27.11.12

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"Hitchin-Kobayashi correspondence on nearly singular conics"

Σ compact connected Riemann surface

$\nu \in \Omega^2(\Sigma)$ volume form

X compact Kähler manifold, $\mathbb{C}^* \circlearrowright X$ holomorphic

$$\cup$$

$$d\mu = \iota_{\mathfrak{X}} \omega_X \qquad S^1 \circlearrowright X \xrightarrow{\mu} i\mathbb{R}$$

$Q \to \Sigma$ holomorphic $\mathbb{C}^\times$-principal bundle.

Reduction of $Q \longleftrightarrow h : \Sigma \to Q/S^1 \xleftarrow{p} Q$

$h \rightsquigarrow \quad Q_h = p^{-1}(h(\Sigma))$ S^1-principal bundle

A_h connection on Q_h

If $\varphi : Q \xrightarrow[\text{hol.}]{} X$ satisfies $\varphi(q\theta) = \theta^{-1}\varphi(q)$

$(\Leftrightarrow \varphi \in \Gamma(Q \times_{\mathbb{C}^\times} X))$,

then $\mu_h(\varphi) := \mu \circ \varphi \circ h : \Sigma \to i\mathbb{R}$

(Q, φ) is stable if for some (or equivalently, any) h

$$\lim_{t \to -\infty} \int_\Sigma -\frac{i}{2\pi}\mu_h(e^{-t}\varphi)\nu < \deg P < \lim_{t \to \infty} \int_\Sigma -\frac{i}{2\pi}\mu_h(e^{-t}\varphi)\nu$$

THM (HK correspondence): If (Q, φ) is stable, then $\exists!\, h$:

$$\text{curvature of } A_h \longrightarrow F_h + \nu\, \mu_h(\varphi) = 0 \qquad (*)$$

If $(*)$ holds, then $(Q_h, A_h, \varphi_h = \varphi|_{Q_h})$ is a twisted holomorphic map (or vortex) from (Σ, ν) to X.

PROOF: Let h_0 be a reduction s.t. $F_{h_0} = \left(\frac{i2\pi d}{vol(\nu)}\right)\nu$ $(d = \deg P)$

Then, if $h = h_0 e^{u}$ $(u : \Sigma \to \mathbb{R})$.

$$(*) \iff \Delta u = \frac{4\pi d}{vol(\nu)} + 2i\mu_{h_0}(e^{u} q)$$

$(\Delta = \text{analyst's laplacian} = -d^*d)$

To prove existence of u, use continuity method for

$$\Delta u = t\left(\frac{4\pi d}{vol(\nu)} + 2i\mu \cdots\right)$$

□

For $\tau \in \mathbb{R}_{\geq 0}$, let $C_\tau = \{[x:y:z] \mid xy = \tau z^2\}$

$\omega_{FS} \in \Omega^2(\mathbb{CP}^2)$ Fubini-Study symplectic form

C_τ

if $\tau > 0$, let

$\iota_\tau : \mathbb{CP}^1 \longrightarrow C_\tau$ parametrization

$[a:b] \longmapsto [\sqrt{\tau}\, a^2 : \sqrt{\tau}\, b^2 : ab]$

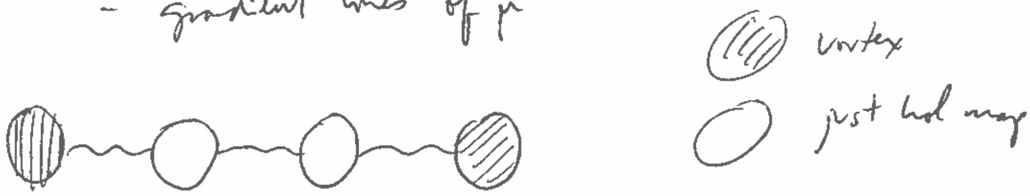
$\nu_\tau := \iota_\tau^* \omega_{FS} \in \Omega^2(\mathbb{CP}^1)$

Let $\{(P_j, A_j, \varphi_j)\}$ be a sequence of THM's on $(\mathbb{CP}^1, \nu_{\tau_j})$ and $\tau_j \to 0$.

