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" Seiberg-Witten theory and Lagrangian correspondences between vortex moduli spaces

'94 : Witten introduced the Seiberg-Witten equations

X compact, oriented, Riemannian 4-manifold

PDE (involving connection and spinor), gauge group $= U(1)$

counting solutions $\rightsquigarrow$ smooth invariants of X

($b_2^+(X) > 1$ assumed)

Y compact, oriented, Riemannian 3-manifold. Plan:

1) SW-theory on Y
2) Dimensional reduction to abelian vortices ($Y = S^1 \times \Sigma$)
3) Y has cylindrical ends $\rightsquigarrow$ moduli space of monopoles as an immersed Lagrangian $\Rightarrow$ correspondence between vortex moduli spaces at ∞

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1) $$SO(3) \hookrightarrow P_{frame} \to Y$$

$$\mathrm{Spin}^c(n) = \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1)$$

$$\mathrm{Spin}^c(3) = SU(2) \times_{\mathbb{Z}_2} U(1) = U(2)$$

DEF: A spin^c-structure is a lift of P_{frame} to a Spin^c-bundle.

i.e.

$$\begin{array}{ccccc} U(1) & \to & U(2) & \hookrightarrow & \tilde{P} \\ & & \downarrow & & \downarrow \\ & & SO(3) & \hookrightarrow & P_{frame} \\ & & & & \downarrow \\ & & & & Y \end{array}$$

Fact: Every $\begin{smallmatrix}3-\\4-\end{smallmatrix}$ manifold admits a spin^c-structure.

$\{\text{Spin}^c\text{-structures on } X\} \leftrightarrows H^2(X;\mathbb{Z})$ classifies line bundles

Spin^c-structure $\rightsquigarrow$ associated spinor bundle

$$S = \tilde{P} \times_{Spin^c(3)} \mathbb{C}^2$$

Clifford multiplication

$$\rho : TY \longrightarrow End(S)$$

$$\rho(u)\rho(v) + \rho(v)\rho(u) = -2(u,v)$$

Riemannian metric

$$\rho(dV) = id$$

In local o.n. frame e_1, e_2, e_3:

$$\rho(e_1) = I = \begin{pmatrix} i & \\ & -i \end{pmatrix}$$

$$\rho(e_2) = J = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

$$\rho(e_3) = -K = \begin{pmatrix} & i \\ i & \end{pmatrix}$$

DEF.: A spin^c-connection ∇ is one for which ρ is parallel ($\nabla\rho = 0$).

$\Rightarrow$ ∇ determined by ∇_{LC} (Levi-Civita) and a connection on a line bundle, $L = \det(\tilde{P}) \to Y$.

$\Rightarrow$ Spin^c-connections form an affine space over $\Omega^1(Y; i\mathbb{R})$.

Dirac operator:

$$\underbrace{\nabla : \Gamma(S) \longrightarrow \Omega^1(Y) \otimes \Gamma(S) \xrightarrow{\rho} \Gamma(S)}_{D = \rho \circ \nabla}$$

Seiberg-Witten equations on Y:

Fix a spin^c-structure, B = spin^c-connection

$\Psi \in \Gamma(S)$ spinor

$$\begin{cases} \frac{1}{2} * F_B = \rho^{-1}(\Psi \otimes \Psi^*)_0 \\ D_B \Psi = 0 \end{cases}$$

- F_B = curvature of the induced connection on det line $\in \Omega^2(Y; i\mathbb{R})$
- $\Psi \otimes \Psi^* \in \mathrm{End}(S)$, $\Psi \otimes \bar{\Psi}^* = \Psi(\Psi, \cdot)$

$(\Psi \otimes \Psi^*)_0$ means trace-free part

((But a 3-manifold is always spin, unlike 4-manifolds ...))

