27.11.12

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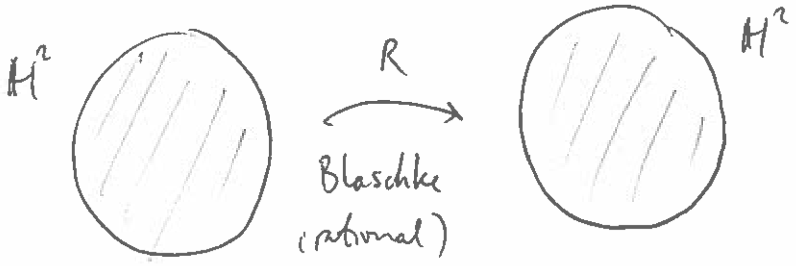
"Vortices on hyperbolic surfaces, and in the dissolving limit"

0. Vortices on $\mathbb{H}^2$ (Witten),

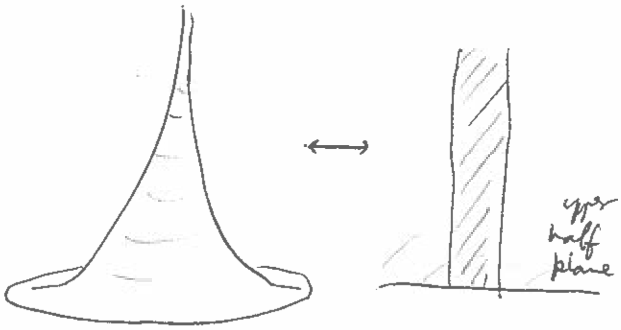


If R has degree n, then the # of ramification points ($R'=0$) is (?) $2n-2$

Vortex # $= N = n-1$ points inside the disc

$$e^u = \frac{R'(z)\,\overline{R'(z)}\,(1-z\bar{z})^2}{(1-R(z)\overline{R(z)})^2} = |\phi|^2$$

$$\phi(z) = \frac{R'(z)\,(1-z\bar{z})}{(1-R\bar{R})}$$



1. Another example: punctured $\mathbb{H}^2$

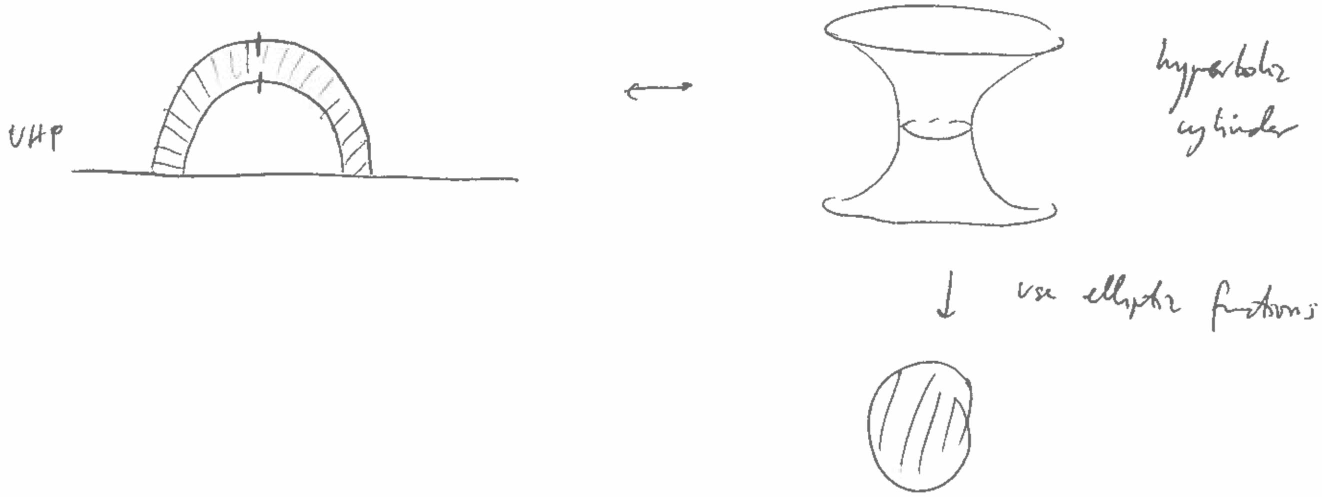
$$ds^2 = \frac{8\,dz\,d\bar{z}}{z\bar{z}\left(\log(z\bar{z})\right)^2}$$

Use a map:

e.g. Blaschke (any)

$$e^u = \frac{R'(z)\,\overline{R'(z)}}{(1-R(z)\overline{R(z)})^2}\, z\bar{z}\left(\log z\bar{z}\right)^2 = |\phi|^2$$ is a vortex solution

2. Solutions on hyperbolic cylinder



Challenge: find vortex solution on surface of genus $g=2$.
using "Fuchsian groups" or "modular forms" technology.