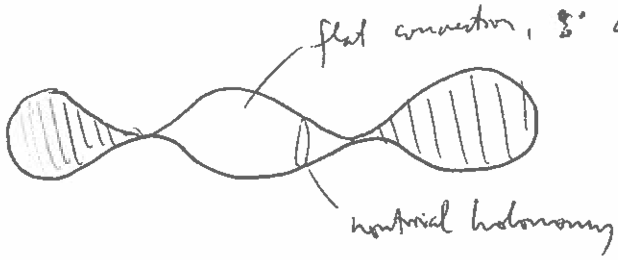
THM (M, Tian): If $\{(P_j, A_j, \varphi_j)\}$ have bounded energy, then up to passing to a subsequence and rearranging, have convergence to limit (P, A, φ) on C_0; the following <u>may</u> occur:

- bubbling away from the node (cf. pseudoholomorphic curves)
- A is smooth away from the node, but the limit holonomy at node is: $\lim \mathrm{Hol}(A_j, P_j) \in S^1$
- φ extends continuously to normalization of C_0.

If $\mathrm{Hol}(A, \text{node})$ is 1, then φ extends smoothly, but there may appear:
- bubbles
- gradient lines of μ



If $\mathrm{Hol}(A, \text{node})$ is $\neq 1$, then get meromorphic bubbles



(NB: In the limit, need to introduce parabolic structures)

<u>Our example:</u>

$X = \mathbb{CP}^1 \qquad \theta \cdot [x:y] = [\theta x : y]$

$$\mu([x:y]) = -i\left(\frac{|x|^2}{|x|^2+|y|^2} - 1\right)$$

$Q = \underbrace{\mathbb{CP}^1}_{\Sigma} \times \mathbb{C}^\times$, $\varphi : \underbrace{\mathbb{CP}^1}_{\Sigma} \longrightarrow \underbrace{\mathbb{CP}^1}_{X}$ of degree 1

Problem: Let $\{\varphi_j\}$ be a sequence, $\tau_j \to 0$

Let h_j be solution of HK for (ℓ, φ_j) on $(\mathbb{CP}^1, \nu_{\tau_j})$. What are the possible limits of the corresponding vortices?

Up to $\mathbb{C}^\times \subset X$, φ is characterized by

$$n_j = \varphi^{-1}([1:0]) \quad , \quad s_j = \varphi^{-1}([0:1])$$

North South $n_j \neq s_j$

<u>Symmetric vortices</u>

$\varphi = id$

$\rightsquigarrow$ $h = h_0 e^u$ (or u) ivariant under rotations $\underbrace{\theta \cdot [x:y]}_{\mathbb{CP}^1 = \Sigma} = [\theta x : y] \quad \theta \in S^1$

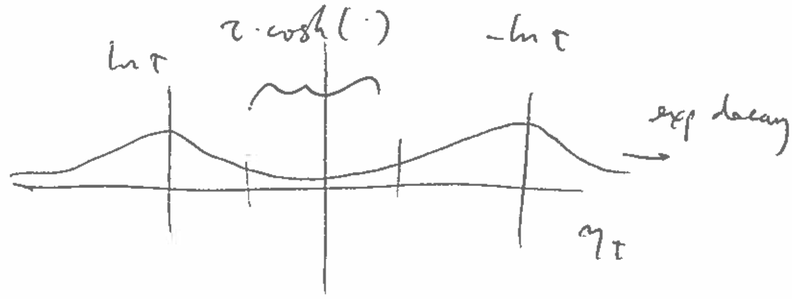
Cylindrical coordinates:

$$(t, \theta) \longmapsto [e^{t+i\theta} : 1] \in \mathbb{CP}^1$$

$u(t,\theta) = u(t)$

$\nu_\tau = \eta_\tau \, dt \wedge d\theta$

$\eta_\tau(t,\theta) = \eta_\tau(t)$



$$\Delta_\tau u = 2i\mu(e^u \cdot Id) \iff u'' = \zeta \eta_\tau \left(\frac{2}{1+e^{-2(t+u)}} - 1 \right)$$

$\zeta \in \mathbb{R}_{>0}$ constant

$$u'(t) \xrightarrow[t \to \pm\infty]{} 0$$

Given $\lambda \in \mathbb{R}$, let $u_\lambda : \mathbb{R} \to \mathbb{R}$ satisfy:

$$\begin{cases} u_\lambda(0) = 0 \\ u_\lambda'(0) = \lambda \\ u_\lambda'' = 3\eta_\tau \left(\dfrac{2}{1+e^{-2(t+u_\lambda)}} - 1 \right) \end{cases}$$