$Y = S^1 \times \Sigma$ product metric, product Spin^c-structure

Σ is spin:

$$\begin{array}{c} Spin(2) = SO(2) \\ \downarrow \\ SO(2) \end{array}$$

$$\begin{array}{c} S = K^{1/2} \oplus K^{-1/2} \\ \downarrow \\ \Sigma \end{array} \qquad K = K_\Sigma \text{ canonical bundle}$$

In general, $S = (K^{1/2} \otimes L) \oplus (K^{-1/2} \otimes L) \longrightarrow Y$

L arbitrary line bundle

coordinates (t, x, y) on $S^1 \times \Sigma$, $z = x + iy$

$$F = F_{t,x}\, dt \wedge dx + F_{t,y}\, dt \wedge dy + F_{x,y}\, dx \wedge dy$$

$$\Phi = \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}$$

$$\begin{cases} \left(F_{x,y} + \frac{i}{2}\left(|\Phi_-|^2 - |\Phi_+|^2 \right) \right) dt = 0 \\ \frac{1}{2}\left(\cancel{F_{y,t} - i F_{x,t}} \right) d\bar{z} + \bar{\Phi}_+ \Phi_-\, d\bar{z} = 0 \\ \begin{pmatrix} \cancel{i \nabla_{B,\partial_t}} & \sqrt{2}\, \bar{\partial}_B^* \\ \sqrt{2}\, \bar{\partial}_B & \cancel{-i \nabla_{B,\partial_t}} \end{pmatrix} \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix} = 0 \end{cases}$$

Dimensional reduction: no time dependence

connection has no dt - component

$\rightsquigarrow$ some terms disappear.

$\bar{\Phi}_+ \Phi_- = 0 \longrightarrow$ either Φ_+ or Φ_- vanishes

$\Rightarrow$ either 1) $-(g-1) \leq \deg L < 0$, vortex number $k_+ = g-1+\deg L$

or 2) $0 < \deg L \leq g-1$, antivortex number $k_- = -(g-1)+\deg L$

e.g. 1) $\rightsquigarrow$

$$\begin{cases} F_{B \otimes C} - \frac{i}{2}|\Phi_+|^2 = F_C & C \text{ connection on } K^{1/2} \\ \bar{\partial}_B \Phi_+ = 0 & \frac{i}{2\pi}\int_\Sigma F_C = g-1 \end{cases}$$

$$\Phi_+ \in \Gamma(K^{1/2} \otimes L)$$

If we substitute F_C by the average curvature on Σ, get the usual vortex equations.

Perturbations: $\frac{1}{2} * F_B - \rho^{-1}(\Phi \otimes \Phi^*)_0 = \eta$

$D_B \Phi = 0$ $\qquad$ η imaginary coclosed 1-form

$\deg L = \frac{1}{2}\langle c_1(\tilde{P}), \Sigma\rangle$

Then get shift:

1) $-(g-1) \leq \deg L < \frac{i}{2\pi}\int_\Sigma \eta^0$ $\qquad$ moduli space of solutions $\equiv$ vortices

2) $\frac{i}{2\pi}\int \eta^0 < \deg L \leq g-1$ $\qquad$ here, antivortices

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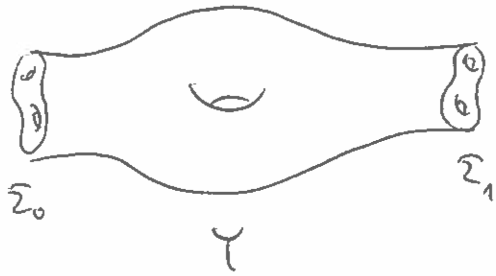
Consider finite-energy monopoles on Y:



THM: $\mathcal{M}(Y)$ smooth, compact, orientable

$\partial_\infty : \mathcal{M}(Y) \longrightarrow \mathcal{V}(\Sigma_0) \times \mathcal{V}(\Sigma_1)$

Image of ∂_∞ is an immersed Lagrangian submanifold.

THM: $SW(Y) = \partial_{\infty *}[\mathcal{M}(Y)] \cap [\text{graph of } h]$

when $\Sigma_0 \xrightarrow[h]{\cong} \Sigma_1$

Crucial: Signs agree.
