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"Classification of affine vortices"

(+ Chris Woodward)

- $\begin{array}{c} P \\ \downarrow \\ \Sigma \end{array}$ principal K-bundle, Riemann surface. K compact Lie group, $K \subseteq K^{\mathbb{C}} =: G$

- (X, ω) Kähler manifold with Hamiltonian action of K

$$\Phi : X \longrightarrow k^{\vee} \cong k$$

K-action extends to a holomorphic action of G

- Gauged holomorphic map from P to X: (A, u) where

A = connection on P

u = holomorphic section $\in \Gamma(\Sigma, P \times_K X)$

$\bar\partial_A u = 0$

- Symplectic vortex is (A, u) s.t. $\quad * F_A + \Phi(u) = 0$

- Energy $\quad E(A, u) = \int_\Sigma |F_A|^2 + |d_A u|^2 + |\Phi(u)|^2$

- $K(P)$ = {gauge transformations}

$\mathcal{G}(P)$ = {complexified gauge transformations}

If Σ is compact, stability condition is dependent on $\tau \in \mathbb{R}$.

Not so when $\Sigma = \mathbb{C}$ with $\omega = dx \wedge dy$.

Jaffe-Taubes: $\Sigma = \mathbb{C}$, $X = \mathbb{C}$, $K = U(1) \curvearrowright X$ linearly

Moduli space of finite-energy vortices with vortex number N is

$$\mathcal{M}_N \cong \mathbb{C}^N / \mathfrak{S}_N \quad , \quad N = \frac{i}{2\pi} \int_\Sigma F_A$$

$$\begin{array}{ccccc} \mathbb{C} \times K & & P & & \\ \downarrow & \text{extends to} & \downarrow & \text{with} & c_1(P) = N \\ \mathbb{C} & & \mathbb{P}^1 & & \end{array}$$

and (A, u) extend to a gauged holomorphic map $\begin{array}{c} P \longrightarrow X \\ \downarrow \\ \mathbb{P}^1 . \end{array}$

Our work: Generalizing Jaffe-Taubes for K = any compact Lie group and $\Sigma = \mathbb{C}$.

$X =$
- compact Kähler manifold with Hamiltonian K-action
- $X = \mathbb{C}^n$ with linear $\underbrace{\text{convex}}_{\text{means: } \Phi \text{ is proper}}$ action of K

We assume $X^s = X^{ss}$, i.e.

G acts on X^{ss} with finite stabilizers

i.e.

K acts on $\Phi^{-1}(0)$ with finite stabilizers

$$X^{ss} = G \cdot \Phi^{-1}(0) \qquad \text{if } X \text{ is Kähler}$$

(stability condition $u(\infty) \in X^{ss}$.)

Main THM: Suppose n is an integer s.t. $\forall_{x \in X^{ss}}$ $|G_x|$ divides n.

a) Suppose
$$\begin{tikzcd} \bar{P} \arrow[r, "(A,u)"] \arrow[d] & X \\ \mathbb{P}(1,n) & \end{tikzcd}$$
s.t. $u(\infty) \in X^{ss}$

Then by applying a complex gauge transformation $g \in \mathcal{G}_{\mathbb{C}}(\bar{P})$, $g(A,u)|_{\mathbb{C}}$ is a finite-energy vortex.

b) Conversely, for any finite energy vortex (A,u) over $\mathbb{C}$, after applying $g \in \mathcal{G}(P)$, $g \cdot (A,u)$ extends to some
$$\begin{tikzcd} \bar{P} \arrow[d] \\ \mathbb{P}(1,n) \end{tikzcd}$$

NB: Jaffe–Taubes:
$$\left\{ \begin{matrix} \text{degree } N \\ \text{vortices} \end{matrix} \right\} \Big/ K \longleftrightarrow \left\{ \begin{tikzcd} \tilde{P} \arrow[r] \arrow[d] & X = \mathbb{C} \\ \mathbb{P}^1 & \end{tikzcd} \right\} \Big/ \mathcal{G}_{\mathbb{C}}$$
$$= \left\{ \begin{matrix} \text{sections } u \text{ on a degree } N \\ \text{line bundle given by zeroes of } u \end{matrix} \right\}$$

Outline of PROOF:

(a)
$$\begin{tikzcd} \bar{P} \arrow[r, "(A,u)"] \arrow[d] & X \\ \mathbb{P}(1,n) & \end{tikzcd}$$

After applying $g \in \mathcal{G}(P)$, $A = d + \lambda\, d\theta$ on $\mathbb{C} \setminus B_R$, $\exists_{R>0}$

$$\lim_{r \to \infty} e^{-\lambda\theta} u(r,\theta) = x_0$$

$e^{2\pi\lambda}$ stabilizes x_0, so it has finite order

Step 1: i) By using another $g' \in \mathcal{G}(P)$

$$\|{*F_A} + \Phi(u)\|_{L^2(\mathbb{C})} < \infty$$

ii) By an implicit function theorem argument for large $R>0$, we find $\xi : \mathbb{C} \setminus B_R \to k$ s.t. $e^{i\xi}(A,u)$ is a vortex on $\mathbb{C} \setminus B_R$.

