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"Gauged Gromov-Witten invariants and applications"

§1. Symplectic vortices

(after Mundet, Salamon, ...)

Equivariant generalization of pseudoholomorphic curves

K = compact connected group $\quad \mathfrak{k} = \mathrm{Lie}(K)$

$G = K^{\mathbb{C}} \quad \mathfrak{g} = \mathrm{Lie}(G)$

X Kähler Hamiltonian K-manifold

$\downarrow \Phi$ = moment map

$\mathfrak{k}^\vee$

$$\underset{\text{symplectic quotient}}{X/\!\!/K} := \Phi^{-1}(0)/K \underset{\text{Kempf-Ness}}{\cong} X^{ss}/G =: X/\!\!/G \quad \text{GIT quotient}$$

assume finite stabilizers

Ex: $X = \mathbb{C}^2$, $K = S^1$, $G = \mathbb{C}^\times$

$$\Phi(z_1, z_2) = 1 - \frac{1}{2}(|z_1|^2 + |z_2|^2)$$

$$X/\!\!/K = S^3/S^1 \cong \mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times$$

What objects "upstairs" in X correspond to holomorphic maps

complex curve $C \longrightarrow X/\!\!/K$?

$P \to C$ principal K-bundle

$\mathcal{A}(P)$ = {connections on P} $\ni A$

$\mathcal{S}$ = {sections of $P \times_K X$} $\ni u$

Twisted Cauchy-Riemann operator

$$\mathcal{S} \longrightarrow \bigcup_{u \in \mathcal{S}} \Omega^{0,1}(u^* T^{vert} P \times_K X)$$

$$u \longmapsto \bar{\partial}_A u$$

$$\mathcal{H} = \{ (A,u) \in \mathcal{A} \times \mathcal{S} \mid \bar{\partial}_A u = 0 \} \qquad (A,u)$$

$\circlearrowleft$

$\mathcal{K} = \left\{ \begin{matrix} K\text{-gauge} \\ \text{transformations} \end{matrix} \right\}$

$vol_C \in \Omega^2(C)$ area form

$k^\vee \cong k$ (choices)

$$\mathcal{H} \longrightarrow \Omega^2(C, P \times_K k), \qquad (A,u) \longmapsto \underbrace{F_A}_{\text{curvature}} + \Phi(u)\, vol_C$$

formal moment map for $\mathcal{K}$-action

$$\Phi(u) : C \longrightarrow P \times_K X \longrightarrow P \times_K k$$

$$M^K(C,X) = \bigcup_{(P)} \mathcal{H}/\mathcal{K} = \{ F_A + \Phi(u)\, vol_C = 0 \text{ solutions} \}/\mathcal{K}$$

$\downarrow$

moduli of symplectic vortices

Want: gauged Gromov-Witten invariants as integrals over $M^K(C,X)$

Problems:
- regularity
- compactness
- classification

§2. Mundet stability

X smooth projective G-variety, polarized

$$\left. \begin{matrix} P \quad G\text{-bundle} \\ u : C \to P \times_G X \quad \text{section} \end{matrix} \right\} \text{algebraic}$$

C projective curve over $\mathbb{C}$, smooth

$G \supseteq R$ parabolic subgroup $\qquad \sigma : C \to P/R$ reduction

$\sigma^* P \quad R$-bundle

1-parameter subgroup $z \mapsto z^{-\lambda}$, λ coweight for R

$P^z = \sigma^* P \times_R G$ where R-action on 2nd factor is twisted by $z^{-\lambda}$.

$Gr(P) = \lim_{z \to 0} P^z$ exists if λ is dominant

$u^z = z^{-\lambda} u$ section of $P^z \times_G X$

$Gr(u)$ Gromov limit of u^z as $z \to 0$

map $\hat{C} \to Gr(P) \times_G X$

$$\mu(\sigma, \lambda) = \int_C \underbrace{c_1 \left(Gr(P) \times_R \mathbb{C}_\lambda \right)}_{\text{Ramanathan term}} + \int_{\hat{C}} \underbrace{\langle \Phi(Gr(u)), \lambda \rangle_{Gr(u)} \pi^* vol_C}_{\text{Hilbert-Mumford term}}$$

(P, u) is Mundet stable (semistable) if $\mu(\sigma, \lambda) \leq 0$ ($<$) for all (σ, λ) s.t. $Gr(P)$, $Gr(u)$ exist.

