28.11.12

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"Hamiltonian Gromov–Witten invariants of nodal curves"

(joint with G. Tian)

(X,ω) compact symplectic manifold

$S^1 \circlearrowright (X,\omega) \xrightarrow{\mu} i\mathbb{R}$

$I \in C^\infty(\mathrm{End}\, TX)^{S^1}$ compatible with ω

$(g,n) \in \mathbb{N}\times\mathbb{N}$ $\quad 3g-3+n \geq 0$

$B \in H_2^{S^1}(X,\mathbb{Z})$ $\quad \underline{x} = (x_1,\dots,x_n) \in C^n$

$$\mathcal{M} = \left\{ (C,\underline{x},P,A,\varphi) \;\middle|\; \begin{array}{ll} [C,\underline{x}] \in \mathcal{M}_{g,n} & (P,\varphi)_*[C] = B \\ P \to C \text{ principal } S^1\text{-bundle} & \bar\partial_A \varphi = 0 \\ A \text{ connection on } P & \\ \varphi \in \Gamma(P\times_{S^1} X) & F_A + \upsilon_{(C,\underline{x})}\,\mu(\varphi) = 0 \end{array} \right\} \Big/ \text{iso}$$

Sketch of the definition of Hamiltonian GW-invariants:

1) Compactify $\mathcal{M} \longrightarrow \overline{\mathcal{M}}$ $\qquad ES^1\times_{S^1}X$ Borel construction

2) Evaluation map: $\overline{\mathcal{M}} \xrightarrow{e} (X_{S^1})^n$ (evaluation)

(Poincaré bundles)

$\iota: (X_{fin})^n \hookrightarrow (X_{S^1})^n$, $\quad e_{fin}: \overline{\mathcal{M}} \to (X_{fin})^n$ — compact manifold; here, using truncation of Borel construction

3) Linear theory

Work on each of the natural strata of $\overline{\mathcal{M}}$ (not necessary to consider what happens across strata).

4) Given subvarieties $A = (A_1,\dots,A_n)$ of X_{fin}, $\overline{\mathcal{M}}(A) = e_{fin}^{-1}(A)$

Perturb parameters in such a way that, if dimensions of A_i are appropriate, $\#\overline{\mathcal{M}}(A) < \infty$

5) Define $HGW^{g,n}_{B}(A) = \sum_{t \in \mathcal{M}} \underbrace{\sigma(t)}_{\in \{-1,1\}}$

6) Splitting axiom ⟶ need to use equivariant version of diagonal class (cf. GW-theory)

Zero class axiom ⟶ Chen-Ruan orbifold product

Degree axiom

In this talk, focus will be on 1) and 6) (in particular, equivariant version of diagonal class).

Difficulties:

i) (2)) Get Poincaré orbibundles rather than bundles.

This is dealt with by working on certain orbifold resolutions of $\bar{\mathcal{M}}$. But then all objects will be decorated by weights.

ii) Perturbations have to be compatible with compactification.

iii) Resolutions ⟶ $HGW \in \mathbb{Q}$

On 1) compactification:

$\bar{\mathcal{M}}_{g,n+1}$ ← universal curve (orbifold sense)

↓

$\bar{\mathcal{M}}_{g,n}$ Pick a smooth Riemannian metric g

$g_{[C,\underline{x}]} = g|_{\pi^{-1}([C,\underline{x}])}$

Then take $v_{(C,\underline{x})}$ = volume form associated to $g_{(C,\underline{x})}$

Consequence: the degenerations to nodes are modelled by

$$\{xy = \varepsilon\} \subset (\mathbb{C}^2, \text{ smooth metric})$$
$$\varepsilon \to 0$$

$$(P, \varphi)_*(C) = B \Rightarrow \|F_A\|^2_{L^2} + \|d_A \varphi\|^2_{L^2} + \|\mu(\varphi)\|^2_{L^2} \leq (\text{const})\, B$$

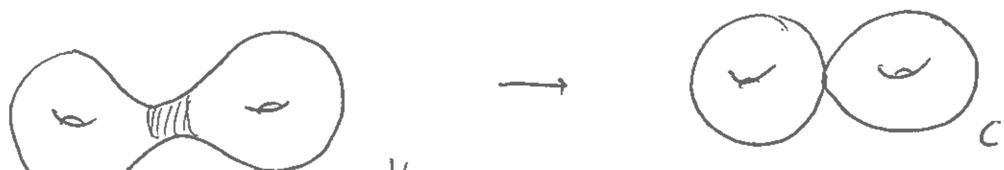
$$\{[C_i, \underline{x}_i, P_i, A_i, \varphi_i]\} \subset \mathcal{M}$$