We rely on:

$$\mathrm{Id} + d^*d : W^{2,p}_\delta(\mathbb{C}\setminus B_1, \mathbb{C}) \to L^p(\mathbb{C}\setminus B_1, \mathbb{C})$$

is an isomorphism

Step 2: We use:

THM 1(V.) (Hitchin–Kobayashi correspondence for symplectic vortices on Σ with boundary):

Σ compact

Given a gauged holomorphic map
$$\begin{array}{ccc} \Sigma \times K & \xrightarrow{(A,u)} & X \\ \downarrow & & \\ \Sigma & & \partial\Sigma \neq \emptyset \end{array}$$

with $*F_A + \Phi(u) = 0$ on $\partial\Sigma$, then

$\exists\ \xi \in W^{2,p}_\delta(\Sigma, k)$ s.t. $e^{i\xi}(A,u)$ is a vortex on Σ.

$\xi|_{\partial\Sigma} = 0$

NB: ξ does not extend to $W^{2,p}_{loc}(\mathbb{C})$.

Apply above theorem to a sequence $B_i \subseteq B_{i+1} \subseteq \cdots \subseteq \mathbb{C}$, $i > R$ to get vortices (A_i, u_i) on B_i.

We show that limit of (A_i, u_i) is a vortex on $\mathbb{C}$, with no bubbling.

(6) Converse: relies on asymptotic behaviour of finite-energy vortices [Gaio-Salamon, Ziltener] $\square$

((NB: n appears as degree of cover of orbifold quotient:
$u(\infty) \in [X/\!\!/G] \longrightarrow$ need degree n cover))

THM 1 is proved using gradient flow of $\| * F_A + \Phi(u) \|^2_{L^2}$ for gauged holomorphic maps on compact Σ with boundary.

Finite dimensions: gradient vector of $|\Phi|^2$ is $J\Phi(x)_X$; this preserves G-orbits.

Flow equations for $(A, u) \longmapsto \| \underbrace{* F_A + \Phi(u)}_{=: F} \|^2_{L^2}$

$$\frac{dA}{dt} = * d_A F$$

$$\frac{du}{dt} = J F_u$$

If Σ has boundary, $F(0)|_{\partial\Sigma} = 0$ is imposed.

Initial conditions: $A(0) = A_0$, $u(0) = u_0$, $\bar\partial_{A_0} u_0 = 0$.

THM 2: a) The above system has a solution for all time:
$t \mapsto (A_t, u_t) \in C^\infty([0,\infty), H^1 \times C^0)$.

b) After applying $k_t \in H^2(K)$, $k_t(A_t, \mu_t)$ is smooth on $[0,\infty) \times \Sigma$.

Convergence behavior:

THM 3: Let $(A_t, \mu_t) \in C^\infty$ be the gradient flow modulo gauge. Then $\exists\, t_i \to \infty$, $k_i \in K_{H^2}$,

$(A_\infty, \mu_\infty) \in \mathcal{A}(P)_{H^2} \times \Gamma(P(X))_{C^1}$ s.t.

a) $k_i A_i \xrightarrow{H^2} A_\infty$

b) If Σ has no boundary: $k_i \mu_{t_i}$ Gromov-converges.

μ_∞: principal component

$\Sigma \supset Z$ = finite bubbling set

$k_i \mu_{t_i} \to \mu_\infty$ in $C^1(K)$, $K \subseteq \Sigma \setminus Z$
compact subset

c) If Σ has boundary: $Z = \emptyset$

$k_i \mu_{t_i} \longrightarrow \mu_\infty$ in $C^1(\Sigma)$

d) (A_∞, μ_∞) is a critical point of $\|F\|^2_{L^2}$, i.e.

$d_{\tilde{A}_\infty} F_\infty = 0$ and $(F_\infty)_{\mu_\infty} = 0$.

when $\partial\Sigma \neq \emptyset$, $F_\infty = 0$

$(A_\infty, \mu_\infty) \in \mathrm{cl}_{\mathcal{G}}(A_0, \mu_0)$.

((NB: $F_i \xrightarrow{L^p} 0 \rightsquigarrow d^* d : W^{2,p} \to L^p$ invertible if Σ compact.))