(Also works for some vector spaces X.)

$$M^G(C, X) := \left\{ \begin{array}{l} \text{semistable pairs} \\ \text{up to isomorphism} \end{array} \right\} \xrightarrow[\text{bijection}]{\text{(Mundet's PhD thesis)}} M^K(C, X)$$

Schmitt's GIT construction ((coarse moduli space; initially defined as category))

If stable = semistable, $M^G(C, X)$ is a Deligne-Mumford stack (i.e. automorphisms are finite)

Special cases:

- $X = \{pt\}$ $\Rightarrow$ $M^G(C, X)$ moduli of semistable G-bundles
- $G = \{e\}$ $\Rightarrow$ $M^G(C, X)$ holomorphic maps $C \to X$.

EXAMPLE: $X = \mathbb{C}^2$, $G = \mathbb{C}^\times$, $C = \mathbb{P}^1$, $P = \mathcal{O}_{\mathbb{P}^1}(d)$, $d \in \mathbb{Z}$

$u : C \to P \times_G X$ section of $\mathcal{O}_{\mathbb{P}^1}(d)^{\oplus 2}$

stability condition: $R = \mathbb{C}^\times$, σ trivial, λ

then: $Gr(P) = P$

$$\int_C \mathrm{vol}_C \overset{!}{=} 1 \qquad Gr(u) = \begin{cases} 0 & u \equiv 0 \\ 0 & \lambda < 0 \\ \text{does not exist} & \lambda > 0 \end{cases} \quad \Big\} \, u \not\equiv 0$$

$$\int_C \langle c_1(P), \lambda \rangle + \int_{\hat{C}} \langle \Phi(Gr(u)), \lambda \rangle \; Gr(u)^* \pi^* \mathrm{vol}_C$$

$\hat{C} \longrightarrow$ here $\hat{C} = C$ (X affine)

$$= \begin{cases} \langle d, \lambda \rangle + \lambda \int_C \mathrm{vol}_C < 0 & \text{if } \lambda < 0,\ u \neq 0 \quad \text{semistable} \\ \langle d, \lambda \rangle + \lambda \int_C \mathrm{vol}_C > 0 & \text{if } \lambda > 0,\ u = 0 \quad \text{unstable} \end{cases}$$

$$M^G(C, X) = \left(H^0\left(\mathcal{O}_{\mathbb{P}}(d)^{\oplus 2}\right) \setminus \{0\} \right) / \mathbb{C}^\times$$
$$\cong \left(\mathbb{C}^{2(d+1)} \setminus \{0\} \right) / \mathbb{C}^\times$$
$$= \mathbb{P}^{2d+1}$$

much easier space than $M(C, X/\!\!/G)$

Compactifications:

- Schmitt quot-scheme compactification $\overline{M}^{quot, G}(C, X)$
  projective coarse moduli space
- Kontsevich style compactification $\overline{M}^G(C, X)$
  allows stable sections $u : \hat{C} \longrightarrow P \times_G X$

$$\begin{array}{ccc} \hat{C} & \longrightarrow & P \times_G X \\ & \searrow & \downarrow \\ & & C \end{array}$$

class $[C]$

This is a proper Deligne–Mumford stack once homology class of (P, u) is fixed:

$$C \longrightarrow P \times_G X \longrightarrow EG \times_G X \quad \text{is fixed}$$

$$[C] \longmapsto d$$

1st proof: "algebraic"

$$\overline{M}^G(C,X) \xrightarrow[\substack{\text{Givental} \\ \text{morphism} \quad (\text{proper})}]{} \overline{M}^{quot}_G(C,X) \quad \text{proper}$$

2nd proof: "symplectic"

$$\overline{M}^G(C,X) \xrightarrow{\sim} \overline{M}^K(C,X) \quad \text{compact Hausdorff by e.g. Ott + bit more}$$

Open question: Does $\overline{M}^G(C,X)$ have a GIT construction?