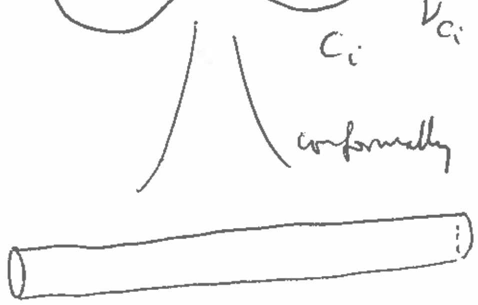
$$(C_i, \underline{x}_i) \longrightarrow (C, \underline{x}) \in \overline{\mathcal{M}}_{g,n}$$

Suppose wlog that each C_i is smooth and C has a node.

Away from the node, the geometry of C_i can be uniformly controlled, so have uniform elliptic estimates. Standard techniques allow to study what happens there.

Near the node:

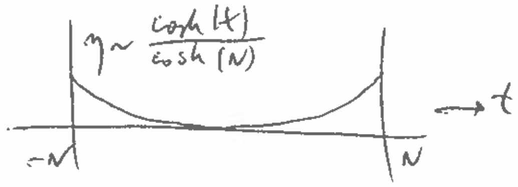




Analysis after discarding (i.e. studying) standard GW bubbling off.

$$C_N = [-N, N] \times S^1 \qquad N \to \infty \text{ as } C_i \to C$$

$$V_{C_i}|_{C_N} = \eta \, dt \wedge d\theta$$



$$F_A = -\mu(\varphi) V_{C_i} \qquad d_A = d + a$$

Assume $P \cong C_N \times S^1$, A s.t. $d_A = d + a$

$$|a(t,\theta)| \le c\, e^{+\lambda(|t|-N)} \qquad \lambda > 0$$

$$\varphi : C_N \longrightarrow X$$

$$|d_A \varphi|_{C^0(C_N)} < \varepsilon$$

Let $H := \mathrm{Hol}(A, \{0\} \times S^1) \in S^1$

$$\Theta_{crit} = \{\theta \in S^1 \mid \exists x \in X \setminus X^{S^1},\ \theta \cdot x = x\} \subset S^1$$

$$\Theta_{gen} = S^1 \setminus \Theta_{crit}$$

THM: If $H \notin \Theta_{crit}$ then:

$$|d_A \varphi(t,\theta)| \le C e^{\lambda'(|t|-N)}$$

where c, λ' depend on $d(H, \Theta_{crit})$

(For Θ_{crit}, have usual exponential decay, as in GW-theory.)

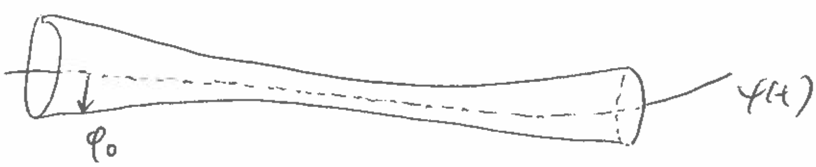
THM: If $\delta = d(H, \Theta_{crit})$ is very small,

$\psi(t)$ = center of mass of $\varphi(\{t\} \times S^1)$

$\varphi_0(t,\theta)$ defined by $\varphi(t,\theta) = \exp_{\psi(t)} \varphi_0(t,\theta)$.

$$\int \varphi_0(t,\theta)\, d\theta = 0.$$

Estimate:
$$|\psi' - \delta \nabla \mu(\psi)| \le C e^{\lambda''(|t|-N)} \qquad \text{of gradient flow}$$
$$|\varphi_0(t,\theta)| \le C e^{\lambda''(|t|-N)}$$



convergence to gradient flow lines

In the limit, near the node:

- A meromorphic
- φ extends
  - as an orbisection of $\mathbb{H} \in \Theta_{crit}$ (→ limit holonomy)
  - as a continuous section of $\mathbb{H} \in \Theta_{crit}$

- gradient lines
- bubbles

Comments on 6) equivariant diagonal class:

$$X \times \mathbb{R} \times S^1 \xrightarrow{\ D\ } X \times X$$

$$(x, t, \theta) \longmapsto (x, \theta \cdot \xi_t(x))$$

(ξ_t ↑ flow of $\nabla \mu$)

THM: D is a pseudocycle $\rightsquigarrow [D] \in H_{S^1 \times S^1}(X \times X)$

Need for multivalued perturbations (of I).