t)} \simeq \mathbf{A}_x$$

and it is easily seen that, under this identification, it results $\alpha(a) = a$.

So, it remains to consider the case $t = 0$. The stalk of $i_* \pi^* \mathbf{A}$ at $(x,0)$ can be identified with $\mathbf{A}_x$ through the isomorphism

$$\begin{aligned} (i_* \pi^* \mathbf{A})_x &\simeq \lim_{\substack{U \ni x \\ \delta \to 0}} \Gamma(U \times [0,\delta), i_* \pi^* \mathbf{A}) \\ &\simeq \lim_{\substack{U \ni x \\ \delta \to 0}} \Gamma(U \times (0,\delta), \pi^* \mathbf{A}) \\ &\simeq \lim_{U \ni x} \Gamma(U, \mathbf{A}) \\ &\simeq \mathbf{A}_x. \end{aligned}$$

Let $a \in (p^* \mathbf{A})_{(x,0)} \simeq \mathbf{A}_x$ and consider a section $\sigma : U \to \mathbf{A}$ which extends a on an open neighborhood of x in X. Using σ we define

$$s : U \times [0,+\varepsilon) \longrightarrow p^* \mathbf{A}$$

$$(y,t) \longmapsto (\sigma(y), t)$$

With this notation $\alpha(a)$ is defined as the germ of $s|_{U\times(0,+\varepsilon)}$ at $(x,0)$. Now, under the identification above, it results

$$\alpha(a) = \lim_{\substack{V\ni x \\ \delta\to\infty}} s|_{V\times(0,+\delta)} \mapsto \lim_{U\ni x} \sigma|_V = a$$

and consequently it follows that α is an isomorphism. ■

Lemma 3.28. *If $\mathbf{A}^\bullet \to \mathbf{I}^\bullet$ is the canonical injective resolution of a complex of sheaves $\mathbf{A}^\bullet \in D^b(X)$, then the sheaves of the form $\pi^*\mathbf{I}$ are i_*-acyclic. In particular, the resolution $\pi^*\mathbf{A}^\bullet \to \pi^*\mathbf{I}^\bullet$ can be used to compute $Ri_*\pi^*\mathbf{A}^\bullet$.*

Proof. The canonical injective resolution $\mathbf{L}^\bullet \to \mathbf{I}^\bullet$ consists by construction of sheaves of the form

$$\mathbf{I} = \prod_{x\in X} \mathbf{B}_x$$

where $\mathbf{B}_x$ is a sheaf concentrated over x. An argument analogous to that used in the proof of ([Bor], V,3.13), shows that there is an isomorphism

$$\pi^*(\prod_{x\in X} \mathbf{B}_x) \simeq \prod_{x\in X} \pi^*\mathbf{B}_x.$$

Now, if $\pi^*\mathbf{B}_x \to \mathbf{J}_x^\bullet$ is the canonical injective resolution of $\pi^*\mathbf{B}_x$, then it follows that the morphism

$$\pi^*\mathbf{I} \simeq \pi^*(\prod_{x\in X} \mathbf{B}_x) \simeq \prod_{x\in X} \pi^*\mathbf{B}_x \longrightarrow \prod_{x\in X} \mathbf{J}_x^\bullet$$

is an injective resolution of $\pi^*\mathbf{I}$.

What we want to prove is that the sequence

$$i_*(\prod_{x\in X} \pi^*\mathbf{B}_x) \longrightarrow i_*(\prod_{x\in X} \mathbf{J}_x^0) \longrightarrow i_*(\prod_{x\in X} \mathbf{J}_x^1) \longrightarrow \cdots$$

is exact. Since direct images commute with direct products, this is equivalent to show the exactness of the sequence

$$i_*\pi^*\mathbf{B}_x \longrightarrow i_*\mathbf{J}_x^0 \longrightarrow i_*\mathbf{J}_x^1 \longrightarrow \cdots$$

for every $x \in X$. On the other hand, it is known that exactness has to be checked on the stalks. The exactness on the stalk over (y,t) is clearly satisfied for $y \neq x$ or $t \neq 0$, and thus we only have to consider the case $(y,t) = (x,0)$.

For any open neighborhood U in X of the point x, the diagram of vector spaces

$$\begin{array}{ccccc}
\Gamma(U\times[0,+\varepsilon), i_*\pi^*\mathbf{B}_x) & \longrightarrow & \Gamma(U\times[0,+\varepsilon), i_*\mathbf{J}^0_x) & \longrightarrow & \ldots \\
\| & & \| & & \\
\Gamma(U\times(0,+\varepsilon), \pi^*\mathbf{B}_x) & \longrightarrow & \Gamma(X\times(0,\varepsilon), \mathbf{J}^0_x) & \longrightarrow & \ldots
\end{array}$$

is commutative and thus the p-th cohomology space of the upper row is isomorphic to $H^p(X\times(0,\varepsilon),\pi^*\mathbf{B}_x)$. Now, applying the Vietoris-Begle theorem, we get an isomorphism

$$H^p(X\times(0,\varepsilon),\pi^*\mathbf{B}_x)\simeq H^p(X,\mathbf{B}_x).$$

Finally, since $\mathbf{B}_x$ is an acyclic sheaf, it results

$$H^p(X,\mathbf{B}_x)=\begin{cases}\Gamma(X,\mathbf{B}_x) & \text{for } p=0\\ 0 & \text{for } p>0\end{cases}$$

and therefore the sequence above is exact. ■

Let $\mathbf{A}^\bullet\in D^b(X)$ be a complex of sheaves and denote by $\mathbf{A}^\bullet\xrightarrow{\simeq}\mathbf{I}^\bullet$ its canonical injective resolution. According to Lemma 3.28, the resolution $\pi^*\mathbf{A}^\bullet\xrightarrow{\simeq}\pi^*\mathbf{I}^\bullet$ consists of i_*-acyclic sheaves and can be therefore used to compute Ri_*. So, it follows that there is an isomorphism

$$j^*Ri_*\pi^*\mathbf{A}^\bullet\simeq j^*i_*\pi^*\mathbf{I}^\bullet.$$

We have showed in Lemma 3.27 that the natural map $j^*i_*\pi^*\mathbf{I}^\bullet\to\mathbf{I}^\bullet$ is an isomorphism, and consequently we get an isomorphism

$$j^*Ri_*\pi^*\mathbf{A}^\bullet\simeq\mathbf{I}^\bullet\simeq\mathbf{A}^\bullet.$$

Now, it is showed in ([Ba1], Lemma 4.1), that there is an equivalence of functors

$$j^*Ri_*\pi^*(-)\simeq j^!Ri_!\pi^*(-)[1]$$

and thus, for any complex of sheaves $\mathbf{A}^\bullet\in D^b(X)$, we get an isomorphism

$$\mathbf{A}^\bullet\xrightarrow{\ \simeq\ } j^!Ri_!\pi^!\mathbf{A}^\bullet.$$

In particular

Corollary 3.29. *Let X be a collared H-stratifold with boundary $(\partial X,\partial\mathcal{S})$ and denote by i and j the inclusions of $\mathring{X}$ and of ∂X in X. If $\mathcal{S}=(\mathbf{A}^\bullet,\nu)$ is the H-structure on $\mathring{X}$, then there is an isomorphism*

$$(\partial X,\partial\mathcal{S})\simeq(\partial X,\delta\mathcal{S})$$

where $\delta\mathcal{S}=(j^!Ri_!\mathbf{A}^\bullet, j_2^!Ri_{2!}(\nu))$ is the H-structure on the boundary defined in ([Ba1], Section 4.2).

3.5.1 Gluing two H-stratifolds along the boundary

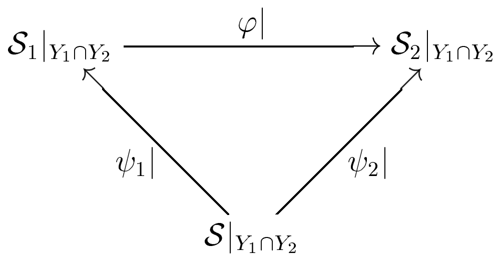
We show here how to glue two H-stratifolds whose boundary is isomorphic. As known from bordism theory, such a construction plays a fundamental role in the proof of the transitivity of bordism.

Let X be an n-dimensional oriented stratifold, Y_1 and Y_2 be two open subsets such that $X = Y_1 \cup Y_2$ and assume Y_1 and Y_2 endowed with the induced orientations.

Lemma 3.30. *Let $\mathcal{S}_1 = (\mathbf{B}_1^\bullet, \nu_1)$ and $\mathcal{S}_2 = (\mathbf{B}_2^\bullet, \nu_2)$ be two H-structures over Y_1 and Y_2 and furthermore suppose that there is an isomorphism of H-stratifolds*

$$\varphi : (Y_1 \cap Y_2, \mathcal{S}_1|_{Y_1 \cap Y_2}) \longrightarrow (Y_1 \cap Y_2, \mathcal{S}_2|_{Y_1 \cap Y_2}).$$

Under these assumptions there is up to isomorphism a unique H-structure $\mathcal{S} = (\mathbf{A}^\bullet, \nu)$ over X together with isomorphisms $\psi_j : \mathcal{S}|_{Y_j} \to \mathcal{S}_j$ which make the diagram



commute.

Proof. An H-structure consists essentially of a self-dual complex of sheaves and we will show how to define such a complex of sheaves by constructing inductively a sequence of complexes $\mathbf{A}_k^\bullet \in \mathrm{SD}(U_k)$ together with isomorphisms

$$(\psi_j)_{U_k} : \mathbf{A}_k^\bullet|_{U_k \cap Y_j} \xrightarrow{\cong} \mathbf{B}_j^\bullet|_{U_k(Y_j)}.$$

Let us write ψ_j instead of $(\psi_j)_{U_k}$ in order to simplify the notation.

For $k = 2$, let us set

$$\mathbf{A}_2^\bullet := \mathbb{R}_{U_2}[n] \in \mathrm{SD}(U_2).$$

The isomorphisms ψ_1 and ψ_2 can be easily defined setting $\psi_j := \nu_j^{-1}$ and the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{B}_1^\bullet|_{U_2(Y_1\cap Y_2)} & \xrightarrow{\ \varphi|\ } & \mathbf{B}_2^\bullet|_{U_2(Y_1\cap Y_2)} \\ & {\scriptstyle \psi_1|}\nwarrow \quad \nearrow{\scriptstyle \psi_2|} & \\ & \mathbf{A}^\bullet|_{Y_1\cap Y_2} & \end{array}$$

is now just a consequence of the definition of isomorphism of H-structures.

Now, assume to have already defined $\mathbf{A}_k^\bullet \in \mathrm{SD}(U_k)$, ψ_1, and ψ_2, and consider the inclusions

$$i : U_k \hookrightarrow U_{k+1}$$
$$i' : U_k(Y_1 \cap Y_2) \hookrightarrow U_{k+1}(Y_1 \cap Y_2)$$
$$j : \Sigma = U_{k+1} - U_k \hookrightarrow U_{k+1}$$

In order to define $\mathbf{A}_{k+1}^\bullet$ we have to distinguish two cases.

• For k even, we set

$$\mathbf{A}_{k+1}^\bullet := \tau_{\leq \overline{m}(k)-n} Ri_* \mathbf{A}_k^\bullet \in \mathrm{SD}(U_{k+1}).$$

The functor $\tau_{\leq \overline{m}(k)-n} Ri_*(-)$ is by ([GM2]) the inverse of $i^* : \mathrm{SD}(U_{k+1}) \to \mathrm{SD}(U_k)$, and consequently $\mathbf{A}_{k+1}^\bullet$ is a self-dual complex of sheaves over U_{k+1}. The isomorphisms $(\psi_j)_{U_{k+1}}$ are defined through the composition

$$\mathbf{A}_{k+1}^\bullet|_{Y_j} = (\tau_{\leq \overline{m}(k)-n} Ri_* \mathbf{B}^\bullet)|_{U_{k+1}(Y_j)} \xrightarrow[\simeq]{\tau_{\leq \overline{m}(k)-n} Ri_*(\psi_j)} \mathbf{B}_j^\bullet|_{U_{k+1}(Y_j)}.$$

• Consider now the case k odd. By Banagl's main result (see Theorem 3.11), there are two natural isomorphisms

$$\mathbf{B}_1^\bullet|_{U_{k+1}(Y_1)} \simeq \mathbf{B}_1^\bullet|_{U_k(Y_1)} \boxplus \Lambda(\mathbf{B}_1^\bullet) \quad \text{and} \quad \mathbf{B}_2^\bullet|_{U_{k+1}(Y_2)} \simeq \mathbf{B}_2^\bullet|_{U_k(Y_2)} \boxplus \Lambda(\mathbf{B}_2^\bullet)$$

where $\Lambda(\mathbf{B}_j^\bullet) = (\mathcal{L}_j, \phi_j)$ is the "canonical" Lagrangian structure over $(\mathbf{B}_j^\bullet)|_{U_k}$. Identifying $\mathbf{B}_j^\bullet|_{U_k}$ with $\mathbf{A}^\bullet|_{U_k(Y_j)}$ through the isomorphism ψ_j, one has

$$(\mathbf{B}_j^\bullet)|_{U_{k+1}} \simeq (\mathbf{A}^\bullet|_{U_k(Y_j)}) \boxplus \Lambda(\mathbf{B}_j^\bullet).$$

On the other hand, restricting φ to $U_{k+1}(Y_1 \cap Y_2)$ one obtains an isomorphism

$$\mathbf{B}_1^\bullet|_{U_k(Y_1\cap Y_2)} \boxplus \Lambda(\mathbf{B}_1^\bullet)|_{U_{k+1}(Y_1\cap Y_2)} \xrightarrow[\simeq]{(i'^*\varphi, \Lambda(\phi))} \mathbf{B}_2^\bullet|_{U_k(Y_1\cap Y_2)} \boxplus \Lambda(\mathbf{B}_2^\bullet)|_{U_{k+1}(Y_1\cap Y_2)}.$$

By the inductive assumption, the diagram

$$\begin{array}{ccc} \mathbf{B}_1^\bullet|_{U_k(Y_1\cap Y_2)} & \xrightarrow{\;i'^*\varphi\;} & \mathbf{B}_2^\bullet|_{U_k(Y_1\cap Y_2)} \\ & {\scriptstyle \psi_1|}\nwarrow \quad \nearrow{\scriptstyle \psi_2|} & \\ & \mathbf{A}_k^\bullet|_{U_k(Y_1\cap Y_2)} & \end{array}$$

commutes, and this allows to identify $\varphi|_{U_{k+1}(Y-1\cap Y_2)}$ with the isomorphism

$$\begin{array}{ccc} \mathcal{L}_1|_{U_{k+1}(Y_1\cap Y_2)} & \xrightarrow{\phi_1} & \mathcal{O}(\mathbf{A}_k^\bullet|_{Y_1\cap Y_2}) \\ {\scriptstyle \alpha}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle 1} \\ \mathcal{L}_2|_{U_{k+1}(Y_1\cap Y_2)} & \xrightarrow[\phi_2]{} & \mathcal{O}(\mathbf{A}_k^\bullet|_{Y_1\cap Y_2}) \end{array}$$

If we set $\mathbf{H} = \mathbf{H}^s j^* \mathcal{O}(\mathbf{A}^\bullet|_k)$ and $\mathbf{E}_j = \mathbf{H}^s j^*(\mathcal{L}_j)|_{U_{k+1}(Y_1\cap Y_2)}$, then we get a diagram

$$\begin{array}{ccc} \mathbf{E}_1|_{Y_1\cap Y_2} & \xrightarrow{\gamma_1} & \mathbf{H}|_{Y_1\cap Y_2} \\ {\scriptstyle \beta}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle 1} \\ \mathbf{E}_2|_{Y_1\cap Y_2} & \xrightarrow[\gamma_2]{} & \mathbf{H}|_{Y_1\cap Y_2} \end{array}$$

Now, we can glue the shaves $\mathbf{E}_1$ and $\mathbf{E}_2$ through β and we obtain thus a sheaf $\mathbf{E}$; the maps γ_1 and γ_2 extend to an injection $\gamma : \mathbf{E} \hookrightarrow \mathbf{H}$. The image $\gamma(\mathbf{E})$ is a Lagrangian subsheaf and, by Lemma 3.10, this determines a Lagrangian structure $(\mathcal{L}, \phi)$ over $\mathbf{A}_k^\bullet$ such that, for $j = 1, 2$, there is an isomorphism $(\mathcal{L}, \phi)|_{U_{k+1}(Y_j)} \simeq \Lambda(\mathbf{B}_j^\bullet)$. The complex $\mathbf{A}_{k+1}^\bullet$ is finally defined by setting

$$\mathbf{A}_{k+1}^\bullet := \mathbf{A}_k^\bullet \boxplus (\mathcal{L}, \phi).$$

The isomorphisms

$$(\psi_j)_{U_{k+1}} : \mathbf{A}_{k+1}^\bullet|_{Y_j} \xrightarrow{\simeq} \mathbf{B}_j^\bullet|_{U_{k+1}(Y_j)}$$

are defined through the compositions

$$\mathbf{A}_{k+1}^\bullet|_{Y_j} \simeq \mathbf{A}^\bullet|_{U_k(Y_j)} \boxplus (\mathcal{L}, \phi)|_{U_{k+1}(Y_j)} \simeq \mathbf{B}_j^\bullet|_{U_k(Y_j)} \boxplus \Lambda(\mathbf{B}_j) \simeq \mathbf{B}_j^\bullet|_{U_{k+1}(Y_j)}.$$

■

The preceding lemma allows to prove the following result.

Proposition 3.31. *Let $(X, \mathcal{S})$ and $(X', \mathcal{S}')$ be two H-stratifolds, and suppose that there is an orientation-reversing isomorphism*

$$\varphi_j : (\partial X, \partial \mathcal{S}) \xrightarrow{\simeq} (\partial X', \partial \mathcal{S}').$$

Then there is up to isomorphism a unique H-structure over $X \underset{\partial X \equiv \partial X'}{\cup} X'$ which restricts to $\mathcal{S}$ over X and to $\mathcal{S}'$ over X'.

Proof. The stratifold $X \cup X'$ is naturally decomposed in the union of the two open sets

$$Y_1 := \overset{\circ}{X} \cup \overset{\circ}{X}'$$

$$Y_2 := \partial X \times (-\varepsilon, +\varepsilon).$$

Both Y_1 and Y_2 are naturally endowed with an H-structure and these two structures are isomorphic if restricted on the intersection. Applying the previous lemma, we can thus obtain an H-structure over $X \cup X'$ which extends $\mathcal{S}$ and $\mathcal{S}'$. ■

3.6 The product of two H-stratifolds

As we have showed in Section 3.3.3, the product of an H-stratifold with an oriented manifold defines a function

$$\mathscr{M} \times \mathscr{C} \longrightarrow \mathscr{C}.$$

The aim of this section is to show that this construction can be extended to a multiplication

$$\mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}.$$

Consider two oriented stratifolds X_1 and X_2 of dimension m and n respectively, and let π_1, π_2 denote the projections of $X_1 \times X_2$ to the first and second factors. For $i = 1, 2$, consider furthermore the map p_i defined by the restriction of π_i to the top stratum $U_2(X_1 \times X_2) = U_2(X_1) \times U_2(X_2)$.

A central role in the definition of the product-structure is played by the tensor product of complexes of sheaves. It is perhaps convenient to recall here that, if $\mathbf{A}^\bullet$ and $\mathbf{B}^\bullet$ are complexes of sheaves of real vector spaces, then, by ([GM2], Section 1.9) (see also Section 2.3 of this thesis), there is an isomorphism

$$\mathbf{A}^\bullet \overset{L}{\otimes} \mathbf{B}^\bullet \simeq \mathbf{A}^\bullet \otimes \mathbf{B}^\bullet.$$

Using ([Bor], Corollary V,10.26), an orientation of $X_1 \times X_2$ is induced by the orientations of X_1 and X_2 through the isomorphism:

$$\mathbb{D}^\bullet_{X_1 \times X_2} \simeq \pi_1^* \mathbb{D}^\bullet_{X_1} \overset{L}{\otimes} \pi_2^* \mathbb{D}^\bullet_{X_2}.$$

In fact, if $\mathfrak{o}_1$ and $\mathfrak{o}_2$ are the orientations of X_1 and X_2 respectively, then $p_1^* \mathfrak{o}_1 \overset{L}{\otimes} p_2^* \mathfrak{o}_2$ induces an isomorphism

$$\begin{aligned}
(\mathbb{D}^\bullet_{X_1 \times X_2})|_{U_2(X_1 \times X_2)} &\simeq p_1^*(\mathbb{D}^\bullet_{X_1}|_{U_2(X_1)}) \overset{L}{\otimes} p_1^*(\mathbb{D}^\bullet_{U_2(X_2)}) \\
&\overset{\simeq}{\to} p_1^*(\mathbb{R}_{U_2(X_1)}[m]) \overset{L}{\otimes} p_2^*(\mathbb{R}_{U_2(X_2)}[n]) \\
&\simeq \mathbb{R}_{U_2(X_1 \times X_2)}[m] \overset{L}{\otimes} \mathbb{R}_{U_2(X_1 \times X_2)}[n] \\
&\simeq \mathbb{R}_{U_2(X_1 \times X_2)}[m+n].
\end{aligned}$$

Now, let $\mathcal{S}_1 = (\mathbf{A}_1^\bullet, \nu_1)$ and $\mathcal{S}_2 = (\mathbf{A}_2^\bullet, \nu_2)$ be two H-structures over X_1 and X_2 respectively.

Lemma/Definition 3.32. *The pair*

$$\mathcal{S}_1 \times \mathcal{S}_2 := (\pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet, p_1^* \nu_1 \overset{L}{\otimes} p_2^* \nu_2)$$

is an H-structure over $X_1 \times X_2$ which is called the ***product H-structure****. The H-stratifold*

$$(X_1 \times X_2, \mathcal{S}_1 \times S_2)$$

is called the ***product*** *of $(X_1, \mathcal{S}_1)$ with $(X_2, \mathcal{S}_2)$.*

Proof. Observe first of all that, for $p = (x_1, x_2) \in X_1 \times X_2$, it holds

$$\begin{aligned}
(\mathbf{H}^i(\pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet))_p &\simeq \mathbf{H}^i((\pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet)_p) \\
&\simeq \mathbf{H}^i((\pi_1^* \mathbf{A}_1^\bullet)_p \overset{L}{\otimes} (\pi_2^* \mathbf{A}_2^\bullet)_p) \\
&\simeq \mathbf{H}^i((\mathbf{A}_1^\bullet)_{x_1} \overset{L}{\otimes} (\mathbf{A}_2^\bullet)_{x_2}) \\
&\simeq \bigoplus_{a+b=j} \mathbf{H}^a((\mathbf{A}_1^\bullet)_{x_1}) \otimes \mathbf{H}^b((\mathbf{A}_2^\bullet)_{x_2})
\end{aligned} \tag{3.1}$$

where the last step is a consequence of the algebraic Künneth formula. According to ([Bor], V,10.25), if $\mathbf{L}_1^\bullet \in D^b(X_1)$ and $\mathbf{L}_2^\bullet \in D^b(X_2)$ are two constructible complexes of sheaves, then there is an isomorphism

$$\pi_1^* \mathcal{D}_X \mathbf{A}^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{B}^\bullet \simeq R\mathbf{Hom}^\bullet(\pi_1^* \mathbf{A}^\bullet, \pi_2^! \mathbf{B}^\bullet).$$

Now, let us show that $\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet \in D^b(X_1 \times X_2)$ is a self-dual complex of sheaves. Axiom **(SD1)** is of course satisfied with the normalization

$$p_1^*\nu_1 \overset{L}{\otimes} p_2^*\nu_2.$$

Axioms **(SD2)** and **(SD3)** can be checked looking at the stalks and using Formula 3.1. Since both $\mathbf{A}_1^\bullet$ and $\mathbf{A}_2^\bullet$ satisfy **(SD2)**, it follows, for $i < -m-n$,

$$\mathbf{H}^i(\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet) = 0,$$

and so $\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet$ satisfies **(SD2)**. In order to show **(SD3)**, let us consider an integer $k \geq 2$ and a point $p = (x_1, x_2) \in U_{k+1}(X_1 \times X_2)$. The structure of stratifold on the product space $X_1 \times X_2$ has the property that for any integer $k \geq 2$ there exists a partition of k of the form $k = \alpha + \beta$, so that $x_1 \in U_\alpha(X_1)$ and $x_2 \in U_\beta(X_2)$. Now, for any $i > \overline{n}(k) - m - n$ and any partition $a + b = i$, it results

$$a + b > \overline{n}(k) - m - n \geq \overline{n}(\alpha) + \overline{n}(\beta) - m - n$$

and so it must also hold

$$a > \overline{n}(\alpha) - m \quad \text{or} \quad b > \overline{n}(\beta) - n.$$

On the other hand, $\mathbf{A}_1^\bullet$ and $\mathbf{A}_2^\bullet$ satisfy **(SD3)**, and consequently it must be

$$(\mathbf{H}^a(\mathbf{A}_1^\bullet))_{x_1} = 0 \quad \text{or} \quad (\mathbf{H}^b(\mathbf{A}_2^\bullet))_{x_2} = 0.$$

Finally, using Formula 3.1, we obtain:

$$(\mathbf{H}^i(\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet))_p \simeq \bigoplus_{a+b=i} (\mathbf{H}^a(\mathbf{A}_1^\bullet))_{x_1} \otimes (\mathbf{H}^b(\mathbf{A}_2^\bullet))_{x_2} = 0.$$

The last to point to prove is the existence of a self-duality isomorphism

$$d : \mathcal{D}(\pi_1^*\mathbf{A}^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet)[m+n] \longrightarrow \pi_1^*\mathbf{A}^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet.$$

Using (among other facts) the identity provided by ([Bor], Theorem V,10.25), we define d as the composition of isomorphisms

$$\begin{aligned}
\mathcal{D}(\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet) &= R\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet, \mathbb{D}^\bullet_{X_1\times X_2}) \\
&\simeq R\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet, \pi_1^*\mathbb{D}^\bullet_{X_1} \overset{L}{\otimes} \pi_2^*\mathbb{D}^\bullet_{X_2}) \\
&\simeq R\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}_1^\bullet, R\mathbf{Hom}^\bullet(\pi_2^*\mathbf{A}_2^\bullet, \pi_1^*\mathbb{D}^\bullet_{X_1} \overset{L}{\otimes} \pi_2^*\mathbb{D}^\bullet_{X_2})) \\
&\simeq R\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}_1^\bullet, R\mathbf{Hom}^\bullet(\pi_2^*\mathbf{A}_2^\bullet, \pi_1^*(\mathcal{D}_{X_1}\mathbb{R}_{X_1}) \overset{L}{\otimes} \pi_2^*\mathbb{D}^\bullet_{X_2})) \\
&\simeq R\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}_1^\bullet, R\mathbf{Hom}^\bullet(\pi_2^*\mathbf{A}_2^\bullet, R\mathbf{Hom}^\bullet(\pi_1^*\mathbb{R}_{X_1}, \pi_2^!\mathbb{D}^\bullet_{X_2}))) \\
&\simeq R\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}_1^\bullet, R\mathbf{Hom}^\bullet(\pi_2^*\mathbf{A}_2^\bullet, R\mathbf{Hom}^\bullet(\mathbb{R}_{X_1\times X_2}, \pi_2^!\mathbb{D}_{X_2}))) \\
&\simeq R\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}_1^\bullet, R\mathbf{Hom}^\bullet(\pi_2^*\mathbf{A}_2^\bullet, \pi_2^!\mathbb{D}^\bullet_{X_2})) \simeq \\
&\simeq R\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}_1^\bullet, \pi_2^! R\mathbf{Hom}^\bullet(\mathbf{A}_2^\bullet, \mathbb{D}^\bullet_{X_2})) \simeq \\
&\simeq R\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}_1^\bullet, \pi_2^!(\mathcal{D}_{X_2}\mathbf{A}_2^\bullet[n]))[-n] \\
&\xrightarrow{\simeq} R\mathbf{Hom}^\bullet(\pi_1^*\mathbf{A}_1^\bullet, \pi_2^!\mathbf{A}_2^\bullet)[-n] \\
&\simeq \pi_1^*(\mathcal{D}_{X_1}(\mathbf{A}_1^\bullet)[m]) \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet[-m-n] \\
&\xrightarrow{\simeq} \pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet[-m-n].
\end{aligned} \tag{3.2}$$

The compatibility of d with the orientation of $X_1 \times X_2$ and with the normalization of $\pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*\mathbf{A}_2^\bullet$ follows from the naturality of the construction. ■

Remark. The product defined above is really an extension of the construction given in Section 3.3.3. In fact, if the second factor X_2 is an oriented n-dimensional manifold (with the trivial sheaf as a self-dual complex of sheaves), then the identification $\pi_1^! \simeq \pi_1^*[n]$ provides an isomorphism

$$\pi_1^!\mathbf{A}_1^\bullet \simeq \pi_1^*\mathbf{A}_1^\bullet[n] \simeq \pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \mathbb{R}_{X_1\times X_2}[n] \simeq \pi_1^*\mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^*(\mathbb{R}_{X_2}[n])$$

and the statement is consequently proved.