THM: 1) $\forall_\tau \ \exists! \ \lambda(\tau)$ s.t. $u_{\lambda(\tau)}$ satisfies

$$u'' = 3\eta_\tau \left(\frac{2}{1+e^{-2(t+u)}} - 1 \right)$$

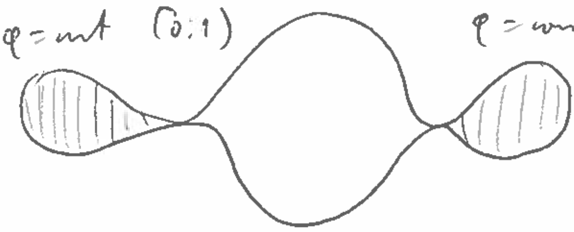
$$u'(t) \xrightarrow[t \to \pm\infty]{} 0$$

2) let $V_\tau = 3\int_{\mathbb{R}} \eta_\tau$

$$\lim_{\tau \to 0} \lambda(\tau) = \begin{cases} -\dfrac{V_\tau}{2} & \text{if } \dfrac{V_\tau}{2} < 1 \qquad (\mathrm{I}) \\ -1 & \text{if } \dfrac{V_\tau}{2} \geq 1 \qquad (\mathrm{II}) \end{cases}$$

$$(e^u \cdot \mathrm{Id})(t,\theta) = \left(e^{t+u(t)+i\theta} : 1 \right)$$

limit case (I):



(II):

Nonsymmetric case:

$$\Delta_\tau u = 2i\mu(e^u \varphi)$$

a-priori bounds on u? RHS is C^0-bounded

THM: Let $\lambda_0 = 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$ be the spectrum of $(-\Delta_\tau)$. Then $\exists\ C, C', C''$

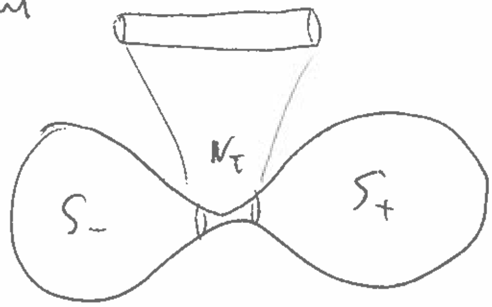
$$C(-\ln \tau)^{-1} \le \lambda_1 \le C' (-\ln\tau)^{-1}$$

$$C'' \le \lambda_2 \quad \text{(no problem here)}$$

Let $\{f_i\}$ be a topological orthonormal basis of $L^2(\mathbb{CP}^1, \kappa_\tau)$ s.t. $-\Delta_\tau f_i = \lambda_i f_i$.

f_0 = constant

LEMMA:



- $f_1|_{S_\pm}$ has oscillation $\le ((-\ln\tau)^{-1})$
- $f_1|_{N_\tau}$ is almost linear in cylindrical coordinates.

THM: $\tau > 0$ small, $\Delta_\tau u = 2i\mu(e^u \cdot \varphi)$

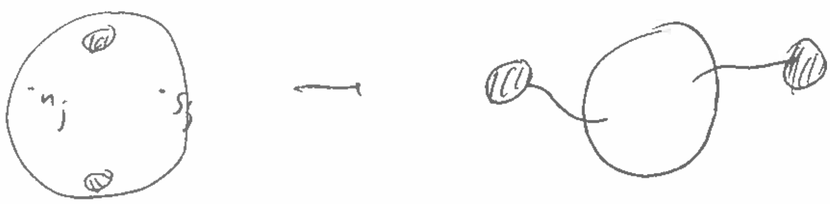
1) $u|_{S_\pm}$ has bounded oscillation (i.e. almost constant on each)

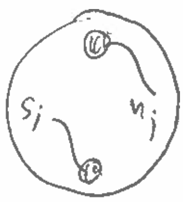
2) Let $u = a_0 f_0 + a_1 f_1 + \sum_{i \ge 2} a_i f_i$

$$u|_{S^\pm} = a_0 \pm (-\ln t) \underbrace{\frac{1}{2\pi}\int_{S^1} \partial_t u(0,\theta)\,d\theta}_{\mathrm{Hol}(A,\ell_\tau)} + O(1)$$

3) $$\underbrace{\int_{S^1} \partial_t u(0,\theta)\,d\theta}_{\text{A restrictions on this}} + \int_{t\geq 0}\int_{S^1} \xi\, \eta_\tau\, 2i\mu(e^u q) = 0$$

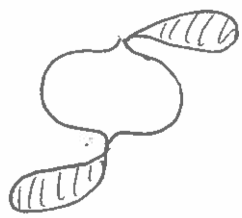
$\frac{\sqrt{\tau}}{2} < 1$





slow:

fast:



faster: