30.11.12

Bumsig Kim

"Quasimap invariants and mirror maps"

(+ I. Ciocan-Fontanine) — C-F

$$[W/G] \supset [W^s(\theta)/G] =: W/\!/_\theta G \qquad \text{character } \theta : G \to \mathbb{C}^\times$$

W: affine variety, G: reductive algebraic group

$W^s(\theta) = W^{ss}(\theta)$ — assumption

G acts freely on $W^{ss}(\theta)$

((There is ? a generalisation for W projective, but things will then be more complicated.))

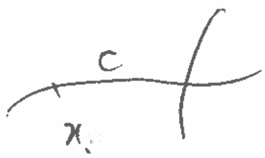
$$\mathcal{M}_{g,k}([W/G], \beta) = \Big\{ (C, \underline{x}, [u]) ; \ \underbrace{[u] : C \to [W/G]}_{\text{prestable with } [u]_*(C) = \beta} \Big\} \Big/ \sim$$

$(C, \underline{x})$: genus g, k-marked curve, $\underline{x} \in C^k$

$\mathcal{M}_{g,k}([W/G], \beta)$: Artin stack

$\Updownarrow$

(P, u), $P \downarrow C$ G-bundle, u a section of $P \times_G W \to C$



$\varepsilon \in \mathbb{Q}_{>0}$

DEF: $(C, \underline{x}, [u])$ is called an ε-stable quasimap if $[u]^{-1}([W^{un}/G])$

* consists of finite smooth points, non-marked
* $\omega_C(\sum x_i) \otimes [u]^* L_\theta^\varepsilon$ is ample, $L_\theta = [W \times \mathbb{C}_\theta / G] \to [W/G]$
* $l(p)\,\varepsilon \le 1$

$\ell(p) =$ " $u \cdot w^{\mathrm{un}}$ " at p $\quad \forall p \in C$

((maybe ε is related to parameter τ in vortex equations))

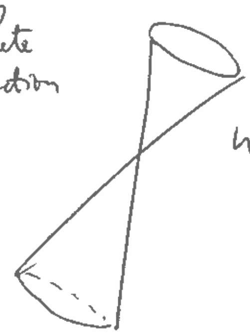
- $\varepsilon > 2$: ε-stable $\Longleftrightarrow$ usual stability for maps to $W/\!\!/_\theta G$

THM (C-F, K, Maulik):

The moduli stack $Q^\varepsilon_{g,n}(W/\!\!/_\theta G, \beta)$ of ε-stable quasimaps is a finite type proper DM-stack over $\operatorname{Spec} \mathbb{C}[W]^G =: W/_{\text{affine}} G$

locally complete intersection

W^s smooth, W lci



W

$Q^\varepsilon_{g,n}$ has a natural perfect obstruction theory.

$\curvearrowright$ GW-type invariants

$$\begin{array}{ccc} Q^\varepsilon_{g,n} & \xrightarrow{ev_i} & [W/G] \\ & \searrow^{ev_i} & \cup \\ & & W/\!\!/G \end{array}$$

Splitting axiom (cf. Maulik) $\quad \Delta \in H^{\dim_{\mathbb{R}} W/\!\!/G}(W/\!\!/_\theta G)^{\otimes 2}$

$\curvearrowright$ associative ring

In general, usual 1 is not a unit.

Cohomological field theory $\langle \psi_1^{a_1}\gamma_1, \dots, \psi_n^{a_n}\gamma_n \rangle^\varepsilon_{g,n,\beta}$

$\gamma_i \in H(W/\!\!/G) \qquad \psi_i = c_1(T^*_{x_i}C)$

- $0 < \varepsilon \ll 1$

$$\varepsilon\text{-stability} \iff \begin{cases} \star \text{ holds} \\ |\mathrm{Aut}(C, \underline{x}, (u))| < \infty \\ \forall\, C_0 \subset C \text{ carries at least two special points} \end{cases}$$

C_0: genus 0 component

i.e.

$$\left| C_0 \cap \left\{ \begin{matrix} \text{nodes of } C \\ x_1, \dots, x_n \end{matrix} \right\} \right| \geq 2$$

Fix a curve C, say $\mathbb{P}^1$

$$\mathrm{Mor}(\mathbb{P}^1, [W/G])$$

$$\cup$$

$$\mathrm{Qmap}_\beta(\mathbb{P}^1, W/\!\!/_\theta G) = \left\{ \mathbb{P}^1 \xrightarrow{(u)} [W/G] \text{ s.t. } (u) \text{ touches a stable point, } \deg = \beta \right\}$$

$$\downarrow$$

$$W/_{\mathrm{aff}} G$$

$$\cup$$

$\mathrm{Qmap}_\beta(\mathbb{C}^1, W/\!\!/_\theta G)$ $\quad$ ($\sim \mathcal{V}_\beta$, cf. Gukov's talk)

$$\shortparallel$$

$\mathbb{C}^* \circlearrowright \{ (u)(\infty) \in [W^{st}/G] \}$ $\quad$ $t \in H^\cdot(W/\!\!/G)$

$\downarrow$ quasimap $\rightsquigarrow$ $0 < \varepsilon \ll 1$

$W/\!\!/_\theta G$ $\quad (t)$ $\quad$ $\left(\overset{?}{\underset{\text{K-theory}}{\rightsquigarrow}} \ ch_{\mathbb{C}^*}(\mathcal{V}_\beta) \quad \text{cf. Gukov's talk} \right)$

$(\triangleq I_{W/\!\!/_\theta G}$ under mirror map$)$

e.g. $W = \mathrm{Hom}(\underline{\mathbb{C}}^r, \mathbb{C}^r)$

$\circlearrowleft$

$G = GL_r$

$$\mathrm{Qmap}_d(\mathbb{P}^1, W/\!\!/_\theta G) = \mathrm{Quot}_{\mathbb{P}^1}(\mathbb{C}^r \otimes \mathcal{O}_{\mathbb{P}^1}, \mathrm{rk}\ 0, \deg d)$$

w/ Bertram, C-F '05

Explicit description of fixed loci

$Fl(m_1, \ldots, m_r, r)$ and $N_{Fl/Quot}$

NB: $Q^{\varepsilon}_{g,k}\left(\mathrm{Grass} = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^n)/_{GL_r}\right)$

studied by Marian - Oprea - Pandharipande

under the name of stable quotient.

CONJECTURE: $GW(W/\!\!/_\theta G) \sim \mathrm{Qmap}^{\varepsilon}\, GW(W/\!\!/_\theta G)\quad \forall_\varepsilon$

THMs (w/ C-F): 1) $W = \mathbb{C}^N/\!\!/T$ smooth projective toric variety

Suppose W is Fano; then

$$GW = \mathrm{Qmap}^{\varepsilon}\, GW \quad \forall_\varepsilon$$

2) $W \circlearrowleft T \times G$ lci (not necessarily Fano)

and $(W/\!\!/G)^T$ isolated

(quantum cohomology) $\Rightarrow J^{\varepsilon > 2}_{W/\!\!/_\theta G}(\tau) = J^{\varepsilon \rightsquigarrow \text{any}}_{W/\!\!/_\theta G}(t)$ after the mirror transformation

$GW_{g=0}(W/\!\!/_\theta G) \longrightarrow$

$t \in H^{\cdot}(W/\!\!/_\theta G)$

3) Calabi-Yau case $[W/G]$

$$J(\tau) = \frac{J^{\varepsilon}(t)}{J_0^{\varepsilon}(t)}$$

$J_0^{\varepsilon}(t) \longrightarrow$ the coefficient of 1 in J^{ε}

$$\tau(t) = t + \sum_{i,\, \beta \neq 0} q^{\beta \cdot t} \langle \gamma_i, 1 \rangle^{\varepsilon}_{0,2,\beta} \gamma^i$$

$t \in H^2(W/\!/G)$

$\{\gamma_i\}$ basis in $H^{*}(W/\!/G)$

$\{\gamma^i\}$ Poincaré-dual basis

in many examples, this yields (the inverse of) mirror map.

i.e.: Jinzenji's conjecture is true for case 2).

Coroll: For Nakajima quiver variety, we can prove that the abelianization conjecture is true.

$J_{W/\!/G} \sim J_{W/\!/S}$, twisting $T_{W/\!/S / W/\!/G}$ $\quad S \subset G$ maximal torus

2) $J^{\text{quasimap}}_{W/\!/G} \sim J^{\text{quasimap}}_{W/\!/S;\, \text{twist}}$