The product of two H-stratifolds can be extended to the case that one of the factors is a collared H-stratifold.

Lemma/Definition 3.33. *Let $(X_1, \mathcal{S}_1)$ be an H-stratifold and let $(X_2, \partial X_2)$ be a collared H-stratifold. Then the* ***product*** *$X_1 \times X_2$ is naturally endowed with the structure of a collared H-stratifold whose boundary is isomorphic to $(X_1, \mathcal{S}_1) \times (\partial X_2, \partial \mathcal{S}_2)$.*

Finally, we state without proof two essential properties of the product of H-stratifolds.

Lemma 3.34. *Let $(X_1, \mathcal{S}_1)$, $(X_2, \mathcal{S}_2)$ and $(X_3, \mathcal{S}_3)$ be three H-stratifolds (with possibly non-empty boundary).*

- *The switch map induces an isomorphism*

$$(X_1, \mathcal{S}_1) \times (X_2, \mathcal{S}_2) \xrightarrow{\simeq} (-1)^{\dim X_1 \cdot \dim X_2} (X_2, \mathcal{S}_2) \times (X_1, \mathcal{S}_1).$$

- *There is a canonical isomorphism*

$$((X_1 \times \mathcal{S}_1) \times (X_2, \mathcal{S}_2)) \times (X_3, \mathcal{S}_3) \xrightarrow{\simeq} (X_1, \mathcal{S}_1) \times ((X_2, \mathcal{S}_2) \times (X_3, \mathcal{S}_3)).$$

Chapter 4

Hirzebruch homology

This chapter is devoted to the construction of the Hirzebruch homology functor $Hh_*(-)$ and to the investigation of some of its properties. In particular we show that $Hh_*(-)$ is a multiplicative homology theory and we compute its coefficients.

4.1 The functor $Hh_*(-)$

The functor $Hh_*(-)$ is defined in this section as the bordism theory associated to the class of H-stratifolds. As we have seen in the previous chapter, an H-stratifold is a pair $(X, \mathcal{S})$ where X is a stratifold and $\mathcal{S}$ is an H-structure over X. From now on, it is convenient to simplify the notation, and we do this by indicating an H-stratifold by its underlying stratifold.

Let X be any topological space.

Definition 4.1. *An n-dimensional* ***singular H-stratifold*** *over X is a pair (S, f) where S is an n-dimensional closed H-stratifold and*

$$f : S \longrightarrow X$$

is a continuous map.

For any integer n, we denote by $\mathscr{C}^n(X)$ the class of all n-dimensional singular H-stratifolds over X.

Definition 4.2. *Two elements $(S, f), (S', f') \in \mathscr{C}^n(X)$ are said to be* **isomorphic** *if there is an isomorphism of H-stratifolds*

$$\varphi : S \xrightarrow{\simeq} S'$$

such that, if φ_1 denotes the first component of φ (see Definition 3.17), then the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{\varphi_1} & S' \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle f'} & \\ X & & \end{array}$$

If (S, f) is a singular H-stratifold, then we denote by $-(S, f)$ the singular H-stratifold $(-S, f)$, where $-S$ is the H-stratifold obtained reversing the orientation of S.

The disjoint union of H-stratifolds can be extended to the class of the singular H-stratifolds setting:

$$+ : \mathscr{C}^n(X) \times \mathscr{C}^n(X) \longrightarrow \mathscr{C}^n(X)$$

$$((S, f), (S', f')) \longmapsto (S + S', f + f')$$

This notation allows to introduce the following definition.

Definition 4.3. *Two singular H-stratifolds $(S, f), (S', f') \in \mathscr{C}^n(X)$ are called* ***bordant****, if there exists a pair (T, g) where T is a collared compact H-stratifold and g is a map $T \to X$ so that*

$$(\partial T, g|_{\partial T}) \simeq (S, f) + (-S', f').$$

If (S, f) and (S', f') are two bordant pairs, then we write

$$(S, f) \sim (S', f').$$

Lemma 4.4. *$\sim$ defines an equivalence relation on $\mathscr{C}^n(X)$.*

Proof. All tools used to prove that the bordism relation is an equivalence relation in the classical case of manifolds, are available in the category of H-stratifolds and therefore the usual argument can be applied. ∎

Remark. Observe that isomorphic pairs are automatically bordant.

Since $\sim$ is an equivalence relation, it makes sense to consider the quotient class

$$Hh_n(X) := \{[S, f] \mid (S, f) \in \mathscr{C}^n(X)\}$$

where $[S, f]$ is the equivalence class determined by (S, f). By Lemma 3.24, for any element $(S, f) \in \mathscr{C}^n(X)$ there exists a pair $(S', f') \simeq (S, f)$ with $S' \in \mathscr{C}_0$. In particular the function

$$\{(S, f) \mid S \in \mathscr{C}_0\} \twoheadrightarrow Hh_n(X)$$

is surjective and, since the source class is a set, it results that $Hh_n(X)$ is a set too.

Lemma 4.5. *$Hh_n(X)$ is a commutative group with the operation defined by the disjoint union.*

Proof. The disjoint union defines an operation

$$+ : Hh_n(X) \times Hh_n(X) \longrightarrow Hh_n(X)$$

$$([S, f], [S', f']) \longmapsto [S + S', f + f']$$

which is associative and commutative. The neutral element of the group is the bordism class $[\emptyset, \emptyset]$, while the inverse of a class $[S, f]$ is provided by $-[S, f] := [-S, f]$. ■

Definition 4.6. *The Abelian group $Hh_n(X)$ is called the n-th **Hirzebruch homology** group of X.*

If $g : X \to Y$ is any continuous map, then we can associate to g a group homomorphism

$$g_* : Hh_n(X) \longrightarrow Hh_n(Y)$$

$$[S, f] \longmapsto [S, g \circ f].$$

In particular, this assignment allows to define a functor

$$Hh_*(-) : \mathrm{Top} \longrightarrow \mathrm{Ab}^{\mathbb{Z}}$$

$$X \longmapsto Hh_*(X) := \bigoplus_n Hh_n(X)$$

where $\mathrm{Ab}^{\mathbb{Z}}$ denotes the category of graded Abelian groups.

Furthermore, observe that the functor

$$\mathscr{M} \longrightarrow \mathscr{C}$$

from Section 3.2 defines a natural transformation

$$\Omega_*^{TOP}(-) \longrightarrow Hh_*(-).$$

In the final part of this section we want to discuss the existence of a multiplicative structure on the functor $Hh_*(-)$.

If X and Y are two topological spaces, then the product of H-stratifolds defined in Section 3.6 allows to construct a natural homomorphism

$$\times : Hh_m(X) \otimes Hh_n(Y) \longrightarrow Hh_{m+n}(X \times Y)$$

$$[S, f] \otimes [S', f'] \longmapsto [S \times S', f \times f']$$

The commutativity (in graded sense), the associativity of $\times$ and the distributivity of $\times$ with respect to $+$ are consequences of analogous properties of the product of H-stratifolds (see Lemma 3.34). In particular, $Hh_*(\mathrm{pt})$ is a graded commutative ring with unit. Finally, observe that the multiplicative structure of $Hh_*(-)$ is compatible with the natural transformation $\Omega_*^{TOP}(-) \to Hh_*(-)$ described above.

4.2 Some properties of $Hh_*(-)$

It has been proved by Kreck that the bordism of smooth oriented stratifolds is a homology theory. Using the machinery developed in the previous chapter, we can reproduce Kreck's argument in the category of H-stratifolds. In particular, we show that the functor $Hh_*(-)$ is a homology theory too.

We begin by discussing the homotopy invariance of $Hh_*(-)$. Let X and Y be topological spaces.

Lemma 4.7. *If $g, g' : X \to Y$ are homotopic maps, then it results*

$$g_* = g'_* : Hh_*(X) \longrightarrow Hh_*(Y).$$

The proof is identical to the usual one, and will be therefore omitted.

The next point which we need to consider is the construction of the boundary operator for the Mayer-Vietoris sequence.

Let U and V be open subsets of a space X and set $A := X - V$ and $B := X - U$. For a singular H-stratifold $(S, f) \in \mathscr{C}^n(X)$, the subspaces

$A_S := f^{-1}(A)$ and $B_S := f^{-1}(B)$ are closed and disjoint, and for this reason there is a morphism

$$\rho : S \longrightarrow \mathbb{R}$$

with $\rho(A_S) = +1$ and $\rho(B_S) = -1$.

Applying the transversality theorem (see Theorem 1.31), one gets a homotopy of ρ relative to $A_S \cup B_S$

$$h : S \times [0,1] \longrightarrow \mathbb{R}$$

with the property that 0 is a regular value of $\sigma = h(-,1)$. The subset $T := \sigma^{-1}(0) \subset S$ is an $(n-1)$-dimensional stratifold and there is a bicollar

$$i : Z \times (-\varepsilon, +\varepsilon) \hookrightarrow S$$

such that $\sigma(i(x,t)) = t$. We indicate by $j : Z \to Z \times (-\varepsilon, +\varepsilon)$ the inclusion $x \mapsto (x,0)$ and by $\pi : Z \times (-\varepsilon, +\varepsilon) \to Z$ the projection on the first factor.

Since i is an open embedding, the H-structure of S can be pulled back to an H-structure $i^*\mathcal{S}$ over $Z \times (-\varepsilon, +\varepsilon)$ (see Definitions 3.16 and 3.19) and, by Lemma 3.22, there exists a unique H-structure $\mathcal{T} = j^! i^* \mathcal{S}$ over Z so that it holds

$$\pi^! \mathcal{T} \simeq i^* \mathcal{S}.$$

Finally, if Z denotes the H-stratifold $(Z, \mathcal{T})$, we define

$$d : Hh_n(U \cup V) \longrightarrow Hh_{n-1}(U \cap V)$$

$$[S, f] \longmapsto [Z, f|_Z].$$

Lemma 4.8. *The boundary operator d is well defined.*

Proof. Consider a bordism (T, g) from (S, f) to (S', f') and define $A_T := g^{-1}(A)$ and $B_T := g^{-1}(B)$. The morphisms ρ and ρ' can be extended to a c-function $\eta : T \to \mathbb{R}$ mapping A_T to $+1$ and B_T to -1. Furthermore we can extend h and h' to a homotopy $H : S \times [0,1] \to \mathbb{R}$. By the transversality theorem we get now a homotopy

$$K : T \times [0,1] \longrightarrow \mathbb{R}$$

of $H(-,1)$ (relative to $\partial T \cup A_T \cup B_W$), so that 0 is a regular value of $\xi := K(-,1)$. Finally, a bordism between $(Z, f|_Z)$ and $(Z', f'|_{Z'})$ is provided by $\xi^{-1}(0)$. ∎

The boundary operator commutes by construction with the induced maps. Moreover it holds the

Proposition 4.9 (Mayer-Vietoris sequence). *The sequence of commutative groups*

$$\cdots \to Hh_n(U \cap V) \to Hh_n(U) \oplus Hh_n(V) \to Hh_n(U \cup V) \to Hh_{n-1}(U \cap V) \to \cdots$$

is exact.

The proof of the exactness of the Mayer-Vietoris sequence is completely analogous to that in [Kr1], and will be therefore omitted.

A consequence of the preceding result and of the homotopy invariance of $Hh_*(-)$ is the following

Corollary 4.10. *The functor $Hh_*(-)$ is a multiplicative homology theory and there is a natural transformation of multiplicative homology theories*

$$\Omega_*^{TOP}(-) \longrightarrow Hh_*(-).$$

4.3 The coefficients of $Hh_*(-)$

In this section we show that the signature of an H-stratifold defined by Banagl allows to construct a ring isomorphism $Hh_*(\text{pt}) \simeq \mathbb{Z}[t]$, were the degree of the variable t is equal to 4.

Let S be a compact $(4k)$-dimensional H-stratifold and denote by $\mathcal{S} = (\mathbf{A}^\bullet, \nu)$ the H-structure of S.

The self-duality isomorphism $d : \mathcal{D}\mathbf{A}^\bullet[4k] \xrightarrow{\simeq} \mathbf{A}^\bullet$ induces by Verdier duality an isomorphism in hypercohomology

$$\mathcal{H}^{-2k}(S, \mathbf{A}^\bullet) \simeq \mathcal{H}^{-2k}(S, \mathcal{D}\mathbf{A}^\bullet[4k]) \simeq \mathcal{H}^{2k}(S, \mathcal{D}\mathbf{A}^\bullet) \simeq \mathrm{Hom}(\mathcal{H}^{-2k}(S, \mathbf{A}^\bullet), \mathbb{R})$$

or, equivalently, a non-degenerate symmetric bilinear form

$$\mathcal{H}^{-2k}(S, \mathbf{A}^\bullet) \otimes \mathcal{H}^{-2k}(X, \mathbf{A}^\bullet) \longrightarrow \mathbb{R}.$$

Following Banagl, we call the index of this pairing the **signature** of S. If the dimension of S is not divisible by 4, one sets $\mathrm{sig}(S) = 0$.

The following three properties of the signature can be easily deduced from the definitions.

1. $\mathrm{sig}(S + S') = \mathrm{sig}(S) + \mathrm{sig}(S')$;
2. $\mathrm{sig}(-S) = -\mathrm{sig}(S)$;

3. if S and S' are isomorphic H-stratifolds, it results

$$\mathrm{sig}(S) = \mathrm{sig}(S').$$

Moreover, the signature is multiplicative with respect to the product of H-stratifolds.

