29.11.12

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"Quiver gauge theories and nonabelian vortices"

§1. Dimensional reduction

* Kaluza-Klein: $M_4 \times S^1 \longrightarrow M_4$

  Einstein $\qquad$ Einstein + Maxwell

* Coset-space dimensional reduction:

  (Witten, Forgács & Manton, ...)

  $M \times G/H \longrightarrow M$

  Yang-Mills $\qquad$ Yang-Mills-Higgs

  gauge group $\mathcal{G}$ $\qquad$ gauge group $K \subset \mathcal{G}$, $[K, H] = 0$

  $H \hookrightarrow \mathcal{G}$

* String theory:

  Heterotic strings $\quad M_{10} \longrightarrow \mathbb{R}^{1,3}$

  (X, Ω) Kähler

  $$\begin{array}{ll} \mathcal{E} & A \text{ connection on } \mathcal{E} \\ \downarrow & \\ X & F_A = \text{curvature} \end{array}$$

  HYM equations: $\quad \Omega \lrcorner F_A = \lambda$

  $$F_A^{0,2} = 0$$

  Donaldson-Uhlenbeck-Yau:

  irreducible HYM connections $\longleftrightarrow$ μ-stable holomorphic vector bundles $\mathcal{E}$

Condition enforced by supersymmetry: $\lambda = 0$
DUY equations

Flux compactifications: non-Kähler deformations

§2. Equivariant bundles and quiver gauge theories

Equivariant dimensional reduction (cf. García-Prada's talk):

$$\underbrace{M \times G/H}_{\substack{\text{reducible representation of } H \hookrightarrow \text{ irrep of } G}} \longrightarrow M \qquad G \text{ reductive}$$

YM theory $\rightsquigarrow$ quiver gauge theory

G-equivariant bundles on G/H:

$$\begin{array}{ccc} \mathcal{L}_q & = & G \times_H V_q \\ \downarrow & & \wr\downarrow \\ G/H & & \text{irrep of } H \end{array}$$

$$\begin{array}{ccccc} \xi & G\text{-equivariant} & \longleftrightarrow & E & H\text{-equivariant} \\ \downarrow & & \xleftarrow{\text{induction}} & \downarrow & \\ X = M \times G/H & & \xrightarrow[\text{restriction}]{} & M & \end{array}$$

G, H act trivially on M

$$\xi = G \times_H E = \bigoplus_{r=0}^{m} E_q \boxtimes \mathcal{L}_{q_r}$$

$$\begin{array}{c} E_q \\ \downarrow \\ M \end{array} \quad \text{bundles with trivial } H\text{-action.}$$

Take V_{q_r} to descend from irrep $\hat{V}$ of G: $\hat{V}|_H = \bigoplus_{r=0}^{m} V_{q_r}$

$$\mathfrak{h} = SU(k) \longrightarrow \prod_{r=0}^{m} U(k_r)/U(1) \quad , \quad k = \sum_{r=0}^{m} k_r d_r$$

$$k_r = \mathrm{rk}(E_r) \quad , \quad d_r = \dim(V_{q_r})$$

Coset space geometry:

$$h = \dim_{\mathbb{R}}(G/H)$$

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \qquad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$$

$$\mathfrak{m}^* = T_0^*(G/H)$$

Basis $\quad I_A \quad , \quad A = 1, \dots, \dim G$

$$\hookrightarrow \quad I_a \quad , \quad a = 1, \dots, h \qquad \text{basis for } \mathfrak{m}$$

$$I_i \quad , \quad i = h+1, \dots, \dim G \qquad \text{basis for } \mathfrak{h}$$

$$[I_A, I_B] = f_{AB}^{\ C}\, I_C$$

1-forms $\quad e^a \ , \ e^i$

e^a: basis for $T_0^*(G/H)$

$e^i \rightsquigarrow a^0 = e^i I_i$ canonical connection on H-bundle $G \to G/H$

$$h = 2n, \quad \theta^\alpha := e^{2\alpha} + i e^{2\alpha - 1} \qquad \alpha = 1, \dots, n$$

$$J\theta^\alpha = i\theta^\alpha \qquad \text{assume complex structure}$$

and $\quad \omega_{G/H} = \frac{i}{2} \sum_\alpha \theta^\alpha \wedge \bar{\theta}^\alpha \ , \quad g = \sum_\alpha \theta^\alpha \otimes \bar{\theta}^\alpha \quad$ Kähler