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Bogomolnyi vortices from SDYM on $\Sigma \times S^2$

symmetry $SO(3)$ $\qquad$ $SU(2)$ gauge group

"Popov's vortices" $\qquad S^2 \times H^2$

SDYM

$SU(1,1)$ symmetry, gauge group $SU(1,1)$

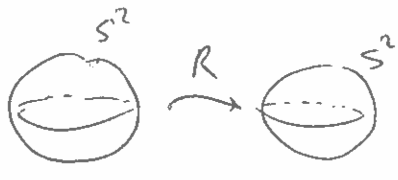
"Vortex" eqs on S^2

$$\begin{cases} \partial_z \phi - i a_z \phi = 0 & \leftarrow \ \partial\phi, \text{ not } \bar\partial\phi \\ F_{z\bar z} = \frac{i}{4}\Omega(1-\phi\bar\phi) \end{cases}$$

Claim: there are stationary points (clearly not minima).
These equations are integrable on S^2 with curvature $\frac{1}{2}$.

$\ulcorner$ $2K \pm iw = 2K' \pm iw'$ ... $\lrcorner$

Solutions as ratio of S^2 metrics



$$e^u = \phi\bar\phi = \frac{R'(z)\overline{R'(z)}\,(1+z\bar z)^2}{(1+R(z)\overline{R(z)})^2}$$

$$\phi = \frac{\overline{R'(z)}\,(1+z\bar z)}{1+R\bar R}$$

vortex number $N \le 2$ (can be negative!)

- Solution with $N=2$, $\phi=0$
- Solution with $N=0$ $\begin{cases} |\phi|=1 \\ F=0 \end{cases}$

$N = 2-2n$, $n = \deg R$

$SO(3)$ - Möbius transformations R are isometries and give "vacuum",

but there are other ones: e.g. $R(z) = cz$ $(c>1)$

$$\int_{S^2} \phi\bar\phi = 8\pi - 4\pi N$$

not true that $|\phi| \le 1$ everywhere
(unlike vortices, by maximum principle)

In limit $A = 4\pi$, $N=1$: $\phi = 0$, $F_{z\bar z} = \frac{i}{4}\Omega$

Solve this (e.g. Kähler potential for Ω) to get a^0.

General solution: $a = a^0 + \alpha$ with $d\alpha = 0$ $\therefore$ α harmonic
$d{*}\alpha = 0$ (Coulomb gauge)

$\exists$ g-torus of harmonic forms

$J = \mathbb{C}^g/\Lambda$ $\Lambda = (I, \Pi)$ symmetric, positive imaginary part

$\hookrightarrow$ coords X_j (mod Λ) periods of holomorphic differentials ξ_i

$$\alpha = -i\pi \left(\operatorname{Im} \Pi^{-1}\right)_{ij} \left(\bar{x}_j \xi_i + x_j \bar{\xi}_i \right)$$

Electric field $\quad e = \dot{\alpha} \quad$ (in Coulomb gauge)

$$\begin{matrix}\text{Energy} \\ \text{(kinetic)}\end{matrix} = \frac{1}{2} \int_\Sigma e \wedge * e = 2\pi^2 \left(\operatorname{Im} \Pi^{-1}\right)_{ij} \dot{x}_i \dot{\bar{x}}_j$$

Read off metric on $\mathcal{M} = \operatorname{Jac}(\Sigma) = 2\pi \left(\operatorname{Im} \Pi^{-1}\right)_{ij} dx_i d\bar{x}_j$

For $A \gtrsim 4\pi$; 1 vortex at $z(t)$

A point in J with coordinates

$$x_j = \int_{z_0}^{z} \xi_j$$

$\mathcal{M}_1 \cong \Sigma \subset J \qquad$ what is metric?

Moving vortex: $\quad \dot{x}_j = \xi_j(z)\, \dot{z}$

Energy $= 2\pi^2 \left(\operatorname{Im} \Pi^{-1}\right)_{jk} \xi_j \bar{\xi}_k \dot{z} \dot{\bar{z}}$

Metric on $\Sigma \subset J$ is Bergman metric.

$$ds^2 = (2\pi)^2 \left(\operatorname{Im} \Pi\right)^{-1}_{jk} \xi_j(z) \overline{\xi_k(z)}\, dz\, d\bar{z}$$

Area $\mathcal{M}_1 = (2\pi)^2 g$

cf. $\operatorname{Vol}(J) = (2\pi)^{2g}$

$N = 2$ on surface with area a little greater than 8π.

Abel-Jacobi map $\quad \mathcal{M}_2 \longrightarrow \operatorname{Jac}$

Image of Abel–Jacobi is smooth if Σ not hyperelliptic.
Otherwise, M_2 has conical singularities (in limit) when the vortices coincide at fixed points of hyperelliptic involution.

cf. arXiv: 0912.2058 M + Rink
1010.0644 M + Romão
1211.4352 M