Proposition 4.11. *If S_1 and S_2 are two H-stratifolds then it holds*

$$\mathrm{sig}(S_1 \times S_2) = \mathrm{sig}(S_1) \cdot \mathrm{sig}(S_2).$$

Proof. Let $\mathcal{S}_1 = (\mathbf{A}_1^\bullet, \nu_1)$ and $\mathcal{S}_2 = (\mathbf{A}_2^\bullet, \nu_2)$ denote the H-structure on S_1 and S_2 respectively. It follows from ([Bor], Theorem V,10.19) that there is an isomorphism of complexes of real vector spaces

$$\Gamma(S_1; \mathbf{A}_1^\bullet) \overset{L}{\otimes} \Gamma(S_2; \mathbf{A}_2^\bullet) \simeq \Gamma(S_1 \times S_2; \pi_1^* \mathbf{A}_1^\bullet \overset{L}{\otimes} \pi_2^* \mathbf{A}_2^\bullet),$$

and so, by the algebraic Künneth formula, there is an isomorphism

$$\mathcal{H}^k(S_1 \times S_2, \pi_1^* \mathbf{A}_1^\bullet \otimes \pi_2^* \mathbf{A}_2^\bullet) \simeq \bigoplus_{i+j=k} \mathcal{H}^i(S_1, \mathbf{A}_1^\bullet) \otimes \mathcal{H}^j(S_2; \mathbf{A}_2^\bullet).$$

Finally, one can apply the usual argument used to show the multiplicativity of the signature of a manifold. ∎

Another fundamental property of the signature is given by the next proposition.

Proposition 4.12. *If S is a $(4k+1)$-dimensional H-stratifold, then the signature of ∂S is zero.*

Proof. Let $\mathcal{S}$ and $\partial\mathcal{S}$ denote the H-structure of $\mathring{S}$ and ∂S respectively. By Lemma 3.29, there is an isomorphism of H-stratifolds

$$(\partial S, \partial\mathcal{S}) \simeq (\partial S, \delta\mathcal{S})$$

where $\delta\mathcal{S}$ is the H-structure defined in ([Ba1], Section 4.2). In particular, since $\mathrm{sig}(\partial S, \delta\mathcal{S})$ is zero by ([Ba1], Corollary 4.1), it results

$$\mathrm{sig}(\partial S, \partial\mathcal{S}) = \mathrm{sig}(\partial S, \delta\mathcal{S}) = 0.$$

∎

The proposition above has the interesting consequence that the signature can be used to define a homomorphism of graded rings

$$\gamma : Hh_*(\{\mathrm{pt}\}) \longrightarrow \mathbb{Z}[t]$$

$$[S] \longmapsto \mathrm{sig}(S) \cdot t^{\dim S/4}$$

where the degree of the variable t is 4.

Remark. If M is a $4k$-dimensional compact oriented manifold, one has

$$\mathcal{H}^{-2k}(X, \mathbb{R}_M[4k]) = H^{2k}(X, \mathbb{R})$$

and it follows that the signature of M (as an oriented manifold) equals the signature of $(M, \mathbb{R}_M[4k], \mathrm{Id})$ (as an H-stratifold). In particular, if we denote by τ the ring homomorphism

$$\Omega_*^{TOP} \longrightarrow \mathbb{Z}[t]$$

$$[M] \longrightarrow \mathrm{sig}(M) \cdot t^{\dim M/4}$$

where $\mathrm{sig}(M)$ is the signature of M as an oriented manifold, then the diagram

$$\begin{array}{ccc} \Omega_*^{TOP}(\mathrm{pt}) & \longrightarrow & Hh_*(\mathrm{pt}) \\ \tau \big\downarrow & \swarrow \gamma & \\ \mathbb{Z}[t] & & \end{array}$$

commutes.

The ring homomorphism γ plays a fundamental role in the computation of the coefficients of $Hh_*(-)$. In fact we have the

Proposition 4.13. *The ring homomorphism γ is an isomorphism.*

Proof. We begin by discussing the surjectivity of γ. According to the remark above, the signature of $[\mathbb{C}P^{2n}]$ is equal to 1 and therefore it results

$$\gamma([\mathbb{C}P^{2n}]) = t^n$$

for any $n \in \mathbb{N}$.

So, it remains to prove that γ is injective. The strategy is to show that, if S is a compact n-dimensional H-stratifold with $\gamma(S) = 0$, then there exists an H-structure on the cone over S, so that $S \simeq \partial(CS)$.

The condition $\gamma(S) = 0$ is equivalent to $\mathrm{sig}(S) = 0$ and furthermore, since the case $n = 0$ is trivial, we can assume the dimension of S to be strictly positive. Let v be the vertex point of CS, and denote by π, i and j the maps indicated in the diagram:

$$\begin{array}{ccccc} S \times (0,1) & \xrightarrow{\ i\ } & \mathring{C}S & \xleftarrow{\ j\ } & \{v\} \\ \Big\downarrow{\scriptstyle \pi} & & & & \\ S & & & & \end{array}$$

Using the notation introduced in Section 3.1, one has

$$U_{n+2} = S \times (0,1) \quad \text{and} \quad U_{n+3} = \mathring{C}S.$$

We have already seen in Section 3.5 that the problem is to extend the product structure

$$(\pi^! \mathbf{A}^\bullet, p^! \nu)$$

over $S \times (0,1)$ to an H-structure over $\mathring{C}S$. If n is odd, this can always be done applying Theorem 3.7 and so it is enough to consider the case $n = 2m$.

Now, since we have supposed $\mathrm{sig}(S) = 0$, it follows that there exists a Lagrangian subspace

$$L \subset \mathcal{H}^{-m}(S, \mathbf{A}^\bullet)$$

and we have to show how such a Lagrangian subspace gives rise to a Lagrangian structure on $\pi^! \mathbf{A}^\bullet$. Note that a Lagrangian subspace always exists if $4 \nmid n$.

As we have seen in Lemma 3.10, a Lagrangian structure over $\pi^! \mathbf{A}^\bullet$ is the same as a Lagrangian subsheaf

$$\mathbf{H} := \mathbf{H}^{\overline{n}(n+1)-(n+1)}(j^* R i_* \pi^! \mathbf{A}^\bullet).$$

In our case, however, $\mathbf{H}$ is a sheaf over $\{v\}$ and so it can be identified with the vector space $\mathbf{H}_v$. Since we have assumed $n = 2m$, it results

$$\overline{n}(n+1) - (n+1) = [\frac{2m+1-1}{2}] - 2m - 1 = -m - 1.$$

Substituting this expression and the canonical identification $\pi^! \simeq \pi^*[1]$, we can also write

$$\mathbf{H} \simeq \mathbf{H}^{-m-1}(j^* Ri_* \pi^* \mathbf{A}^\bullet[1]) \simeq \mathbf{H}^{-m}(j^* Ri_* \pi^* \mathbf{A}^\bullet).$$

We make now the following assumption, which will be proved in the lemma below: if $\mathbf{A}^\bullet \overset{\simeq}{\to} \mathbf{I}^\bullet$ is the canonical injective resolution of $\mathbf{A}^\bullet$, then the resolution

$$\pi^* \mathbf{A}^\bullet \xrightarrow{\ \simeq\ } \pi^* \mathbf{I}^\bullet$$

can be used to compute Ri_*. Using this fact, the vector space $\mathbf{H}$ can be identified with the stalk at v of the sheaf $\mathbf{H}^{-m}(i_* \pi^* \mathbf{I}^\bullet)$.

On the other hand, the stalk at v of $\mathbf{H}^{-m}(i_* \pi^* \mathbf{I}^\bullet)$ is isomorphic to the $(-m)$-th cohomology of the complex of vector spaces $(i_* \pi^* \mathbf{I}^\bullet)_v$. The latter is by definition equal to

$$\lim_{U \ni v} \Gamma(U, i_* \pi^* \mathbf{I}^\bullet) = \lim_{U \ni v} \Gamma(i^{-1}(U), \pi^* \mathbf{I}^\bullet) \simeq \lim_{\varepsilon \to 0} \Gamma(S \times (0, \varepsilon), \pi^* \mathbf{I}^\bullet),$$

where the last isomorphism follows from the compactness of S.

Since $\pi^* \mathbf{I}^\bullet$ is constant on the fibres, one has

$$\lim_{\varepsilon \to 0} \Gamma(S \times (0, \varepsilon), \pi^* \mathbf{I}^\bullet) \simeq \lim_{\varepsilon \to 0} \Gamma(S, \mathbf{I}^\bullet) \simeq \Gamma(S, \mathbf{I}^\bullet).$$

In particular, this computation shows that we can identify $\mathbf{H}$ with the $(-m)$-th cohomology space of the complex $\Gamma(S, \mathbf{I}^\bullet)$ or, in other words, that there is an isomorphism

$$\mathbf{H} \simeq \mathcal{H}^{-m}(S, \mathbf{A}^\bullet).$$

It follows from the definition of the bilinear form on $\mathbf{H}$ (see [Ba1], Lemma 2.4 and [GM2], Section 5.2), that the diagram

$$\begin{array}{ccc} \mathbf{H} \otimes \mathbf{H} & \longrightarrow & \mathcal{H}^{-m}(S, \mathbf{A}^\bullet) \otimes \mathcal{H}^{-m}(S, \mathbf{A}^\bullet) \\ \downarrow & \swarrow & \\ \mathbb{R} & & \end{array}$$

commutes. This means that the Lagrangian subspace $L \subset \mathcal{H}^{-m}(S, \mathbf{A}^\bullet)$ induces a Lagrangian subspace $\mathbf{L} \subset \mathbf{H}$ and thus a Lagrangian structure on $\pi^! \mathbf{A}^\bullet$. Finally, by Theorem 3.11, there is an H-structure over $\overset{\circ}{C}S$ extending the product structure over $S \times (0,1)$ and finally it results

$$S = \partial(CS).$$

■

Following the argument used to prove Lemma 3.28, one shows the

Lemma 4.14. *Let* $\mathbf{A}^\bullet \to \mathbf{I}^\bullet$ *be the canonical injective resolution of* $\mathbf{A}^\bullet \in D^b(S)$*. The sheaves of the form* $\pi^*\mathbf{I}$ *are* i_**-acyclic: in particular, the resolution* $\pi^*\mathbf{A}^\bullet \to \pi^*\mathbf{I}^\bullet$ *can be used to compute* $Ri_*\pi^*\mathbf{A}^\bullet$*.*

Proof. The canonical injective resolution $\mathbf{A}^\bullet \to \mathbf{I}^\bullet$ consists by construction of sheaves of the form

$$\mathbf{I} = \prod_{x\in S} \mathbf{B}_x.$$

As in the proof of ([Bor], V,3.13), one has

$$\pi^*(\prod_{x\in S} \mathbf{B}_x) \simeq \prod_{x\in S} \pi^*\mathbf{B}_x$$

and, if $\pi^*\mathbf{B}_x \to \mathbf{J}_x^\bullet$ is the canonical injective resolution of $\pi^*\mathbf{B}_x$, then it follows that

$$\pi^*\mathbf{I} \simeq \pi^*(\prod_{x\in S} \mathbf{B}_x) \simeq \prod_{x\in S} \pi^*\mathbf{B}_x \longrightarrow \prod_{x\in S} \mathbf{J}_x^\bullet$$

is an injective resolution of $\pi^*\mathbf{I}$.

Now, what we have to prove is that the sequence

$$i_*(\prod_{x\in S} \pi^*\mathbf{B}_x) \longrightarrow i_*(\prod_{x\in S} \mathbf{J}_x^0) \longrightarrow i_*(\prod_{x\in S} \mathbf{J}_x^1) \longrightarrow \cdots$$

is also exact. Since direct image commute with direct products, this is equivalent to show the exactness of

$$i_*\pi^*\mathbf{B}_x \longrightarrow i_*\mathbf{J}_x^0 \longrightarrow i_*\mathbf{J}_x^1 \longrightarrow \cdots$$

for every $x \in CS$. On the other hand, exactness has to be checked on the stalks and, since exactness at every point $p \neq v$ is clearly satisfied by construction, it remains to consider the stalk at v. This is equivalent to show that the morphism of complexes of vector spaces

$$\prod_{x\in S} (i_*\pi^*\mathbf{B}_x)_v \longrightarrow \prod_{x\in S} (i_*\mathbf{J}_x^\bullet)_v$$

is a quasi-isomorphism. By a compactness argument, we can consider a conelike open neighborhood U_ε of v. On U_ε one has

$$\begin{array}{ccccc}
\Gamma(U_\varepsilon, i_*\pi^*\mathbf{B}_x) & \longrightarrow & \Gamma(U_\varepsilon, i_*\mathbf{J}_x^0) & \longrightarrow & \ldots \\
\| & & \| & & \\
\Gamma(X\times(0,\varepsilon), \pi^*\mathbf{B}_x) & \longrightarrow & \Gamma(X\times(0,\varepsilon), \mathbf{J}_x^0) & \longrightarrow & \ldots
\end{array}$$

The p-th cohomology space of the bottom row is $H^p(X \times (0,\varepsilon), \pi^* \mathbf{B}_x)$ and, applying the Vietoris-Begle theorem, one has

$$H^p(X \times (0,\varepsilon), \pi^* \mathbf{B}_x) \simeq H^p(X, \mathbf{B}_x).$$

Finally since $\mathbf{B}_x$ is an acyclic sheaf, it results

$$H^p(X, \mathbf{B}_x) = \begin{cases} \Gamma(X, \mathbf{B}_x) & \text{for } p = 0 \\ 0 & \text{for } p > 0 \end{cases}$$

■

Chapter 5

The Hirzebruch fundamental class

In the following pages we show how to use Hirzebruch homology to construct a characteristic class for smooth oriented manifolds, and we explain the connection between the Novikov conjecture and Hirzebruch homology.

5.1 The Hirzebruch fundamental class of a manifold

Let u be the natural transformation defined through the composition

$$\Omega_*^{SO}(-) \longrightarrow \Omega_*^{TOP}(-) \longrightarrow Hh_*(-),$$

and denote by u_n^M the homomorphism

$$\Omega_n^{SO}(M) \longrightarrow Hh_n(M)$$

induced by u for an n-dimensional smooth oriented closed manifold M.

Definition 5.1. *The* ***Hirzebruch fundamental class*** *of M is by definition the element*

$$[M] := u_n^M([M, \mathrm{Id}]) \in Hh_n(M)).$$

Despite the use of the same name and of the same notation, the class $[M] \in Hh_n(M)$ defined here does not coincide with the class $[M] \in hh_n(M)$ defined in the introduction. The use of the same terminology is justified by the fact that, as we will see in Appendix B, there is an isomorphism

$$\varphi : hh_n(M) \xrightarrow{\simeq} Hh_n(M)$$

which carries $[M] \in hh_n(M)$ onto $[M] \in Hh_n(M)$.

The first property of the Hirzebruch fundamental class of a manifold is its topological invariance.

Lemma 5.2. *If $f : N \to M$ is an orientation-preserving homeomorphism, then it results*

$$[M] = f_*([N]).$$

The proof of this fact is straightforward since it holds

$$[M, \mathrm{id}] = [N, f] \in \Omega_n^{TOP}(M)$$

and u_n^M factorizes by construction through $\Omega_n^{TOP}(M)$.

5.2 Rational Hirzebruch homology

In this section we will interpret the information carried by the rational Hirzebruch fundamental class: in particular we will show that for any closed smooth manifold M there is an identification

$$Hh_n(M) \otimes \mathbb{Q} \xrightarrow{\simeq} H_n(M; \mathbb{Q}[t])$$

which maps the Hirzebruch fundamental class to the L-genus.

The ring $Hh_*(\mathrm{pt})$ is an Ω_*^{SO}-module with the multiplication induced by the ring homomorphism

$$u_* : \Omega_*^{SO} \longrightarrow Hh_*(\mathrm{pt}).$$

For this reason, if A is any Ω_*^{SO}-module, then we denote by $A \otimes_{u_*} Hh_*(\mathrm{pt})$ the tensor product in the category of Ω_*^{SO}-modules. For simplicity we also denote by u_* the homomorphism

$$\Omega_*^{SO} \otimes \mathbb{Q} \longrightarrow Hh_*(\mathrm{pt}) \otimes \mathbb{Q}$$

and by $(A \otimes \mathbb{Q}) \otimes_{u_*} (Hh_*(\mathrm{pt}) \otimes \mathbb{Q})$ the tensor product in the category of $(\Omega_*^{SO} \otimes \mathbb{Q})$-modules.

Lemma 5.3. *The functor $(\Omega_*^{SO}(-) \otimes \mathbb{Q}) \otimes_{u_*} (Hh_*(\mathrm{pt}) \otimes \mathbb{Q})$ is a homology theory.*

Proof. For a manifold M, let Δ denote the Poincaré duality isomorphism

$$H^k(M) \longrightarrow H_{n-k}(M)$$

$$x \longmapsto x \cap [M]$$

and let us define a natural transformation setting:

$$\Omega_n^{SO}(X) \otimes \mathbb{Q} \longrightarrow H_n(X) \otimes (\Omega_*^{SO} \otimes \mathbb{Q})$$

$$[M,f] \longmapsto f_*\Big(\sum_{\substack{k \\ i_1,\dots,i_k}} \Delta(p_{i_1}(M)\dots p_{i_k}(M)) \otimes [\mathbb{CP}^{i_1}]\dots[\mathbb{CP}^{i_k}]\Big)$$

According to Thom, this transformation is an isomorphism of homology theories. On the other hand the equality $(A \otimes B) \otimes_B C \simeq A \otimes C$ implies

$$(H_*(-) \otimes \Omega_*^{SO} \otimes \mathbb{Q}) \otimes_{u_*} (Hh_*(\mathrm{pt}) \otimes \mathbb{Q}) \simeq H_*(-) \otimes (Hh_*(\mathrm{pt}) \otimes \mathbb{Q}),$$

and thus we get a natural isomorphism of functors

$$(\Omega_*^{SO}(-) \otimes \mathbb{Q}) \otimes_{u_*} (Hh_*(\mathrm{pt}) \otimes \mathbb{Q}) \simeq H_*(-) \otimes Hh_*(\mathrm{pt}) \otimes \mathbb{Q}.$$

So, in order to prove the statement, it is enough to show that the latter functor is a homology theory, but this is clear since $Hh_*(\mathrm{pt}) \otimes \mathbb{Q}$ is flat. ∎

The product of a singular manifold with an H-stratifold induces a family of natural homomorphisms

$$\Omega_*^{SO}(X) \otimes_{u_*} Hh_*(\mathrm{pt}) \longrightarrow Hh_*(X)$$

$$[M,f] \otimes_{u_*} [S] \longmapsto [M \times S, f \circ \pi_1]$$

and therewith a transformation

$$(\Omega_*^{SO}(-) \otimes \mathbb{Q}) \otimes_{u_*} (Hh_*(\mathrm{pt}) \otimes \mathbb{Q}) \longrightarrow Hh_*(-) \otimes \mathbb{Q}.$$

Since both sides are homology theories, by the comparison theorem we get the following

Lemma 5.4. *The transformation*

$$(\Omega_*^{SO}(-) \otimes \mathbb{Q}) \otimes_{u_*} (Hh_*(\mathrm{pt}) \otimes \mathbb{Q}) \longrightarrow Hh_*(-) \otimes \mathbb{Q}$$

is an isomorphism.

Now, the ring homomorphism

$$\tau : \Omega_*^{SO} \longrightarrow \mathbb{Z}[t]$$

induces an Ω_*^{SO}-module structure on $\mathbb{Z}[t]$. Moreover, the commutative diagram

$$\begin{array}{ccc} \Omega_*^{SO} & \xrightarrow{u_*} & Hh_*(\mathrm{pt}) \\ {\scriptstyle\tau}\downarrow & \swarrow {\scriptstyle \gamma,\ \simeq} & \\ \mathbb{Z}[t] & & \end{array}$$

induces a natural isomorphism

$$\Omega_*^{SO}(X) \otimes_{u_*} Hh_*(\mathrm{pt}) \xrightarrow{\simeq} \Omega_n^{SO}(X) \otimes_\tau \mathbb{Z}[t]$$

$$[M, f] \otimes_{u_*} [S] \longmapsto [M, f] \otimes_\tau \gamma([S])$$

and consequently an equivalence of homology theories:

$$(\Omega_*^{SO}(-) \otimes \mathbb{Q}) \otimes_{u_*} (Hh_*(\mathrm{pt}) \otimes \mathbb{Q}) \xrightarrow{\simeq} \Omega_*^{SO}(-) \otimes_\tau \mathbb{Q}[t].$$

Now we are going to show that the latter functor is isomorphic to singular homology.

Lemma 5.5. *The L-genus defines a natural transformation*

$$\lambda : \Omega_*^{SO}(-) \longrightarrow H_*(; \mathbb{Q}[t])$$

which, for $X = \mathrm{pt}$, realizes the ring homomorphism τ.

Proof. Let

$$L_1(p_1), L_2(p_1, p_2), \ldots, L_k(p_1, \ldots, p_k), \ldots$$

be the multiplicative sequence of rational polynomials defined by the even power series $Q(x) = \frac{x}{\tanh x}$ (see [Hi] or [HBJ]). Substituting the Pontrjagin classes of a manifold M in these polynomials, one gets the L-class

$$L(M) = \sum_{k \geq 0} L_k(M) \cdot t^k \in H^0(M; \mathbb{Q}[t])$$

and, by Poincaré duality, the class $\Delta L(M) \in H_n(M;\mathbb{Q}[t])$. For any space X and any integer n, we define natural homomorphisms setting:

$$\lambda_n^X : \Omega_n^{SO}(X) \longrightarrow H_n(X;\mathbb{Q}[t])$$

$$[M,f] \longmapsto f_*(\Delta L(TM))$$

For $X = \mathrm{pt}$, this is just the homomorphism

$$\lambda_* : \Omega_n^{SO} \longrightarrow \mathbb{Q}[t]$$

$$[M] \longmapsto \varepsilon_*(L(M) \cap [M])$$

where ε denotes the constant map $M \to \{\mathrm{pt}\}$, and by Hirzebruch's signature theorem it holds

$$\varepsilon_*(L(M) \cap [M]) = < L(M), [M] > = \mathrm{sig}(M) \cdot t^{n/4} = \tau(M)$$

■

This construction can be used to prove the

Lemma 5.6. *There is an isomorphism of homology theories*

$$\Omega_*^{SO}(-) \otimes_\tau \mathbb{Q}[t] \xrightarrow{\simeq} H_*(-;\mathbb{Q}[t]).$$

Proof. The transformation λ from Lemma 5.5 induces by multiplication an homomorphism

$$\Omega_*^{SO}(-) \times \mathbb{Q}[t] \longrightarrow H_*(-;\mathbb{Q}[t]).$$

Because of the universal property of the equalizer, there exists a unique transformation

$$\Omega_*^{SO}(-) \otimes_\tau \mathbb{Q}[t] \longrightarrow H_*(-;\mathbb{Q}[t])$$

such that the diagram

$$\begin{array}{ccccc} \Omega_*^{SO}(-) \times \Omega_*^{SO} \times \mathbb{Q}[t] & \rightrightarrows & \Omega_*^{SO}(-) \times \mathbb{Q}[t] & \longrightarrow & \Omega_*^{SO}(-) \otimes_\tau \mathbb{Q}[t] \\ & & \downarrow & \swarrow & \\ & & H_*(-;\mathbb{Q}[t]) & & \end{array}$$

commutes. Finally, it is easily seen that this new transformation is an isomorphism by the comparison theorem. ■

Collecting these results together we get the

Proposition 5.7. *There exists an equivalence*

$$\varphi : Hh_*(-) \otimes \mathbb{Q} \xrightarrow{\simeq} H_*(-; \mathbb{Q}[t])$$

such that the diagram

$$\begin{array}{ccc} \Omega_*^{SO}(-) & \xrightarrow{u} & Hh_*(-) \otimes \mathbb{Q} \\ \lambda \Big\downarrow & \swarrow \varphi & \\ H_*(-; \mathbb{Q}[t]) & & \end{array}$$

commutes.

Proof. If X is a space, then the homomorphism φ_*^X is defined through the composition

$$\begin{array}{ccc} \Omega_*^{SO}(X) \otimes_{u_*} Hh_*(pt) \otimes \mathbb{Q} & \xleftarrow{\simeq} & Hh_*(X) \otimes \mathbb{Q} \\ \Big\downarrow \simeq & & \Big\downarrow \\ \Omega_*^{SO}(X) \otimes_\tau \mathbb{Q}[t] & \xrightarrow[\simeq]{} & H_*(X; \mathbb{Q}[t]) \end{array}$$

■

In particular for a smooth oriented n-dimensional manifold M one has

Corollary 5.8. *If M is an n-dimensional smooth oriented manifold M, then it holds*

$$\varphi_n^M([M]) = \Delta L(M) \in H_m(M; \mathbb{Q}[t]).$$

Since the rational Pontrjagin classes of a manifold M determine and at the same time are determined by the L-class of M, one can also interpret this result as follows.

Meta-Theorem. *The rational Hirzebruch fundamental class of a manifold contains the same information as the rational Pontrjagin classes.*

According to a theorem of Dold and Milnor, the rational Pontrjagin classes of a manifold are not homotopy invariant and so we get in particular

Corollary 5.9. *The Hirzebruch fundamental class is not homotopy invariant.*

5.3 The Novikov conjecture

In this section we want to show that the Novikov conjecture for a group π is equivalent to the homotopy invariance of the rational Hirzebruch fundamental class for singular manifolds over $K(\pi,1)$.

Let π be any discrete group, and let us fix any rational cohomology class $x \in H^*(K(\pi,1);\mathbb{Q})$.

Definition 5.10. *The* ***higher signature*** sig_x *of a singular manifold* (M,α) *over* $K(\pi,1)$ *is the characteristic number*

$$\mathrm{sig}_x(M,\alpha) =< L(M) \cup \alpha^* x, [M] >=< x, \alpha_*(\Delta L(M)) > .$$

The higher signature sig_x is by definition a bordism invariant and thus it defines a group homomorphism

$$\Omega_*^{SO}(K(\pi,1)) \longrightarrow \mathbb{Q}$$

$$[M,\alpha] \longmapsto \mathrm{sig}_x(M,\alpha).$$

Definition 5.11. *The higher signature* sig_x *is said to be* ***homotopy invariant*** *if for singular manifold* (M,α) *and for every orientation-preserving homotopy equivalence* $f : N \to M$*, it holds*

$$\mathrm{sig}_x(M,\alpha) = \mathrm{sig}_x(N,\alpha \circ f)$$

Now we want to connect the Novikov conjecture with the homology theory $Hh_*(-)$.

Definition 5.12. *The rational Hirzebruch fundamental class is said to be* **homotopy invariant for the group** π *if for every pair* (M,α)*, and for every orientation-preserving homotopy equivalence* $f : N \to M$*, it results*

$$[M,\alpha] = [N,\alpha \circ f] \in Hh_n(K(\pi,1)) \otimes \mathbb{Q}.$$

This terminology allows to formulate the following

Proposition 5.13. *The Novikov conjecture for a group* π *is equivalent to the homotopy invariance of the rational Hirzebruch fundamental class for* π*.*

Proof. If (M,α) is a singular manifold (M,α) over $K(\pi,1)$, then one has

$$\mathrm{sig}_x(M,\alpha) =< x, \alpha_*(\Delta(L(M)) >=< x, \varphi_n([M,\alpha]) >$$

and thus it is clear that it results

$$\mathrm{sig}_x(M,\alpha) = \mathrm{sig}_x(N,\alpha\circ f)$$

for any $x \in H^*(K(\pi,1);\mathbb{Q})$, if and only if

$$[M,\alpha] = [N,\alpha\circ f] \in Hh_n(K(\pi,1))\otimes\mathbb{Q}.$$

■

The proposition above suggests that an integral version of the Novikov conjecture can be obtained requiring the homotopy invariance of the (integral) Hirzebruch fundamental class.

Integral Novikov Problem (M. Kreck). *Determine all discrete groups π for which the Hirzebruch fundamental class is homotopy invariant.*

5.4 $\mathbb{Z}[1/2]$-localized Hirzebruch homology

Applying the Landweber exact functor theorem, we show in this section that the $\mathbb{Z}[1/2]$-localization of Hirzebruch homology is isomorphic to connective KO-theory. In particular the Hirzebruch fundamental class of a manifold M differs from the ko-theoretical characteristic classes of M only by elements of 2-torsion.

By the Landweber exact functor theorem (see [La], Example 3.4), the tensor product

$$\Omega_*^{SO}(-)\otimes_\tau \mathbb{Z}[1/2][t,t^{-1}]$$

is a homology theory and therefore u induces an isomorphism

$$\Omega_*^{SO}(-)\otimes_\tau \mathbb{Z}[1/2][t,t^{-1}] \xrightarrow{\simeq} Hh_*(-)\otimes\mathbb{Z}[1/2][t^{-1}].$$

On the other hand, the map of spectra

$$MSpin \longrightarrow MSO$$

is a $\mathbb{Z}[1/2]$-equivalence and so one can define a map $v: MSO \to KO[1/2]$ through the composition

$$MSO[1/2] \longrightarrow MSpin[1/2] \longrightarrow KO[1/2].$$

Here the last map is induced by the Atiyah-Bott-Shapiro $MSpin$-orientation of KO-theory. The map v defines a natural transformation

$$\Omega_*^{SO}(-) \longrightarrow KO_*(-)[1/2]$$

which induces, for $X = \{\mathrm{pt}\}$, the ring homomorphism τ by a theorem of Sullivan (see [MM], page 84). In particular, it results

$$\Omega_*^{SO}(-) \otimes_\tau \mathbb{Z}[1/2][t, t^{-1}] \simeq \Omega_*^{SO}(-) \otimes_{v_*} (KO_*(\mathrm{pt})[1/2]).$$

and so, applying again the Landweber exact functor theorem, we get an isomorphism

$$\Omega_*^{SO}(-) \otimes_{v_*} KO_*(\mathrm{pt})[1/2] \xrightarrow{\simeq} KO_*(-)[1/2].$$

The diagram

$$\begin{array}{ccc} \Omega_*^{SO}(-) \otimes_\tau \mathbb{Z}[1/2][t, t^{-1}] & \xrightarrow{\simeq} & Hh_*(-) \otimes \mathbb{Z}[1/2][t^{-1}] \\ \simeq \downarrow & \swarrow \simeq & \\ KO_*(-)[1/2] & & \end{array}$$

provides an isomorphism

$$Hh_*(-) \otimes \mathbb{Z}[1/2][t^{-1}] \simeq KO_*(X)[1/2]$$

and passing to the connected coverings one concludes the proof of the following

Proposition 5.14. *There is an isomorphism*

$$Hh_*(-) \otimes \mathbb{Z}[1/2] \xrightarrow{\simeq} ko_*(-)[1/2].$$

Appendix A

Hirzebruch cohomology

Following Quillen and Dold (see [Quil] and [Do]), we construct the cohomology theory associated to $Hh_*(-)$ on the category of smooth oriented manifolds and compute its formal group law.

A.1 The functor $Hh^*(-)$

In this section we define the functor $Hh^*(-)$ as the cobordism theory associated to the class of smooth H-stratifolds. The notion of a smooth stratifold is very similar to that of a topological stratifold and, for this reason, the reader is referred to Kreck's book (see [Kr1]) for a precise definition and for the proofs of the main results mentioned here.

Let us denote by $\mathscr{S}$ the category of smooth H-stratifolds and by $Hh'_*(-)$ the associated bordism theory. The arguments used in Chapter 4 to show that $Hh_*(-)$ is a homology theory and to compute the coefficients of $Hh_*(-)$ can be applied with some modifications to $Hh_*(-)$ and so one gets the

Proposition A.15. *The functor $Hh'_*(-)$ is a homology theory and there is a ring isomorphism*

$$\gamma' : Hh'_*(\mathrm{pt}) \xrightarrow{\simeq} \mathbb{Z}[t]$$

so that the diagram

$$\begin{array}{ccc} \Omega_*^{SO} & \longrightarrow & Hh'_*(\mathrm{pt}) \\ \downarrow \tau & \swarrow \gamma' & \\ \mathbb{Z}[t] & & \end{array}$$

commutes.

In particular, it follows from the comparison theorem that the natural transformation

$$Hh'_*(-) \longrightarrow Hh_*(-).$$

induced by the forgetful functor is an equivalence of homology theories.

By a standard argument, if $f : S \to M$ is a continuous map from a smooth stratifold to a smooth manifold, whose restriction to ∂S is a smooth map, then f is homotopic relative ∂S to a smooth map.

A consequence of this fact is that for each smooth manifold M we can consider $Hh'_*(M)$ as the bordism of smooth stratifolds and smooth maps.

Now, let us fix a smooth oriented n-dimensional manifold M without boundary.

Definition A.16. *The p-th **Hirzebruch cohomology group** of M is by definition the group of the bordism classes of pairs (S, f) where S is a smooth $(n-p)$-dimensional H-stratifold and $f : S \to M$ is a proper smooth map.*

If M is compact, then there is an isomorphism

$$Hh_p(M) \simeq Hh'_p(M) = Hh^{n-p}(M)$$

which is called Poincaré duality isomorphism.

The transversality theorem allows to turn $Hh^*(-)$ into a contravariant functor, and by a standard argument one can prove the

Proposition A.17. *The functor $Hh^*(-)$ is a multiplicative cohomology theory whose coefficients are isomorphic to $\mathbb{Z}[t]$ (here the degree of t is equal to -4).*

Actually it should be possible to describe $Hh^*(-)$ already on the category of topological manifolds (the problem here is to prove the needed formulation of the transversality theorem). However, this fact is not relevant for the following and so we will not go into details.

A.2 The formal group law of $Hh^*(-)$

Let $E^*(-)$ be a commutative multiplicative cohomology theory on the category of finite dimensional smooth manifolds.

Definition A.18. *A sequence $t = \{t_n\}$ of elements $t_n \in E^2(\mathbb{C}P^n)$ is called a $\mathbb{C}$-**orientation** of E^*, if the following two properties are satisfied.*

- *The sequence t is natural with respect to the inclusions $j_n : \mathbb{C}P^n \subset \mathbb{C}P^{n+1}$.*

- *t_1 reduces to the canonical generator of $\widetilde{E}^2(\mathbb{C}P^1) = \widetilde{E}^2(S^2)$.*

The **Euler class** of a line bundle ξ over a manifold M is defined as the element

$$e(\xi) := f^*(t_n)$$

where the map $f : M \to \mathbb{C}P^n$ classifies ξ.

The cohomology theory $\Omega^*_U(-)$ has a canonical $\mathbb{C}$-orientation which is given by the elements $[\mathbb{C}P^{n-1}, j_{n-1}] \in \Omega^2_U(\mathbb{C}P^n)$. Moreover, using the transformation

$$\Omega^*_U(-) \longrightarrow Hh^*(-)$$

induced by the forgetful functor, we can define a natural orientation of $Hh^*(-)$ setting:

$$u_n := [\mathbb{C}P^{n-1}, j_{n-1}] \in Hh^2(\mathbb{C}P^n).$$

If $E^*(-)$ is a complex oriented cohomology theory, then, using the Atiyah-Hirzebruch spectral sequence, one can prove that there exists a unique formal group law

$$F(x,y) = \sum_{i,j} a_{ij} x^i y^j$$

with coefficients in the ring $E^* = E^*(\mathrm{pt})$, so that for any two line bundles ξ_1 and ξ_2 over a manifold M it holds

$$e(\xi_1 \otimes \xi_2) = F(e(\xi_1), e(\xi_2)).$$

Recall that a power series $F \in R[[x,y]]$ is called a **commutative formal group law**, if it satisfies the relations

$$\begin{aligned} F(0,y) &= y, \\ F(x,0) &= x, \\ F(x,F(y,z)) &= F(F(x,y),z), \\ F(x,y) &= F(y,x). \end{aligned}$$

According to Quillen (see [Quil]), the formal group law of $\Omega^*_U(-)$ is the series

$$F(x,y) = \frac{\sum_{i,j} h_{ij} x^i y^j}{CP(x)CP(y)}.$$

where h_{ij} denotes the bordism class of the Milnor manifold $H_{i,j}$, and $CP(u)$ is the power series

$$CP(u) = \sum_{n=0}^{\infty} [\mathbb{C}P^n] u^n \in \Omega_U^*(pt)[[u]].$$

Since the $\mathbb{C}$-orientation of Hh^* is induced by the natural transformation $u : \Omega_U^*(-) \to Hh^*(-)$, the formal group law of Hh^* is given by

$$G(x,y) = u_*(F(x,y)) = \frac{\sum_{i,j} u_*(h_{ij}) x^i y^j}{u_*(CP(x)) u_*(CP(y))}.$$

In order to compute $G(x,y)$ explicitly, we can use the isomorphism

$$\gamma : Hh^*(\mathrm{pt}) \xrightarrow{\simeq} \mathbb{Z}[t].$$

The ring homomorphism $\gamma \circ u_*$ coincides with the genus τ, and so what we need to know is the signature of the coefficients of $F(x,y)$. If we set

$$\frac{1}{1-a} := \sum_{n=0}^{\infty} a^n,$$

then the power series $\tau_*(CP(u))$ can be expressed by the formula

$$\tau_*(CP(u)) = \sum_{n=0}^{\infty} \tau([\mathbb{C}P^n]) u^n = \sum_{n=0}^{\infty} t^n u^{2n} = \sum_{n=0}^{\infty} (tu^2)^n = \frac{1}{1-tu^2},$$

and consequently we can write

$$G(x,y) = (\sum_{i,j} \tau(h_{ij}) x^i y^j)(1-tx^2)(1-ty^2).$$

Now, according to Hirzebruch (see [HBJ]), for $i \neq j$ it holds

$$\tau(h_{ij}) = \begin{cases} 0 & \text{for } i \equiv 1 \ (2) \\ 0 & \text{for } i, j \equiv 0 \ (2) \\ t^{\frac{i+j-1}{2}} & \text{if } i \equiv 0, j \equiv 1 \ (2) \end{cases}$$

and, since it results $\tau(h_{ii}) = 0$, one has

$$G(x,y) = (\sum_{i \neq j} \tau(h_{ij}) x^i y^j)(1-tx^2)(1-ty^2).$$

Finally, using the equality $h_{ij} = h_{ji}$, we obtain

$$\begin{aligned}
\sum_{i\neq j}\tau(h_{ij})x^iy^j &= \sum_{i<j}\tau(h_{ij})x^iy^j + \sum_{i<j}\tau(h_{ij})x^jy^i = \\
&= \sum_{k\leq l} t^{\frac{2k+2l+1-1}{2}}x^{2l+1}y^{2k} + \sum_{k\leq l} t^{\frac{2k+2l+1-1}{2}}x^{2k}y^{2l+1} = \\
&= \sum_{k\leq l} t^{k+l}x^{2l+1}y^{2k} + \sum_{k\leq l} t^{k+l}x^{2k}y^{2l+1} = \\
&= x\cdot(\sum_{k\leq l} t^{k+l}x^{2l}y^{2k}) + y\cdot(\sum_{k\leq l} t^{k+l}x^{2k}y^{2l}) = \\
&= x\cdot(\sum_{k} t^{k}y^{2k}(\sum_{l\geq k} t^{l}x^{2l})) + y\cdot(\sum_{k} t^{k+l}x^{2k}(\sum_{l\geq k} t^{l}y^{2l})) = \\
&= x\cdot(\sum_{k} t^{k}y^{2k}\frac{t^kx^2k}{1-tx^2}) + y\cdot(\sum_{k} t^{k}x^{2k}\frac{t^ky^{2k}}{1-ty^2}) = \\
&= \frac{x\cdot(\sum_k t^{2k}x^{2k}y^{2k})}{1-tx^2} + \frac{y\cdot(\sum_k t^{2k}x^{2k}y^{2k})}{1-ty^2} = \\
&= \frac{x}{(1-t^2x^2y^2)(1-tx^2)} + \frac{y}{(1-t^2x^2y^2)(1-ty^2)} = \\
&= \frac{x+y}{(1+txy)(1-tx^2)(1-ty^2)}.
\end{aligned}$$

This concludes the proof of the

Proposition A.19. *The formal group law of $Hh^*(-)$ is*

$$G(x,y) = \frac{x+y}{1+txy}.$$

Appendix B

The homology theory $hh_*(-)$

Appendix B is devoted to the construction of the homology theory $hh_*(-)$ as complex bordism with singularities in the sense of Baas-Sullivan.

B.1 The Baas-Sullivan construction

We begin by summarizing some results from the theory of Baas-Sullivan (unitary) manifolds with singularities. For a more detailed account the reader is referred to Rudyak ([Ru], Chapter VIII) or Botvinnik ([Bot]).

Let us fix a closed k-dimensional unitary manifold F.

Definition B.1. *A* **Baas-Sullivan manifold with singularity** F *is a triple* (R, S, f) *where* R *and* S *are compact unitary manifolds, and* f *is a diffeomorphism* $\partial R \simeq S \times F$ *which respects the unitary structure.*

Slightly modifying this definition one can also introduce the notion of objects with boundary, and therefore consider the associated bordism $MU_*^F(-)$. The forgetful functor defines a natural transformation $\Omega_*^U(-) \to MU_*^F(-)$. Moreover $MU_*^F(-)$ has the following fundamental property:

Proposition B.2. $MU_*^F(-)$ *is a homology theory, and for any space* X *there is a long exact sequence*

$$\cdots \to \Omega_n^U(X) \xrightarrow{\cdot[F]} \Omega_{n+k}^U(X) \to MU_{n+k}^F(X) \to \Omega_{n-1}^U(X) \to \cdots$$

which is called the Bockstein-Baas-Sullivan sequence.

Now let $\Sigma = \{F_1, F_2, \ldots\}$ be an arbitrary sequence of closed unitary manifolds and denote by Σ_m the finite sequence $\{F_1, \ldots, F_m\}$.

Definition B.3. *A* **manifold with singularity** Σ_m *is defined inductively as a triple* (R, S, f) *where* R *and* S *are manifolds with singularity* Σ_{m-1} *and* f *is an isomorphism* $\partial R \simeq S \times F_m$.

We denote by $MU_*^{\Sigma_m}(-)$ the bordism of Baas-Sullivan manifolds with singularity Σ_m. Again we have:

Proposition B.4. $MU_*^{\Sigma_m}(-)$ *is a homology theory and for any* X *there is a Bockstein-Baas-Sullivan long exact sequence:*

$$\cdots \to MU_*^{\Sigma_{m-1}}(X) \to MU_*^{\Sigma_{m-1}}(X) \to MU_*^{\Sigma_m}(X) \to MU_*^{\Sigma_{m-1}}(X) \to \cdots$$

This construction provides a sequence of homology theories and natural transformations

$$\Omega_*^U(-) \to MU_*^{\Sigma_1}(-) \to \cdots \to MU_*^{\Sigma_m}(-) \to \cdots$$

Taking the limit over the diagram above we define:

$$MU_*^{\Sigma}(-) := \lim_{m\to\infty} MU_*^{\Sigma_m}(-).$$

By construction $MU_*^{\Sigma}(-)$ comes with a natural transformation

$$u : \Omega_*^U(-) \longrightarrow \lim_{m\to\infty} MU_*^{\Sigma_m}(-) = MU_*^{\Sigma}(-).$$

and we denote by u_* the Ω_*^U-module homomorphism

$$\Omega_*^U \longrightarrow MU_*^{\Sigma}(\mathrm{pt})$$

induced by u for $X = \mathrm{pt}$.

