A Quantum Kirwan Map and Symplectic Vortices

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partly joint with C. Woodward
and K. L. Nguyen

References:

A Quantum Kirwan Map: Bubbling and Fredholm Theory, 129 pages, arXiv:1209.5866, to appear in Mem. Amer. Math. Soc.

The Invariant Symplectic Action and Decay for Vortices, J. Symplectic Geom. **7** (2009), no. **3**, 357–376.

with Khoa Lu Nguyen and Chris Woodward: Morphisms of CohFT Algebras and Quantization of the Kirwan Map,
arXiv:0903.4459, 32 pages, to appear in Proc. Hayashibara Forum.

Related work:

K. Cieliebak, A.R. Gaio, I. Mundet i Riera,
D. A. Salamon
E. Gonzalez, C. Woodward
I. Mundet and G. Tian
A. Ott
U. Frauenfelder
S. Venugopalan, C. Woodward
A. Givental, B. Kim

Overview:

1. Quantum Kirwan maps
2. Symplectic Vortices
3. Morphisms of Cohomological Field Theories

1. A quantum Kirwan map

We fix:

(M, ω) a symplectic manifold,

G a compact connected Lie group,

$\mathfrak{g}$:= Lie algebra of G

$\mu : M \to \mathfrak{g}^*$ a momentum map

”The introduction of the cipher 0 or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps . . .”

(Alexander Grothendieck)

Example:

$$M := \mathbb{C}^{k\times n}, \quad \omega := \omega_0, \quad G := \mathsf{U}(k),$$

$$\mu : \mathbb{C}^{k\times n} \to \mathfrak{u}(k)^* \cong \mathfrak{u}(k), \quad \mu(\Theta) := \frac{i}{2}(\mathbf{1} - \Theta\Theta^*)$$

standing hypothesis:

(H) G acts freely on $\mu^{-1}(0)$ and μ is proper.

Then symplectic quotient well-defined closed symplectic manifold $(\overline{M}, \overline{\omega})$.

Kirwan homomorphism:

$$\kappa : H_G^*(M) \to H^*(\overline{M})$$

Question: quantum version?

under restrictive conditions constructed by A. R. Gaio and D. A. Salamon

algebraic setup:
recently constructed by C. Woodward

We define

$\mathsf{QH}^*_G(M,\omega)$:= set of maps

$$\alpha : H_2^G(M,\mathbb{Z}) \to H^*_G(M)$$

satisfying the equivariant Novikov condition.

module over the equivariant Novikov ring Λ^{μ}_{ω}

Product $*_G$ counts holomorphic maps into the fibers of $(M \times \mathsf{EG})/G$.
(A. Givental and B. Kim)

We denote

$$\mathsf{QH}^*(\overline{M},\overline{\omega}) := H^*(M) \otimes \Lambda^{\mu}_{\omega}$$

and equip this with the usual quantum product.

A map

$$\varphi : H_2^G(M,\mathbb{Z}) \to \mathsf{Hom}_{\mathbb{Q}}\left(H_G^*(M), H^*(\overline{M})\right)$$

satisfying the equivariant Novikov condition, induces a Λ_ω^μ-module homomorphism

$$\varphi_* : \mathsf{QH}_G^*(M,\omega) \to \mathsf{QH}^*(\overline{M},\overline{\omega}),$$
$$(\varphi_*\alpha)(B) := \sum \varphi(B_1)\alpha(B_2),$$

where the sum is over all pairs $B_1, B_2 \in H_2^G(M,\mathbb{Z})$ satisfying $B_1 + B_2 = B$.

Conjecture Assume that (H) holds and that (M, ω, μ) is convex at ∞ and semipositive. Then there exists a map φ as above, such that φ_* is a surjective ring homomorphism, and

$$\varphi(0) = \kappa,$$
$$\Big\langle [\omega - \mu], B \Big\rangle \leq 0,\ B \neq 0 \Longrightarrow \varphi(B) = 0.$$

without semipositivity: similar conjecture stating the existence of a morphism of cohomological field theories (joint work with K. L. Nguyen and C. Woodward)

2. Idea of proof, symplectic vortices

Idea of proof: We define $\varphi = \mathrm{Q}\kappa_G$ by counting symplectic vortices over $\mathbb{C}$.

Fix $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$,

J: G-invariant and ω-compatible,

Σ, ω_Σ, j.

Consider

$$\widetilde{\mathcal{B}}_\Sigma := \Big\{ w := (P, A, u) \,\Big|\, P \text{ smooth } G\text{-bundle over } \Sigma, \\ A \in \mathcal{A}(P),\ u \in C^\infty_G(P, M) \Big\}.$$

A symplectic vortex is a solution $(P, A, u) \in \widetilde{\mathcal{B}}_\Sigma$ of

$$\begin{cases} \bar\partial_{J,A}(u) & = & 0 \\ F_A + (\mu \circ u)\omega_\Sigma & = & 0. \end{cases} \tag{1}$$

We define

$$\mathcal{B}_\Sigma := \widetilde{\mathcal{B}}_\Sigma / \sim,$$

and the *energy density and energy* of a class $W = [P, A, u] \in \mathcal{B}_\Sigma$ to be

$$e_W := \frac{1}{2}\Big(|d_A u|^2 + |F_A|^2 + |\mu \circ u|^2\Big),$$
$$\mathsf{E}(W) := \int_\Sigma e_W \omega_\Sigma.$$

We define the *image* of W to be the set of orbits of $u(P)$.

Consider $\Sigma := \mathbb{R}^2, \omega_{\mathbb{R}^2} := \omega_0, j := i$.

Fact: Every finite energy vortex class W with compact image naturally carries a class $[W] \in H_2^G(M, \mathbb{Z})$.

Let $B \in H_2^G(M, \mathbb{Z})$.

$$\mathcal{M}_B := \Big\{ \text{vortex class } W \Big| [W] = B \Big\}.$$

Fact: There are natural evaluation maps

$$\mathrm{ev}_z : \mathcal{M}_B \to (M \times \mathrm{EG})/G, \quad \text{for } z \in \mathbb{C},$$
$$\overline{\mathrm{ev}}_\infty : \mathcal{M}_B \to \overline{M} \quad (\text{at } \infty \in \mathbb{C} \cup \{\infty\}).$$

To prove the conjecture, heuristically, we define

$$\varphi : H_2^G(M, \mathbb{Z}) \to \mathsf{Hom}_{\mathbb{Q}}\Big(H_G^*(M), H^*(\overline{M})\Big),$$

$$\Big\langle \varphi(B)\alpha, \bar{b} \Big\rangle := \int_{\mathcal{M}_B} \mathsf{ev}_0^* \, \alpha \smile \overline{\mathsf{ev}}_\infty^* \overline{\mathsf{PD}}(\bar{b}),$$

Reason why Qκ_G intertwines $*_G$ and $\bar{*}$:

Let $B \in H_2^G(M)$, $\alpha_1, \alpha_2 \in H_G^*(M)$, $\bar{\beta} \in H^*(\overline{M})$.

Claim: see blackboard.

Idea of proof of claim:

Set $z_\nu^+ := \nu$, $z_\nu^- := 1/\nu$

Consider a sequence W_ν of gauge classes of vortices such that $[W_\nu]_G = B$ and

$$\mathrm{ev}_{z_\nu^\pm}(W_\nu) \in X_1,\ \mathrm{ev}_{-z_\nu^\pm}(W_\nu) \in X_2,$$
$$\overline{\mathrm{ev}}_\infty(W_\nu) \in \bar{X}.$$

Limits: see blackboard.

Goal: Define Qκ rigorously.

Compactification:

We call (M, ω) *aspherical* iff

$$\int u^*\omega = 0, \quad \forall u \in C^\infty(S^2, M).$$

Consider $\Sigma := \mathbb{C}$.

Theorem (Z.). *Assume (H), (M,ω) is aspherical, and (M,ω,μ,J) is convex at ∞. Let $k \in \mathbb{N}_0$, and for $\nu \in \mathbb{N}$ let $W_\nu \in \mathcal{M}$ be a vortex class and $z_1^\nu, \dots, z_k^\nu \in \mathbb{C}$ be points. Suppose that the closure of the image of each W_ν is compact, and*

$$\mathrm{E}(W_\nu) > 0, \forall \nu \in \mathbb{N}, \quad \sup_{\nu \in \mathbb{N}} \mathrm{E}(W_\nu) < \infty.$$

Then there exists a subsequence of

$$\Big(W_\nu, z_0^\nu := \infty, z_1^\nu, \dots, z_k^\nu\Big)$$

that converges to some genus 0 stable map $(\mathbf{W}, \mathbf{z})$ consisting of vortex classes over $\mathbb{C}$ and pseudo-holomorphic spheres in $\overline{M}$, with $k+1$ marked points.

Proof:

Combines Gromov- and Uhlenbeck-compactness.

Idea: zoom out rapidly, then zoom back in to capture first bubble, . . .

Proof involves an isoperimetric inequality for the invariant symplectic action functional and an upper bound on the "momentum map component" of a vortex.

Main difficulty: $\mathbb{C}$ has infinite area, energy can escape to $\infty \in \mathbb{C} \cup \{\infty\}$.

Fredholm theory:

View LHS of (1) as $\widetilde{\mathcal{S}}(w)$, where $\tilde{\mathcal{S}} : \tilde{\mathcal{B}} \to \tilde{\mathcal{E}}$ is a section.

This descends to

$$\mathcal{S} : \mathcal{B} := \widetilde{\mathcal{B}}/\sim \to \mathcal{E} := \widetilde{\mathcal{E}}/\sim .$$

Vertical differential of $\mathcal{S}$ at $W = [w] \in \mathcal{S}^{-1}(0)$:

$$\mathcal{D}_w : \ker L_w^* \cong T_W\mathcal{B} \to \widetilde{\mathcal{E}}_w \cong \mathcal{E}_W,$$

$$\mathcal{D}_w(v,\alpha) := \begin{pmatrix} \left(\nabla^A v + L_u\alpha\right)^{0,1} + \frac{1}{2}(\nabla_v J)\, d_A u\, j \\ d_A\alpha + \omega_\Sigma\, d\mu(u)v \end{pmatrix}.$$

Explanations:

$L_w : \mathrm{Lie}\,\mathcal{G} \to T_w\widetilde{\mathcal{B}}$.

We denote

$TM^u := (u^*TM)/G \to \Sigma, \quad \mathfrak{g}_P := (P \times \mathfrak{g})/G \to \Sigma,$

$$\begin{aligned} E_u &:= TM^u \oplus \bigwedge^1\Big(\mathfrak{g}_P\Big) \to \Sigma, \\ E'_u &:= \bigwedge^{0,1}(TM^u) \oplus \bigwedge^2(\mathfrak{g}_P) \to \Sigma. \end{aligned}$$

Then $T_w\widetilde{\mathcal{B}} = \Gamma(E_u)$, $\widetilde{\mathcal{E}}_w = \Gamma(E'_u)$.

$\nabla :=$ Levi-Civita connection of $\omega(\cdot, J\cdot)$.

(∇, A) induces a connection ∇^A on E_u.

Consider $\Sigma := \mathbb{C}$.

Fix $p > 2$ and $\lambda \in \mathbb{R}$.

For $f : \mathbb{R}^n \to \mathbb{R}$ we denote

$$\|f\|_{p,\lambda} := \left\| f(1 + |\cdot|^2)^{\frac{\lambda}{2}} \right\|_p .$$

We define

$$\widetilde{\mathcal{B}}^p_\lambda := \Big\{ \quad w := (P, A, u) \in \widetilde{\mathcal{B}}^p_{\mathsf{loc}} \,\Big|\, \overline{u(P)} \text{ compact}, \; \|\sqrt{e_w}\|_{p,\lambda} < \infty \Big\},$$

$$\mathcal{B}^p_\lambda := \widetilde{\mathcal{B}}^p_\lambda / \sim_p,$$

Let $w := (P, A, u) \in \widetilde{\mathcal{B}}^p_\lambda$.

For $\zeta := (v, \alpha) \in W^{1,p}_{\mathrm{loc}}(E_u)$ we define

$$\|\zeta\|_{w,p,\lambda} := \|\zeta\|_\infty + \Big\| |\nabla^A \zeta| + |d\mu(u)v| + |\alpha| \Big\|_{p,\lambda}.$$

We define

$$L_w^* : \Gamma^{1,p}_{\mathrm{loc}}\Big(\bigwedge^1(\mathfrak{g}_P)\oplus TM^u\Big) \to \Gamma^p_{\mathrm{loc}}(\mathfrak{g}_P), \quad L_w^*(\alpha, v) := -d \tag{2}$$

$$\widetilde{\mathcal{X}}_w^{p,\lambda} := \Big\{\zeta \in \Gamma^{1,p}_{\mathrm{loc}}\Big(\bigwedge^1(\mathfrak{g}_P) \oplus TM^u\Big)\,\Big|\, L_w^*\zeta = 0,\ \|\zeta\|_{w,p,\lambda} < \tag{3}$$

$$\widetilde{\mathcal{Y}}_w^{p,\lambda} := \Big\{\zeta' \in \Gamma^{p}_{\mathrm{loc}}\Big(\bigwedge^{0,1}(TM^u) \oplus \bigwedge^2(\mathfrak{g}_P)\Big)\,\Big|\, \|\zeta'\|_{p,\lambda} < \infty\Big\}, \tag{4}$$

$$\mathcal{X}_W^{p,\lambda} := \coprod_{w\in W} \widetilde{\mathcal{X}}_w^{p,\lambda}/\sim_p, \tag{5}$$

$$\mathcal{Y}_W^{p,\lambda} := \coprod_{w\in W} \widetilde{\mathcal{Y}}_w^{p,\lambda}/\sim_p . \tag{6}$$

Theorem (Z., Fredholm property)**.** *Let* $p > 2$, $\lambda \in \mathbb{R}$, *and* $W \in \mathcal{B}^p_\lambda$. *Assume that* $\dim M > 2\dim G$. *Then the following statements hold.*

1. *If* $\lambda > 1-2/p$ *then the normed vector spaces* $\mathcal{X}^{p,\lambda}_W$ *and* $\mathcal{Y}^{p,\lambda}_W$ *are complete.*

2. *If* $1 - 2/p < \lambda < 2 - 2/p$ *then the operator* $\mathcal{D}^{p,\lambda}_W$ *is well-defined and Fredholm of real index*

$$\operatorname{ind} \mathcal{D}^{p,\lambda}_W = \dim M - 2\dim G + 2\left\langle c_1^G(M,\omega), [W]\right\rangle, \tag{7}$$

where $[W]$ *denotes the equivariant homology class of* W.

Proof:

1. Fredholm result for augmented vertical differential and
2. surjectivity of L_w^*.

The proof of 1. is based on:

- Existence of a good trivialization of E_u. (Respects splitting
$$T_{u(p)}M = (\mathrm{im}\, L^{\mathbb{C}})^{\perp} \oplus \mathrm{im}\, L^{\mathbb{C}}_{u(p)}$$
for $p \in \pi^{-1}(z) \subseteq P$, $z \in \mathbb{C} \setminus B_R$ and R large.)

- Work by R. B. Lockhart and R. C. McOwen.

Main difficulty:

Kondrachov compactness fails on $\mathbb{C}$. In fact:

The terms $\alpha \mapsto (L_u\alpha)^{0,1}$ and $v \mapsto \omega_0\, d\mu(u)v$ occuring in $\mathcal{D}^{p,\lambda}$ are not compact.