G-equivariant connections:

$$\hat{V}|_H = \bigoplus_{r=0}^{m} V_{q_r}$$

generators: $\hat{I}_i = \begin{pmatrix} I_i^{q_r} & & 0 \\ 0 & & \end{pmatrix}$ act on V_{q_r}

$$\hat{I}_a = \begin{pmatrix} 0 & \ddots & I_a^{q_{rs}} \\ & \ddots & \\ I_a^{q_{sr}} & \ddots & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} A_r \otimes \mathbb{1}_{q_r} & 0 \\ 0 & \end{pmatrix} \qquad A_r \text{ connection on } E_r \to M$$

General form: $\mathcal{A} = A + \underset{\underset{\tilde{a}\,\circ}{\uparrow}}{\hat{I}_i}\, e^i + X_a e^a$

$$\mathcal{F}_{\mathcal{A}} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

$$\overset{!}{=} F + (dX_a + [A, X_a]) \wedge e^a$$

$$- \frac{1}{2} (\dots) e^a \wedge e^b$$

$$+ \big(\underbrace{[\tilde{I}_i, X_a] - f_{ia}^{\ b} X_b}_{0} \big) e^i \wedge e^a$$

$$\Rightarrow \quad X_a = \begin{pmatrix} 0 & \ddots & \phi_{sr} \otimes I_a^{q_{rs}} \\ & \ddots & \\ \phi_{sr} \otimes I_a^{q_{sr}} & - & 0 \end{pmatrix} \qquad \phi_{rs} \in \mathrm{Hom}(E_r, E_s \qquad \phi_{sr}^{\dagger}$$

This defines a quiver (Álvarez-Cónsul + García-Prada)

Lechtenfeld - Popov - S.

$$Q = (Q_0, Q_1, R)$$

$Q_0 = \{\text{vertices}\}$ irreps V_{q_r}

$Q_1 = \{\text{arrows}\}$ $\beta_{rs} = e^{\alpha} I_{\alpha}^{q_{rs}}$

$$\cdots \; r \xrightarrow{\Phi_{rs}} s \; \cdots$$

$$\Phi_{rs} = \phi_{rs} \otimes \beta_{rs}$$

$r \longrightarrow (A^r, E_r)$

$s \longrightarrow (A^s, E_s)$

R = relations
$\parallel$
"commutative quiver diagram"

NB: $G/H \cong G^{\mathbb{C}}/P$

$P \subset G$ parabolic subgroup

$\mathfrak{p} = \mathfrak{h}^{\mathbb{C}} \oplus \mathfrak{n}$ $\quad \mathfrak{n} = \Lambda^{0,1} T_0^*(G^{\mathbb{C}}/P)$

$[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}$ canonical complex structure on coset.

Q_1 = actions of $\mathfrak{n}$

$R = [\mathfrak{n}, \mathfrak{n}]$

Ex.: $G/H = SU(2)/U(1) = S^2 = \mathbb{P}^1$

q-monopole bundles $\mathcal{L}^q := \mathcal{L}^{\otimes q}$

$$\begin{array}{c} S^1 \hookrightarrow S^3 \\ \downarrow \\ S^2 \end{array} \quad \text{Hopf fibration} \qquad \xi = \bigoplus_{r=0}^{m} E_r \boxtimes \mathcal{L}^{q_r}$$

q-monopole connection f_q, $c_1(\mathcal{L}^q) = q$

$\dim(\hat{V}) = m+1$

$\rightsquigarrow$ decomposition of $U(1)$-ireps with charges $q_r = m - 2i$

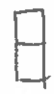
$i = 0, 1, \dots, m$

Chains: $\underset{0}{\circ} \xrightarrow{\Phi_{01}} \underset{1}{\circ} \xrightarrow{\Phi_{12}} \cdots\cdots \longrightarrow \underset{m}{\circ}$