Marked version $\overline{M}^G_n(C,X)$: adds n-tuples $(z_1, \dots, z_n) \in \hat{C}$

$$\begin{array}{ccc} & \overline{M}^G_n(C,X) & \\ {}^{ev}\swarrow & & \searrow^{f} \\ (X/G)^n & & \overline{M}_n(C) \end{array} \qquad \left(\begin{array}{c} \text{i.e.} \\ X = G = pt \end{array} \right)$$

Behrend - Fantechi machinery: virtual fundamental classes

$$[\overline{M}^G_n(C,X,d)] \qquad d \in H_2^G(X,\mathbb{Z})$$

(cf. Olsson)

Another open question: does $\overline{M}^{quot}_G(C,X)$ also have virtual fundamental classes? ((maybe not))

Gauged Gromov-Witten invariants:

$$H_G(X)^{\times n} \times H(\overline{M}_n(C)) \longrightarrow \mathbb{Q}$$

$$(\alpha, \beta) \longmapsto \int_{[\overline{M}^G_n(C,X,d)]} ev^*\alpha \cup f^*\beta = \langle \alpha, \beta \rangle^G_{X,d}$$

Next: computations + relation to GW of $X/\!/G$.

Point of view on $\overline{M}_n(C)$ appears

$\uparrow$ stable maps to C of class $[C]$

Fulton - MacPherson compactification

Old: technical convenience / necessity

New: $\overline{M}_n(C)$ is supposed to appear because it is the complexification of the "cyclohedron"

relates to homotopy traces

Gauged GW-invariants define quantum traces on quantum cohomology.

Goal: relate gauged-GW of X with GW of $X/\!/G$.

EXAMPLE of gauged GW-invariants above (cf. p. $\boxed{4}$):

$X = \mathbb{C}^2$, $G = \mathbb{C}^\times$, $M^G(C, X, d) = \mathbb{P}^{2d+1}$

$C = \mathbb{P}^1$ $\quad M^G_3(C, X, d) = \mathbb{P}^{2d+1} \times \overline{M}_3(C)$

$$\downarrow ev$$

$$(X/G)^3 \simeq (BG)^3 = (\mathbb{P}^\infty)^3$$

$\zeta \in H^2(\mathbb{P}^\infty)$

$\beta \in H^6(\overline{M}_3(C))$ point class

$\alpha = (\zeta^a, \zeta^b, \zeta^c) \in H^2(X/G)^3$

$$\langle \alpha, \beta \rangle^G_{d,X} = \int_{[\mathbb{P}^{2d+1}]} \omega^a \cup \omega^b \cup \omega^c$$

...

$\omega^j \in H^2(\mathbb{P}^{2d+1})$
$= ev^j \zeta \quad j = 1, 2, 3$

$\ldots$

$$= \begin{cases} 1 & \text{if } a+b+c = 2d+1 \\ 0 & \text{otherwise} \end{cases}$$

cf. GW-invariants of $X/\!\!/G = \mathbb{P}^1$:

$$\langle \omega^a, \omega^b, \omega^c ; \beta \rangle_{d, X/\!\!/G} = 1 \quad \begin{cases} a,b,c \leq 1 \\ d=1 \quad a+b+c=3 \\ \text{or } d=0, \; a+b+c=1 \end{cases}$$

$\omega \in H^2(\mathbb{P}^1)$
point class

$\therefore$ unique map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree 1 mapping $0 \mapsto 0$, $1 \mapsto 1$, $\infty \mapsto 1$

Goal: explain discrepancy.

§2. Large area limit, "$\mathrm{vol}_C \to \infty$" Gaio-Salamon

NB: $F_A + \Phi(u)\, \mathrm{vol}_C = 0 \rightsquigarrow \Phi(u) = 0 \rightsquigarrow M^G(C,X)$
$\mathrm{vol}_C \downarrow \infty$ $\qquad$ $M^G(C,X) \downarrow M(C, X/\!\!/G)$

with bounded energy!

