30.11.12

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"Non-local vortices via holonomy perturbations"

Goal: Define GW-invariants for monotone Hamiltonian G-manifolds, G any compact Lie group, extending previous definitions by Mundet-Tian (S^1-actions) and Cieliebak-Gaio-Salamon (G-actions on symplectic aspherical manifolds).

The symplectic vortex equations

$$\begin{array}{ccc} G \circlearrowright P & \xrightarrow{u} & (M, \omega, \mu) \\ \downarrow & & \\ (\Sigma, j_\Sigma) & & \end{array}$$

M compact

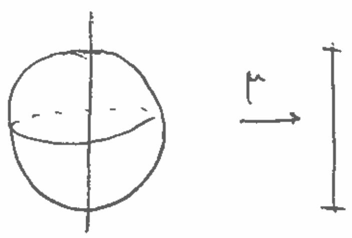
G compact Lie group

Σ compact Riemann surface with fixed complex structure j_Σ

P G-principal bundle over Σ

(M, ω, μ) Hamiltonian G-manifold, $\mu : M \longrightarrow \underset{G\text{-equiv.}}{\mathfrak{g}^* \cong \mathfrak{g}}$

Ex: $U(1) \circlearrowright \mathbb{CP}^1 \cong S^2$ by rotations



$\mu(\underline{x}) = i x_3 \in \mathfrak{u}(1) = i\mathbb{R}$

Fix a G-invariant ω-compatible almost complex structure $J : TM \to TM$

For pairs (A, u), $A \in \Omega^1(P, \mathfrak{g})$ connection on P, $u : P \to M$ G-equivariant, have vortex equations:

$$\begin{cases} \bar\partial_{J,A}(u) := \frac{1}{2}(d_A u + J(u) \circ d_A u \circ j_\Sigma) = 0 \\ F_A + \mu(u)\, dvol_\Sigma = 0 \end{cases}$$

$dvol_\Sigma$ = fixed area form

Basic energy identity:

$$E(A,u) := \frac{1}{2}\int_\Sigma \left(|F_A|^2 + |d_A u|^2 + |\mu(u)|^2 \right) dvol_\Sigma$$

$$= \int_\Sigma \left(|\bar\partial_{J,A}(u)|^2 + \frac{1}{2}|F_A + \mu(u)\, dvol|^2 \right) dvol_\Sigma + \langle \underset{\substack{\cap \\ H^2_G(M)}}{[\omega-\mu]^G}, \underset{\substack{\cap \\ H^G_2(M)}}{[u]_G} \rangle$$

Moduli space:

- fix a class $B \in H^G_2(M;\mathbb{Z}) \rightsquigarrow$ G-bundle $P \to \Sigma$
- fix complex structure j_Σ on Σ.
- gauge transformations $\mathcal{G} = \{$ equivariant maps $g : P \to G \}$ acts on pairs (A,u) by
  $$g^*(A,u) = (g^{-1}Ag + g^{-1}dg,\ g^{-1}u)$$
- $\mathcal{M}(B,J) = \{$ vortices (A,u) s.t. $[u]_G = B \}/\mathcal{G}$

NB: $\mathcal{M}(B,J)$ is in general not compact

THM: (Cieliebak – Gaio – Salamon):

Assume B non-torsion, and that:

- G acts freely on $\mu^{-1}(0)$
- $dvol_\Sigma \underset{B}{\gg} 0$

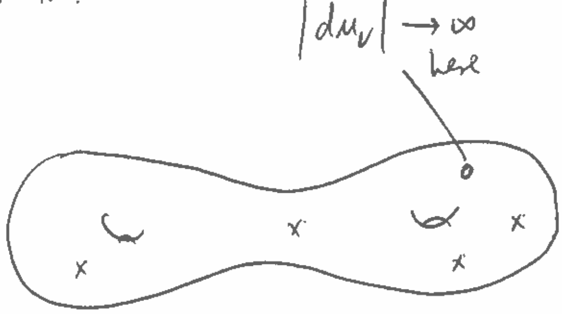
Then, for generic G-invariant J_z $(z \in \Sigma)$, $\mathcal{M}(B,J)$ is a manifold of dimension

$$\left(\frac{1}{2}\dim M - \dim G\right)\chi(\Sigma) + 2\langle c_1^G(TM), B\rangle$$

((NB: such z-dependent perturbations are standard in usual GW theory.))

THM (Mundet–Tian for $G = S^1$; 0):

Every sequence of marked vortices with uniformly bounded energy has a subsequence Gromov-convergent to polystable vortex.