Moreover, since passing to the limit preserves exactness, one has the following

Proposition B.5. $MU_*^{\Sigma}(-)$ *is a homology theory.*

The convergence of the limit above is sure if, for all n, the sequence Σ contains only finitely many manifolds of dimension $\leq n$. In fact if $F_{\sigma(n)}$ is the last element in Σ of dimension $\leq n$, then the homomorphism

$$MU_n^{\Sigma_i}(X) \to MU_n^{\Sigma_{i+1}}(X)$$

is an isomorphism for all $i \geq \sigma(n)$ because of dimensional reasons, and therefore the limit is reached after a finite number of steps.

The coefficients of $MU_*^\Sigma(-)$ are in general quite difficult to compute and so we restrict our attention to the case of a regular sequence Σ. Recall that a sequence $\{x_1, x_2, \ldots\}$ in a commutative ring R is called regular if, for every n, the multiplication by x_n induces injective homomorphisms

$$R \longrightarrow R$$

and

$$R/(x_1, \ldots, x_{n-1}) \longrightarrow R/(x_1, \ldots, x_{n-1})$$

where $(x_1, \ldots, x_n)$ denotes the ideal generated by $x_1, \ldots, x_n$.

By an easy inductive application of the Bockstein-Baas-Sullivan sequence one can show the

Proposition B.6. *If $\Sigma = \{F_1, \ldots, F_n, \ldots\}$ is a regular sequence in Ω_*^U, then u_* factorizes to an Ω_*^U-module isomorphism*

$$\begin{array}{ccc} \Omega_*^U & \xrightarrow{\quad u_* \quad} & MU_*^\Sigma \\ \downarrow & \overset{\simeq}{\nearrow} & \\ \Omega_*^U/([F_1], \ldots, [F_n], \ldots) & & \end{array}$$

B.2 The construction of $hh_*(-)$

According to Milnor (see [Miln]), the bordism ring Ω_*^U is isomorphic to the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n, \ldots]$ with $\deg x_n = 2n$; furthermore a family $\{M^{2n}\}$ of unitary manifolds is a basis sequence of Ω_*^U if and only if it results

$$s_k(c_1, \ldots, c_n)[M^{2n}] = \begin{cases} \pm q & \text{if } n+1 \text{ is a power of the prime } q \\ \pm 1 & \text{if } n+1 \text{ is not a prime power} \end{cases}$$

This characterization can be used to prove the

Lemma B.7. *There exists a basis sequence $\{M^{2n}\}$ of Ω_*^U with the property that*

$$\mathrm{sig}(M^{2n}) = \begin{cases} +1 & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases} \tag{B.1}$$

Proof. Let $\{N^{2n}\}$ be an arbitrary family of free polynomials generators of Ω_*^U and set:

$$\begin{cases} M^4 := [\mathbb{CP}^2] \\ M^{2n} := N^{2n} - \mathrm{sig}(N^{2n}) \cdot (\mathbb{CP}^2)^{(n/2)} & \text{for } n \neq 2 \end{cases}$$

By construction, the family $\{M^{2n}\}$ satisfies Condition B.1 above. Moreover, it is easily seen that $\{M^{2N}\}$ is again a basis sequence: the only thing to show is that $\mathbb{CP}^2$ can be taken as 4-dimensional generator, but this is clear since it holds

$$\begin{cases} s_2(c_1, c_2)[\mathbb{CP}^2] = c_1^2[\mathbb{CP}^2] - 2c_2[\mathbb{CP}^2] = 3 \\ s_{1,1}(c_1, c_2)[\mathbb{CP}^2] = c_2[\mathbb{CP}^2] = 3 \end{cases}$$

■

Now let us choose a basis sequence $\{M^{2n}\}$ of Ω_*^U which satisfies B.1, and denote by Σ the sequence $\{M^{2n}\}_{n\neq 2}$.

Definition B.8. *The homology theory $hh_*(-)$ is defined by setting:*

$$hh_*(-) := MU_*^{\Sigma}(-) = \lim_{m\to\infty} MU_*^{\Sigma_m}(-).$$

Remark. $hh_*(-)$ depends on the choice of the basis sequence $\{M^{2n}\}$.

The fact that the dimensions of the elements in Σ build a strictly increasing sequence has the following consequence:

Lemma B.9. *For any $n \in \mathbb{N}$ there exists another integer $m \in \mathbb{N}$ such that the homomorphism*

$$MU_n^{\Sigma_m}(X) \longrightarrow hh_n(X)$$

is an isomorphism for any topological space X.

Furthermore we have the

Proposition B.10. *The homology theory $hh_*(-)$ has the following properties:*

1. *there is a natural transformation*

$$u : \Omega_*^U(-) \longrightarrow hh_*(-).$$

2. there is an isomorphism

$$\gamma : hh_*(\mathrm{pt}) \xrightarrow{\simeq} \mathbb{Z}[t]$$

such that the composition $\gamma \circ u_*$ *is the multiplicative genus*

$$\tau : \Omega_*^U \longrightarrow \mathbb{Z}[t]$$

$$[N^k] \longmapsto \mathrm{sig}(N) \cdot t^{k/4}$$

Proof. The natural transformation $u : \Omega_*^U(-) \to hh_*(-)$ is given by construction as explained in the previous section.

Since Σ is a proper sequence, u_* induces according to Proposition B.6 an isomorphism

$$\Omega_*^U(-)/([M^2],[M^6]\ldots) \xrightarrow{\simeq} hh_*(\mathrm{pt})$$

and consequently a commutative diagram

$$\begin{array}{ccc} \Omega_*^U & \xrightarrow{u_*} & hh_*(\mathrm{pt}) \\ \downarrow & \swarrow \simeq & \\ \Omega_*^U(-)/([M^2],[M^6]\ldots) & & \end{array}$$

The quotient $\Omega_*^U/([M^2],[M^6],\ldots)$ is isomorphic to the Ω_*^U-module $\mathbb{Z}[t]$ generated by $[M^4]$, and thus we get by composition an isomorphism

$$\gamma : hh_*(\mathrm{pt}) \xrightarrow{\simeq} \Omega_*^U/([M^2],[M^6],\ldots) \xrightarrow{\simeq} \mathbb{Z}[t]$$

which coincides with τ by construction. ■

B.3 The isomorphism $Hh_*(-) \simeq hh_*(-)$

In this section, we want to sketch the construction of a natural transformation

$$hh_*(-) \longrightarrow Hh_*(-)$$

which is an equivalence of homology theories.

Let $\Sigma = \{F_1, F_2, \ldots\}$ denote the sequence of manifolds used in the previous section to construct $hh_*(-)$, and let $\mathcal{E}_m$ denote the category of all Baas-Sullivan manifold (possibly with non-empty boundary) with singularity $\Sigma_m = \{F_1, \ldots, F_m\}$. Since $hh_*(-)$ is defined as the limit of the functors $MU_*^{\Sigma_m}(-)$ for $m \to \infty$, it follows that a natural transformation

$$hh_*(-) \longrightarrow Hh_*(-)$$

is the same as a sequence of natural transformations

$$MU_*^{\Sigma_m}(-) \longrightarrow Hh_*(-)$$

which satisfy a compatibility property.

Such a family of transformations can be defined constructing a sequence of functors (which respect the boundaries and the products)

$$\alpha_m : \mathcal{E}_m \longrightarrow \mathscr{C}$$

where $\mathscr{C}$ denotes the category of all H-stratifolds.

For $m = 0$, $\mathcal{E}_0$ is the class of unitary manifolds and so we can consider the canonical inclusion described in Section 3.2.

Now, we show how to define α_{m+1} from α_m. For simplicity we only consider Baas-Sullivan manifolds with empty boundary. Let (R, S, f) is an element of $\mathcal{E}_{m+1}$ and let us denote by F the manifold $F_{m+1} \in \Sigma_{m+1}$. Moreover let us choose a Lagrangian subspace in the middle cohomology of F and observe that, according to Section 4.3, this choice allows to consider the cone over F as an H-stratifold with boundary F.

The isomorphism $f : \partial R \xrightarrow{\simeq} S \times F$ induces an isomorphism of H-stratifolds

$$\alpha_m(f) : \partial(\alpha_m(R)) \xrightarrow{\simeq} \alpha_m(S) \times F$$

Finally, gluing $\alpha_m(R)$ and $\alpha_m(S) \times CF$ along the boundary, we define

$$\alpha_{m+1}(R, S, f) := \alpha_m(R) \cup (\alpha_m(S) \times CF)$$

The induced transformation

$$\psi : hh_*(-) \longrightarrow Hh_*(-)$$

has the property that the diagram

$$\begin{array}{ccc} \Omega_*^U(-) & & \\ \downarrow & \searrow & \\ hh_*(-) & \longrightarrow & Hh_*(-) \end{array}$$

commutes and by the comparison theorem we get the

Corollary B.11. *The natural transformation* ψ *is an equivalence of homology theories.*

Appendix C

Hirzebruch spectra

A Hirzebruch spectrum is a strict MU-algebra spectrum representing a homology theory which has properties analogous to that of $Hh_*(-)$. In the following we show how to construct a Hirzebruch spectrum and we investigate some of its properties.

C.1 Strict algebra spectra

We begin by reporting some results from [EKMM]. Let R be a commutative S-algebra, and assume $\pi_i(R) = 0$ for odd i.

Theorem C.1. ([EKMM], Theorem V,3.2) *Let X be a regular sequence in $\pi_*(R)$ and let I be the ideal generated by X. If $\pi_*(R)/I$ is concentrated in degrees congruent to zero mod 4, then there is an associative and commutative strict R-algebra spectrum R/X and a natural map of strict R-algebra spectra*

$$u : R \longrightarrow R/X$$

such that

$$u_* : \pi_*(R) \longrightarrow \pi_*(R/X)$$

realizes the homomorphism of $\pi_*(R)$-algebras

$$\pi_*(R) \longrightarrow \pi_*(R)/I$$

Quotients in the category of strict R-modules have the following universal property

Lemma C.2. ([EKMM], Lemma V,1.5) *Let N be a strict R-module, and $x \in \pi_n(R)$ such that $x\cdot : \Sigma^n N \to N$ is zero. Then, for any map of strict R-modules $\alpha : M \to N$, there is a map of strict R-modules $\tilde{\alpha} : M/xM \to N$ such that $\tilde{\alpha} \circ v = \alpha$.*

There are analogous constructions for localizations.

Theorem C.3. ([EKMM], Proposition V,2.3) *Let Y be any sequence of elements of $\pi_*(R)$. If A is a strict R-algebra spectrum, then there is a strict R-algebra spectrum $A[Y^{-1}]$ and a natural map of strict R-algebra spectra*

$$\lambda : A \longrightarrow A[Y^{-1}]$$

such that

$$\lambda_* : A_* \longrightarrow \pi_*(A[Y^{-1}])$$

realizes the morphism

$$A_* \longrightarrow A_*[Y^{-1}]$$

Lemma C.4. ([EKMM], Lemma V,1.13) *Let N be a strict R-module, and $Y \subset \pi_*(R)$ such that $y_i \cdot : \Sigma^{k_i} N \to N$, $\deg y_i = k_i$, is an equivalence for each i. Then, for any map of strict R-modules $\alpha : M \to N$, there is a unique map of strict R-modules $\tilde{\alpha} : M[Y^{-1}] \to N$ such that $\tilde{\alpha} \circ \lambda = \alpha$.*

C.2 Hirzebruch spectra

Let MU denote the unitary Thom spectrum. It is known that MU can be constructed as an E_∞-ring spectrum and thus it makes sense to speak of strict MU-algebra spectra and strict MU-module spectra.

Definition C.5. *A strict MU-algebra spectrum E with unit map $u : MU \to E$, is called a **Hirzebruch-spectrum** if*

- *$\pi_*(E) \simeq \mathbb{Z}[t]$, with $\deg t = 4$;*
- *the ring homomorphism*

  $$\pi_*(u) : \pi_*(MU) \longrightarrow \pi_*(E) \simeq \mathbb{Z}[t]$$

  coincides with τ.

A Hirzebruch spectrum can be constructed applying the results from the previous section. As we have observed in Appendix B, it is possible to choose a basis sequence $\{x_1, x_2, \ldots\}$ of $\pi_*(MU)$ with

$$\mathrm{sig}(x_n) = \begin{cases} 0 & \text{for n} \neq 2 \\ 1 & \text{for n} = 2 \end{cases}$$

Let X indicate the sequence $\{x_1, x_3, x_4, \ldots\}$.

Proposition C.6. *The MU-algebra spectrum MU/X is a Hirzebruch spectrum.*

Proof. The conditions of Theorem C.1 are satisfied since $\pi_*(MU)$ is concentrated in even degrees, and X is a regular sequence in $\pi_*(MU)$. It follows that there is a ring spectrum MU/X together with a MU-algebra map $u : MU \to MU/X$. The assumptions on the signature of the x_i's imply that u induces, for $X = \text{pt}$, the ring homomorphism

$$\tau : \pi_*(MU) \longrightarrow \mathbb{Z}[t]$$

and this proves that MU/X is a Hirzebruch spectrum. ■

The definition above characterizes the notion of a Hirzebruch spectrum up to isomorphism of strict MU-module spectra. In fact it holds the

Proposition C.7. *Every Hirzebruch spectrum E is isomorphic to MU/X as a strict MU-module.*

Proof. Let u be the orientation of E, and denote by v the orientation of MU/X. By definition u realizes, for $X = \text{pt}$, the genus τ. Since we have assumed the signature of x_1 to be zero, the composition $u \circ x_1 : S^2 \to MU \to E$ is null-homotopic. The left-multiplication by $x_1 \cdot : \Sigma^2 E \to E$ is the map

$$S^2 \wedge E \xrightarrow{(u \circ x_1) \wedge 1} E \wedge E \xrightarrow{\mu} E$$

which is therefore zero. Lemma C.2 implies thus that there exists a map of strict MU-modules

$$u_1 : MU/ < x_1 > \longrightarrow E$$

such that the following diagram commutes

$$\begin{array}{ccc} MU & \xrightarrow{u} & E \\ {\scriptstyle v_1}\downarrow & \nearrow{\scriptstyle \tilde{u}_1} & \\ MU/ < x_1 > & & \end{array}$$

An inductive argument shows how to define a sequence of factorizations

$$\begin{array}{ccc} MU & \xrightarrow{u} & E \\ {\scriptstyle v_n}\downarrow & \nearrow{\scriptstyle \tilde{u}_n} & \\ MU/ < x_1, x_3, \ldots, x_n > & & \end{array}$$

which satisfy an obvious compatibility property. The spectrum MU/X is by definition the telescope of the spectra $MU_n = MU/ < x_1, \ldots, x_n >$, and so there is a commutative diagram of strict MU-modules:

$$\begin{array}{ccc} MU & \xrightarrow{\quad u \quad} & E \\ {\scriptstyle v}\Big\downarrow & \overset{\tilde{u}}{\nearrow} & \\ MU/X & & \end{array}$$

To conclude the proof it is enough to observe that the map

$$\tilde{u}_* : \pi_*(MU/X) \longrightarrow E_*$$

sends x_2 to $\tau(x_2) = t$, and that consequently $\pi_*(u)$ is an isomorphism. ■

C.3 Determination of the Hirzebruch spectrum

In this section, we show how to determine the Hirzebruch spectrum E by computing its localizations $E[1/2]$ and $E_{(2)}$. Some of the considerations presented here have been suggested by Neil Strickland.