Suppose $e_\nu \to \infty$, (A_ν, u_ν) vortices for $e_\nu \,\mathrm{vol}_C$ with bounded energy in general: $\underbrace{\sup |d_{A_\nu} u_\nu|}_{c_\nu} < \infty \Rightarrow$ good case

case	limit	c_ν if $c_\nu \to \infty$ rescale, so finite
$e_\nu / c_\nu \to \infty$	hol. map to $X/\!\!/G$	
$e_\nu / c_\nu \to$ finite $\neq 0$	vortex on $\mathbb{C}$ with $\mathrm{vol}_C = dx \wedge dy$	
$e_\nu / c_\nu \to 0$	hol. map to X	

elements of $M^G(C,X)$ converge to $M(C, X/\!\!/G)$

+ vortex bubbles + bubbles in X

Ziltener: $M^K_n(\mathbb{C},X)$ n-marked affine vortices

$$\begin{tikzcd}
M^K_n(\mathbb{C},X) \arrow[d, phantom, "\cap"] & \\
\overline{M}^K_{n,1}(\mathbb{C},X) \arrow[r, "ev"] \arrow[dr, "ev_\infty"'] \arrow[ddr, "f"'] & (X/K)^n \\
& X/\!\!/K \\
& \overline{M}_{n,1} \quad X = K = pt
\end{tikzcd}$$

Algebraic case: Venugopalan + Woodward

$M^K_{n,1}(\mathbb{C},X) \xrightarrow{\text{bijection}} \left(\begin{array}{c}\text{coarse moduli}\\ \text{of}\end{array}\right)$ $M^G_{n,1}(\mathbb{C},X)$

$$M^G_{n,1}(\mathbb{C},X) = \left\{\begin{array}{l} P \to \mathbb{P}^1 \\ u : \mathbb{P}^1 \to P \times_G X \\ u(\infty) \in X/\!\!/G \end{array}\right\} \Big/ \text{equivalence}$$

by Ziltener correspondence algebraic proof?

Proper DM-stack with virtual fundamental classes

$\Rightarrow$ so can define invariants

$$QH^G(X) = H^G(X) \otimes \left(\begin{array}{c}\text{formal series}\\ \text{in } q\end{array}\right)$$

$$K : QH^G(X) \to QH(X/\!\!/G)$$

$$\alpha \longmapsto \sum_{d,n} ev_{\infty,*}\, ev^*(\underbrace{\alpha,\dots,\alpha}_{n \text{ times}}) \frac{q^d}{n!}$$

EXAMPLE: $X = \mathbb{C}^2$, $G = \mathbb{C}^\times$, $P = \mathcal{O}_{\mathbb{P}^1}(d)^\times$ again

$$M_1^G(\mathbb{C},X) = \left\{ \begin{array}{l} u=(u_1,u_2) = (a_d z^d + \text{lower order}, \ b_d z^d + \text{lower order}) \\ \mathbb{C}^{2d} \text{ bundle over } \mathbb{P}^1 \qquad (a_d, b_d) \neq 0 \end{array} \right\} / \mathbb{C}^*$$

cf. Jaffe–Taubes, Guangbo Xu

Properties of quantum Kirwan map K

- $K|_{q=0}$ is the classical Kirwan map $H^G(X,\mathbb{Q}) \to H(X/\!\!/G, \mathbb{Q})$

  surjective

- $DK(\alpha * \beta) = DK(\alpha) * DK(\beta)$ $\qquad \alpha, \beta \in QH_G(X)$

NB: $D_0K : T_0 QH^G(X) \longrightarrow T_{K(0)} QH(X/\!\!/G)$

is defined by integration over $\overline{M}^G_{1,1}(\mathbb{C},X)$

NB: $\overline{M}_{n,1}(\mathbb{C})$ is the complexified "multiplihedron"

$$\begin{array}{ccc} & \overline{M}^G_{1,1}(\mathbb{C},X) & \\ {}^{ev}\swarrow & & \searrow^{ev_\infty} \\ X/G & & X/\!\!/G = \mathbb{P}^1 \\ \wr & & \\ \mathbb{P}^\infty & & \end{array} \qquad G = \mathbb{C}^*, \ X = \mathbb{C}^2$$

$\xi \in H^2(\mathbb{P}^\infty)$ $\qquad ev_1^* \xi^{2d} = \mathrm{Eul}(\mathbb{C}^{2d}_{\text{fibre}})$

$\shortparallel$

$\mathrm{Eul}(\mathbb{C})$

$$\int_{\overline{M}^G_{1,1}(\mathbb{C},X)} \mathrm{Eul}(\mathbb{C}^{2d}) \cup ev_\infty^*(\cdot) \qquad \int_{S^{-1}(0)} ev_\infty^*(\cdot)$$

Define:

$s(u)$ = derivatives of u at 0 of degree $< d$

$$S^{-1}(0) = \{ (a_d z^d, b_d z^d) \} / \mathbb{C}^\times \cong \mathbb{P}^1$$
$$a_d, b_d \neq 0$$

$$DK(\xi^{2d}) = q^d [\mathbb{P}^1]$$
$$DK(\xi^{2d+1}) = q^d [\{pt\}]$$

So can compare now gauged-GW of X with GW of $X/\!\!/G$:

$$\sum_d q^d \langle DK(\xi^a), DK(\xi^b), DK(\xi^c); pt \rangle_{X/\!\!/G, d} = \sum_d q^d \langle \xi^a, \xi^b, \xi^c; pt \rangle^G_{X,d}$$

This equality is a special case of:

Adiabatic limit THM: the diagram

$$\begin{tikzcd} QH_G(X) \arrow[rr, "K"] \arrow[dr, "\tau^G_X"'] & & QH(X/\!\!/G) \arrow[dl, "\tau_{X/\!\!/G}"] \\ & \text{formal power series in } q & \end{tikzcd}$$

$$\tau^G_X(\alpha) = \sum_{d,n} \frac{q^d}{n!} \langle \underbrace{\alpha, \dots, \alpha}_{n}, 1 \rangle^G_{X,d}$$

$$\tau_{X/\!\!/G}(\alpha) = \sum_{d,n} \frac{q^d}{n!} \langle \underbrace{\alpha, \dots \alpha}_{n}, 1 \rangle_{X/\!\!/G, d}$$

commutes in the limit "$\mathrm{vol}_C \to \infty$".

idea of proof: show $\overline{M}^G_n(C, X)$ is cobordant to $\overline{M}_n(C, X/\!\!/G)$ + vortex bubbles

Gaio – Salamon $\qquad c_1 \gg 0$ cases

Givental, Lian-Liu-Yau are in some sense a special case.

((Caveat: unlike ungauged GW-theory, extension to higher genus from $g=0$ case is still uncharted (not automatic)))

I function	J-function	X vector space
$\wr$	$\wr$	G torus
corresponds to τ_X^G	$\tau_{X/\!\!/G}$	$c_1 \geq 0$

K is determined by commutativity $\longrightarrow$ computable!

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§3. Application to quantum cohomology of toric varieties

X vector space

G torus, weights $\beta_1, \dots, \beta_n$ contained in a half space and span $\mathfrak{g}^\vee$

$X/\!\!/G$ proper smooth toric DM-stack

Homomorphism $\qquad D\kappa : QH^G(X) \longrightarrow QH(X/\!\!/G)$

cf. Batyrev's quantum Stanley–Reisner ideal:

$$QSR = \Big\langle \prod_{\langle d, \beta_i\rangle > 0} (\xi^{\beta_i})^{\beta_i(d)} - q^d \prod_{\langle d, \beta_i\rangle < 0} (\xi^{\beta_i})^{-\beta_i(d)} \Big\rangle$$

THM: After suitable completion, $QH^G(X)/_{QSR} \xrightarrow{\sim} QH(X/\!\!/G)$

EXAMPLE $\quad X = \mathbb{C}^2, \quad G = \mathbb{C}^\times, \quad \beta_1 = \beta_2 = 1$

$$QSR = \langle \xi^2 - q \rangle$$

$$QH(\mathbb{P}^1) = \mathbb{Q}[\xi, q] / (\xi^2 - q)$$

Batyrev : monotone case

Givental LLY $\quad c_1 \geq 0$

McDuff – Tolman – Iritani $\quad$ toric varieties

González – W $\left(\begin{array}{c} \text{Iritani} \\ \text{Ciocan-Fontanine} \\ \text{Kim} \end{array} \right)$ toric stacks

Sketch of proof:

(1) DK is surjective — by explicit computation (unknown if true in general)

(2) $QSR \subset \ker DK$ $\quad$ uses adiabatic theorem above

(3) $\dim QH_G(X) / QSR = \dim QH(X /\!\!/ G)$

This is done by reducing to the Fano case.
Here, there is an alternative formulation using GL models that makes use of Kouchnirenko's theorem.

Check both sides change in the same way under flips
(cf. minimal model programme) $\quad$ cf. Delzant polytope
((understand non-displaceable Lagrangian tori))