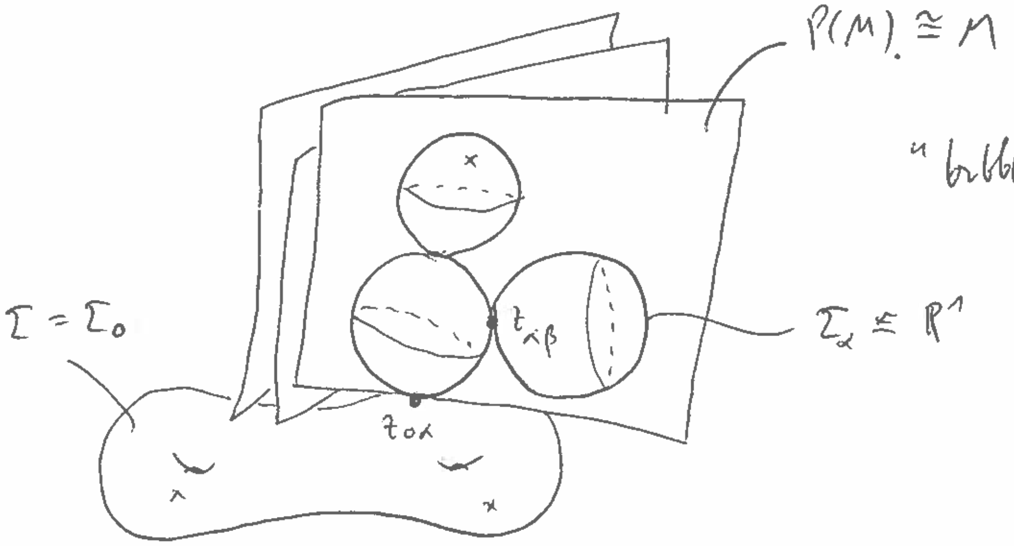


$$P(M) = P \times_G M$$

$$\begin{array}{c} u_\nu \circlearrowleft \ \downarrow \\ \Sigma \end{array}$$

$$(A_\nu, u_\nu, \underline{z}^\nu) \longrightarrow (A, u, \underline{z})$$

$$\underline{z}^\nu = (z_1^\nu, \dots, z_n^\nu)$$



DEF.: A polystable vortex $(A, u, \underline{z})$ consists of:

- normalized nodal curve $\{\Sigma_0 = \Sigma\} \cup \{\Sigma_\alpha \cong \mathbb{P}^1\}_{\alpha \in S}$
- smooth marked points $z_i \in \Sigma_\alpha$, $\alpha \in \{0\} \cup S$
- vortex (A, u) on $P \to \Sigma = \Sigma_0$
- J-holomorphic spheres $u_\alpha : \Sigma_\alpha \to P(M)_{z_{0\alpha}} \cong M$, $\alpha \in S$

s.t.
- $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$
- $u_\alpha = \text{const.}, \alpha \in S \Rightarrow \#(\text{nodal points} + \text{marked points}) \geq 3$

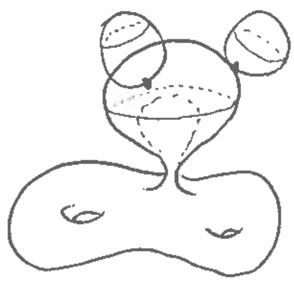
DEF.: Gromov convergence: $\exists$ sequence of gauged transformations $g_\nu \in \mathcal{G}$ s.t.

- $g_\nu^* A_\nu \to A$ on Σ_0
- $g_\nu^{-1} u_\nu \to u$ on compact subsets of $\Sigma_0 \setminus \{\text{nodal points}\}$
- $(g_\nu^{-1} u_\nu, z^\nu_\nu) \to (u, \underline{z})$ usual Gromov convergence in $P(M)$
- $E(A_\nu, u_\nu) \to E(A, u) + \sum_{\alpha \in S} E(u_\alpha)$

Ingredients in proof:

L^p bound on F_{A_ν} $\qquad F_{A_\nu} + \mu(u_\nu)\, d\mathrm{vol}_\Sigma = 0$ $\qquad$ look only at the 2nd eq.

Uhlenbeck compactness $\rightsquigarrow g_\nu, A$



finite number of bubbles

Removal of singularities: can smooth out where bubbles touch $\Sigma = \Sigma_0$

$\square$

Towards the invariants: need <u>evaluation map</u>.

$$\begin{array}{ccccc}
\mathcal{M}_n^{fr}(B,J) = \left\{ \begin{array}{l} \text{framed vortices } (A, u, p_1, \dots, p_n) \\ [u]_G = B \end{array} \right\} \Big/ \mathcal{G} & \xrightarrow{ev^{fr}} & M^n \\
& \xrightarrow{\theta} & EG^n \\
\downarrow \qquad\qquad \downarrow ev & & \downarrow \\
M_n(B,J) = \left\{ \begin{array}{l} \text{marked vortices } (A, u, z_1, \dots, z_n) \\ [u]_G = B \end{array} \right\} \Big/ \mathcal{G} & \xrightarrow[\text{map}]{\text{classifying}} & BG^n
\end{array}$$

$$\mathrm{ev}^{\mathrm{fr}}([A,u,p_1,\ldots,p_n]) = (u(p_1),\ldots,u(p_n)) \in M^n$$

$$\mathrm{ev} := (\theta, \mathrm{ev}^{\mathrm{fr}})/G^n : \mathcal{M}_n(B,J) \longrightarrow (EG \times_G M)^n \cong EG^n \times_{G^n} M^n$$

The invariants:

Assume (M,ω) is monotone.

$$GGW_B : H_G^*(M)^{\otimes n} \longrightarrow \mathbb{Z}$$

$$GGW_B(a_1,\ldots,a_n) \underset{\text{formally}}{=} \int_{\overline{\mathcal{M}}_n(B,J)} \overline{\mathrm{ev}}_1^*(a_1) \cup \cdots \cup \overline{\mathrm{ev}}_n^*(a_n)$$

$f : X \longrightarrow (EG \times_G M)^n$

pseudocycle, Poincaré-dual to $a_1 \otimes \cdots \otimes a_n$ $\quad =: f \cdot \mathrm{ev}$

Want ev to be a pseudocycle.

$\rightsquigarrow$ Need smooth boundary strata of $\overline{\mathcal{M}}_n(B,J)$.

$\rightsquigarrow$ "transversality"

FACT (McDuff, Salamon): Can achieve this for generic J.

Problem: this will not be possible for our G-invariant J!

Will focus on this problem from now on.

THM (O., partly joint with Ziltener):

(a) Can solve this problem by letting J depend on the pair (A,u).

This leads to a nonlocal version of the vortex equations:

$$\begin{cases} \bar\partial_{J,A,\Theta}(u) := \frac{1}{2}\left(d_A u + J_{\Theta(A,u)}(u) \circ d_A u \circ j_\Sigma \right) = 0 \\ F_A + \mu(u)\, d\mathrm{vol}_\Sigma = 0 \end{cases}$$

where now $J : EG^N \longrightarrow \mathcal{J}(M,\omega)$ is an equivariant family and Θ is "regular" $\mathcal{G}$-equivariant classifying map for the $\mathcal{G}$-action on pairs (A,u):

$$\begin{tikzcd} \mathcal{B} = \{\text{pairs } (A,u)\} \arrow[r, "\Theta"] \arrow[d] & E\mathcal{G} = \mathrm{Map}(P, EG)^G \arrow[d] \\ \mathcal{B}/\mathcal{G} \arrow[r] & B\mathcal{G} \end{tikzcd}$$

For (A,u), get $\Theta_{(A,u)} : P \longrightarrow EG$, equivariant

$\curvearrowright$ $J_{\Theta(A,u)} : P \longrightarrow \mathcal{J}(M,\omega)$ but no longer G-invariant

(b) can Gromov-compactify moduli space (and hence define GGW-invariants).

(c) GGW-invariants are independent of J and Θ.

Regular Θ:

- fix $d\mathrm{vol}_\Sigma \gg_{E_B} 0$ $\qquad E_B = \langle [\omega - \mu]^G, B \rangle$
- can define
  $\mathcal{B} = \left\{ \text{pairs } (A,u) \;\middle|\; \int_\Sigma |\mu(u)|^2\, d\mathrm{vol}_\Sigma < E_B \right\}$

This stability condition ensures that $\mathcal{G} \circlearrowright \mathcal{B}$ freely: it forces μ to be $\underbrace{\text{close to } \mu^{-1}(0)}_{\substack{\circlearrowleft \\ G \text{ freely}}}$ at some $p_0 \in P$

$\mu(p_0)$

DEF: Θ is regular if

(a) $\forall_{(A,\mu)}$, $\Theta_{(A,\mu)} : P \longrightarrow EG^N$ + regularity

(b) Θ continuous w.r.t. Gromov convergence

$$\left.\begin{array}{l} A_\nu \to A \text{ on } P \\ \mu_\nu \to \mu \text{ on } \Sigma \setminus \{\text{bubbling points}\} \end{array}\right\} \Rightarrow \Theta_{(A_\nu,\mu_\nu)} \to \Theta_{(A,\mu)}$$

(c) control 1st and 2nd derivatives of $\Theta_{(A,\mu)}$ in terms of F_A.