- $G/H = F_3 = SU(3)/U(1)^2$

Q = weight diagram of $\hat{V}$

$\hat{V}$



$\Phi_{13} = \Phi_{12}\,\Phi_{23}$

§3. Quiver vortex equations

$G/H = \mathbb{CP}^1$, DUY equations on $M \times G/H$

$\rightsquigarrow$ coupled vortex equations on M:

(M, ω_M) Kähler

$\Omega = \omega_M + \omega_{G/H}$

$$\begin{cases} \omega \lrcorner F_r = \frac{1}{2}\left(m - 2r + \phi_r^\dagger \phi_r - \phi_{r+1}\phi_{r+1}^\dagger\right) \\ F_r^{2,0} = 0 \\ \bar{\partial}\phi_{r+1} + A_r^{0,1}\phi_{r+1} - \phi_{r+1}A_{r+1}^{0,1} = 0 \end{cases}$$

- $m = 1 \longrightarrow$ triples
- $\dim_{\mathbb{C}} M = 1$, $k_0 = k_1 = 1 \longrightarrow$ abelian vortex equations
- $\dim_{\mathbb{C}} M = 2$, $k_0 = k_1 = 1 \Rightarrow A_0 = -A_1 =: A \Rightarrow U(1)$

$\longrightarrow$ perturbed SW monopole equations

§4. Nearly Kähler reduction:

$$\begin{array}{ccc} \mathbb{P}^1 & \hookrightarrow & F_3 \\ & & \downarrow \\ & & \mathbb{P}^2 \end{array}$$

twistor space of $\mathbb{P}^2$ (flag manifold)

Frame $\hat{\theta}^\alpha$ on F_3 = pullback from $\mathbb{CP}^2$

Complex structure: $J_+ \theta^\alpha = +i\,\theta^\alpha$

Kähler structure: $\hat{\omega} = \sum_\alpha \hat{\theta}^\alpha \wedge \overline{\hat{\theta}^\alpha}$

Rotate $J_+ \longmapsto J_-$ along $\mathbb{P}^1$ fibre:

$$\theta^{1,2} = \hat{\theta}^{1,2}, \quad \theta^3 = \overline{\hat{\theta}^3}$$

$J_- \theta^\alpha = +i\,\theta^\alpha$ almost complex structure, non-integrable

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus (\mathfrak{m}^+ \oplus \mathfrak{m}^-)$$

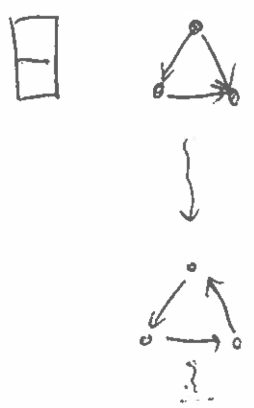
$$\mathfrak{m}^+ = \Lambda^{1,0} T_0^*(G/H)$$

$$[\mathfrak{m}^\pm, \mathfrak{m}^\pm] \subset \mathfrak{m}^\mp$$

non-integrable

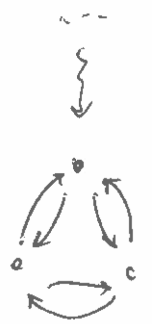
$Q \rightsquigarrow$ double quiver $\overline{Q}$ for every arrow, draw arrow in opposite direction

Ex.:



$$\Phi_{13} = \Phi_{12}\,\Phi_{23}$$

$$\Phi_{13} = \Phi_{23}^\dagger\,\Phi_{12}^\dagger$$



$\bar{R} \equiv$ commutativity of $\bar{Q}$

Nearly Kähler: $$\omega = \frac{i}{2} \sum_{\alpha} \theta^{\alpha} \wedge \bar{\theta}^{\alpha}$$

$$d\omega = \frac{\sqrt{3}}{2} \operatorname{Im} \Omega \quad , \ \Omega = \theta^1 \wedge \theta^2 \wedge \theta^3$$

$$d\Omega = \frac{1}{\sqrt{3}} \omega \wedge \omega$$

$$\begin{array}{c} (E, A) \\ \downarrow \\ M \times F_3 \end{array} \qquad F^{0,2} = 0 \qquad \curvearrowright \qquad \text{pseudoholomorphic bundle } \mathcal{E}$$

there should be a correspondence between homogeneous pseudoholomorphic bundles on G/H and representations of the double quiver $\bar{Q}, \bar{R}$.