Let us begin with the localization $E[1/2]$. The homology theory associated to $E[1/2][t^{-1}]$ satisfies the hypotheses of Landweber exact functor theorem, and so the argument used in 5.4 to show the isomorphism $Hh_*(-)[1/2] \simeq ko_*(-)[1/2]$ can be applied to prove the existence of an isomorphism

$$E[1/2] \simeq ko[1/2].$$

So we only have to understand the $\mathbb{Z}_{(2)}$-localization $E_{(2)}$. By construction, the spectrum $E_{(2)}$ comes with a map $j : E \to E_{(2)}$ and thus it has a natural complex orientation which corresponds to the universal Thom class

$$MU \xrightarrow{\quad u \quad} E \xrightarrow{\quad j \quad} E_{(2)}.$$

The formal group law of $(E_{(2)}, j \circ u)$ is

$$F_{\text{sig}}(x, y) = \frac{x + y}{1 + txy}$$

as a power series over $\pi_*(E_{(2)}) \simeq \mathbb{Z}_{(2)}[t]$.

Every formal group law over the rationals is uniquely strictly isomorphic to the additive one and, for F_{sig}, this isomorphism is given by the power series

$$\log = \sum_{n=0}^{\infty} \frac{t^n}{2n+1} y^{2n+1} \in E_* \otimes \mathbb{Q} \simeq \mathbb{Q}[t].$$

Since all denominators are odd, this series defines a strict isomorphism

$$\log : F_{\mathrm{sig}} = F_{\mathrm{ad}}^{\log} \longrightarrow F_{\mathrm{ad}}$$

already over $\mathbb{Z}_{(2)}$-algebras.

Now it is well known that $\pi_*(MU \wedge MU)$ corepresents the groupoid of all formal group laws and strict isomorphisms (see for example [Mill], Section II.6)). Therefore there exists a unique ring homomorphism

$$\varphi : \pi_*(MU \wedge MU) \longrightarrow \pi_*(E_{(2)})$$

which classifies the strict isomorphism $\log : F_{\mathrm{sig}} \to F_{\mathrm{ad}}$. This means that if $\eta_L, \eta_R : \pi_*(MU) \to \pi_*(MU \wedge MU)$ are the ring homomorphisms induced by

$$MU = MU \wedge S \longrightarrow MU \wedge MU \quad \text{and} \quad MU = S \wedge MU \longrightarrow MU \wedge MU,$$

then $\varphi \circ \eta_L$ and $\varphi \circ \eta_R$ classify F_{ad} and F_{sig} respectively.

Observe that since $\pi_*(E_{(2)})$ is a $\mathbb{Z}_{(2)}$-algebra, φ factorizes as in the diagram:

$$\begin{array}{ccc} \pi_*(MU \wedge MU) & \longrightarrow & \pi_*(E_{(2)}) \\ \downarrow & \nearrow & \\ \pi_*(MU \wedge MU)_{(2)} & & \end{array}$$

In order to understand φ, let us choose an isomorphism

$$\pi_*(MU \wedge MU) \xrightarrow{\simeq} \pi_*(MU)[b_1, b_2, \ldots]$$

and a basis-sequence $\{x_1, x_2, \ldots\}$ of $\pi_*(MU)$ with the usual condition on the signature. This choice defines a system of coordinates

$$\pi_*(MU \wedge MU) \xrightarrow{\simeq} M := \mathbb{Z}[x_1, x_2, \ldots][b_1, b_2, \ldots]$$

in which the homomorphism

$$\eta_L : MU_* \to \pi_*(MU \wedge MU)$$

becomes the natural inclusion

$$\mathbb{Z}[x_1, x_2, \ldots] \hookrightarrow \mathbb{Z}[x_1, x_2, \ldots][b_1, b_2, \ldots].$$

Since $\varphi \circ \eta_L$ classifies F_{ad}, the composition $\varphi \circ \eta_L$ has to coincide with the morphism

$$\pi : \mathbb{Z}[x_1, x_2, \ldots] \longrightarrow \mathbb{Z}_{(2)}[t]$$

which sends every x_i to 0. Furthermore the universal strict isomorphism $f = \sum_{n=0}^{\infty} b_n x^{n+1}$ has to be sent under φ to the power series log, and thus it must hold

$$\varphi(b_n) = \begin{cases} \frac{t^{n/2}}{n+1} & \text{if n=0 (mod 2)} \\ 0 & \text{else} \end{cases}$$

Now use the fact that φ factorizes through the ring

$$M_{(2)} := \mathbb{Z}_{(2)}[x_1, \ldots][b_1, \ldots]$$

and let us set

$$b'_n := \begin{cases} b_n & \text{for } n \text{ odd} \\ 3b_2 & \text{for } n = 2 \\ (n+1)b_n - 3(n-1)b_{n-2} \cdot b_2 & \text{for } n \text{ even and } n > 2 \end{cases}$$

The new elements b'_n provide a basis of $M_{(2)}$ as $\pi_*(MU_{(2)})$-module, and moreover it holds

$$\varphi(b'_n) = \begin{cases} t & \text{for } n = 2 \\ 0 & \text{for } n \neq 2 \end{cases}$$

The kernel of φ is thus generated, as ideal of $M_{(2)}$, by the regular sequence

$$Y := \{x_1, x_2, \ldots, b'_1, b'_3, b'_4, \ldots\}.$$

According to Theorem C.1, there is a $(MU \wedge MU)$-module spectrum $G = (MU \wedge MU)/Y$ and a $(MU \wedge MU)$-module map

$$\phi : MU \wedge MU \longrightarrow G$$

with the following two properties:

- $\pi_*(G) \simeq \mathbb{Z}_{(2)}[t]$;
- the induced homomorphism $\pi_*(\phi)$ coincides with φ.

Since G_* is concentrated in degrees congruent to zero modulo 4, there is a canonical multiplicative structure on G such that ϕ is a $(MU \wedge MU)$-ring morphism.

We use ϕ to induce two different orientations of G. Let u and v be the map indicated in the following diagram

$$v : MU \xrightarrow{1 \wedge \eta} MU \wedge MU \longrightarrow (MU \wedge MU)_{(2)} \xrightarrow{\phi} G$$

$$u : MU \xrightarrow{\eta \wedge 1} MU \wedge MU \longrightarrow (MU \wedge MU)_{(2)} \xrightarrow{\phi} G$$

The $\mathbb{C}$-oriented spectrum (G, u) is by construction the $\mathbb{Z}_{(2)}$-Hirzebruch spectrum and it is thus isomorphic as a strict MU-module to $E_{(2)}$.

Consider now the orientation corresponding to v: the formal group law of (G, v) is the additive one, and thus the classifying map of F_{ad} is the projection π. This fact means that $v_*(x_i) = 0$ for each i, or equivalently that the map

$$x_i \cdot : \Sigma^{2i} G \longrightarrow G$$

is zero. By Lemma C.2 we get a factorization

$$\begin{array}{ccc} MU & \xrightarrow{\quad v \quad} & G \\ \downarrow & \overset{\tilde{v}}{\nearrow} & \\ MU/\{x_1, x_2, ..\} & & \end{array}$$

Using the fact that $MU/\{x_1, \ldots\}$ is canonically isomorphic to the Eilenberg-Mac Lane spectrum $H\mathbb{Z}$, one obtains a map of strict MU-module spectra

$$\tilde{v} : H\mathbb{Z}_{(2)} \longrightarrow G.$$

The map $\tilde{v}$ can be used to define a new map

$$H\mathbb{Z}_{(2)} \wedge G \xrightarrow{\tilde{v} \wedge 1} G \wedge G \longrightarrow G$$

such that the two diagrams

$$\begin{array}{ccccccc} & & MU \wedge G & & & & \\ & \overset{\eta \wedge 1}{\nearrow} & \downarrow & \overset{v \wedge 1}{\searrow} & & & \\ S^0 \wedge G & \xrightarrow{\eta \wedge 1} & H\mathbb{Z}_{(2)} \wedge G & \xrightarrow{\tilde{v} \wedge 1} & G \wedge G & \xrightarrow{m} & G \end{array}$$

and

$$\begin{array}{ccc} H\mathbb{Z}_{(2)} & \longrightarrow & G \\ \uparrow {\scriptstyle \eta \wedge 1} & \nearrow {\scriptstyle 1} & \\ S^0 \wedge G & & \end{array}$$

commute.

Finally, applying ([Ru], Theorem II,7.5), one finds

Proposition C.8. *There is an isomorphism*

$$E_{(2)} \simeq H\mathbb{Z}_{(2)}[t].$$

Epilogue

We want to conclude this thesis by spending some words on two open problems which could be of interest.

Question 1. *Can we use Hirzebruch homology to prove the topological invariance of rational Pontrjagin classes?*

Since the Hirzebruch fundamental class of a manifold corresponds over the rationals to the L-class, one could be tempted to use the topological invariance of the former to prove the topological invariance of rational Pontrjagin classes. Unfortunately the latter property is needed to prove transversality in the topological category and this is in turn needed to show that $Hh_*(-)$ is a homology theory, so that this argument does not work. For this reason it would be interesting to look for an alternative construction of the boundary operator which makes no use of transversality.

Question 2. *Is it possible to represent $Hh_*(-)$ by a strict MU-ring spectrum?*

The solution of this problem could have a very interesting application. Since all coefficients of the L-polynomials lie in the ring $\mathbb{Z}_{(2)}$, the L-class of a smooth manifold M can be seen as an element of $H_*(M;\mathbb{Z}_{(2)}[t])$ and in particular, Now, if the answer to the question above *were* positive, then, according to the results of Appendix C, there would be an isomorphism

$$Hh_*(M) \otimes \mathbb{Z}_2 \xrightarrow{\simeq} H_*(M;\mathbb{Z}_2[t])$$

mapping the Hirzebruch fundamental class of a smooth manifold M to the L-class of M, and, in particular, one could prove the topological invariance of the $\mathbb{Z}_{(2)}$-local L-class.

Bibliography

[Ad] Adams J.F., *Stable Homotopy and Generalized Homology*, Chicago Lectures in Mathematics. Chicago - London: The University of Chicago Press (1974)

[Ba1] Banagl M., *Extending Intersection Homology Type Invariants to non-Witt Spaces*, Mem. Am. Math. Soc. 760 (2002)

[Ba2] Banagl M., *Sheaf Theory, Derived Categories, and Intersection Homology*, Preprint

[Bi] Bing R.H., *The Cartesian Product of a certain nonmanifold and a line is E^4*, Ann. Math.(2) 70, 399-412 (1959)

[Bor] Borel et al., *Intersection Cohomology*, Progress in Mathematics, Vol. 50, Birkhäuser

[Bot] Botvinnik B.I., *Manifolds with Singularities and the Adams-Novikov Spectral Sequence*, LMS Lecture Notes Series 170 (1992)

[Do] Dold A., *Geometric Cobordism and the Fixed Point Transfer*, Algebr. Topol., Proc. Vancouver 1977, Lect. Notes Math. 673, 32-87 (1978)

[EKMM] Elmendorf A.D., Kriz I., Mandell M.A., May J.P., *Rings, Modules, and Algebras in Stable Homotopy Theory*, Mathematical Surveys and Monographs 47, AMS (1997).

[GM1] Goresky M., MacPherson R., *Intersection Homology Theory*, Topology 19, 135-165 (1980)

[GM2] Goreski M. and MacPherson R., *Intersection Homology II.*, Invent.Math. 72, 77-129 (1983)

[Hi] Hirzebruch F., *Topological Methods in Algebraic Geometry*, Springer Verlag (1978)

[HBJ] Hirzebruch F., Berger T., Jung R., *Manifolds and Modular Forms*, Aspects of Mathematics, Vieweg (1992)

[KaSc] Kashiwara M. and Schapira P., *Sheaves on Manifolds*, Springer Verlag (1990)

[KiSi] Kirby R., Siebenmann L., *Foundational essays on topological manifolds, smoothings and triangulations*, Ann. of Maths. Studies Princeton University Press (1977)

[Kr1] Kreck M., *Differential Algebraic Topology*, Preprint

[Kr2] Kreck M., *Stratifolds*, Preprint

[La] Landweber P.S., *Homological Properties of Comodules over* $MU_*(MU)$ *and* $BP_*(BP)$, Am. J. Math. 98, 591-610 (1976)

[Ma] Marin A., *La transversalité topologique*, Ann. of Maths. 106, 269-293 (1977)

[MM] Madsen I. and Milgram R.J., *The classifying spaces for surgery and cobordism of manifolds*, Ann. of Maths. Studies Princeton University Press

[Mill] Miller H., *Notes on Cobordism*, unpublished MIT Lectures Notes (1994)

[Miln] Milnor J., *On the Cobordism Ring* Ω^* *and a complex analogue.* I., Am. J. Math. 82, 505-521 (1960)

[MS] Milnor J. and Stasheff J., *Characteristic Classes*, Annals of Mathematics Studies Princeton University Press

[Quil] Quillen D., *Elementary Proofs of Some Results of Cobordism Theory Using Steenrod Operations*, Adv. Math. 7, 29-56 (1971)

[Quin] Quinn F., *Topological transversality holds in all dimensions*, Bull. Amer. Math. Soc. 18, 145-148 (1988)

[Ru] Rudyak Y.B., *On Thom Spectra, Orientability, and Cobordism*, Springer Monographs in Mathematics (1998)

[Si] Siegel P.H., *Witt spaces: A geometric cycle theory for KO-homology at odd primes*, Am. J. Math 105, 1067-1105 (1983)

[Sp] Spanier E., *Algebraic Topology*, Springer Verlag (1966)

[St] Steenrod N., *The Topology of Fibre Bundles*, Princeton University Press (1999)