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"From vortex counting to knot homologies"

I. What

II. Why

I.

Abelian vortex equations

A = connection on $U(1)$-bundle on $D = \mathbb{R}^2$ ← but more generally, any Riemann surface

$$\begin{cases} * F_A = i\phi\bar{\phi} - it \\ \bar{\partial}_A \phi = 0 \end{cases} \qquad \text{vortex equations}$$

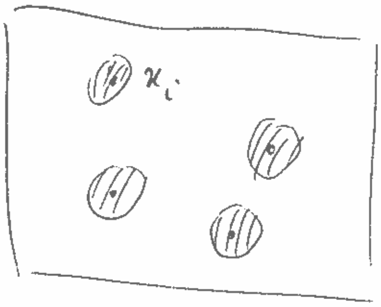
$$m = \frac{1}{2\pi}\int_D F_A = \text{vortex number } (= \text{monopole number, later})$$

$\underset{\underset{U(1)}{\parallel}}{V_m}(\underset{\parallel}{G},\ \underset{\underset{\mathbb{R}^2}{\parallel}}{D})$:= moduli space of solutions to these equations up to gauge equivalence

Properties:

- $\dim_{\mathbb{R}} V_m = 2m$
- V_m is Kähler, $\omega = \frac{i}{2\pi}\int_D d^2 w\, (\delta A_w \wedge \delta A_{\bar{w}} + \delta\phi \wedge \delta\bar{\phi})$
- $V_m = \mathrm{Sym}^m(D) = \mathrm{Sym}^m(\mathbb{C}) := \mathbb{C}^m / \mathfrak{S}_m \simeq \mathbb{C}^m$

$$f(x) = \prod_{i=1}^{m} (x - x_i) = x^m + a_1 x^{m-1} + \dots + a_m$$



$D = \mathbb{C}$

$U(1)_q \circlearrowright D = \mathbb{R}^2 \simeq \mathbb{C} \qquad x \mapsto e^{i\theta} x \qquad$ rotation

$U(1)_q \circlearrowright V_m \qquad x_k \mapsto e^{i\theta} x_k$

$\qquad\qquad\qquad\qquad a_k \mapsto e^{ik\theta} a_k \qquad$ $w_k = k$ weights

$$ch_q(V) := Tr_V(q) = \frac{1}{(1-q^{w_1})\cdots(1-q^{w_m})}$$

equivariant character

representation space for $U(1)_q$

specialize $V = V_m$

i.e. m abelian vortices on $D = \mathbb{R}^2$

$$= \frac{1}{(1-q)(1-q^2)\cdots(1-q^m)}$$

$$H_{U(1)_q}(pt) = \mathbb{C}[q] \rightsquigarrow K_{U(1)}(pt)$$

$$ch_q(\mathbb{C}) = \frac{1}{1-q}$$

equivariant cohomology

" integration

This is familiar in knot theory:

$ch_q(V_m)$ = "Bottom row of m-coloured HOMFLY polynomial of $\bigcirc$"

$P_m(\bigcirc; a, q)$

$$P_m(\bigcirc; a, q) = \frac{(1-a)(1-qa)\cdots(1-q^m a)}{(1-q)(1-q^2)\cdots(1-q^m)}$$

"bottom row" means $a \overset{!}{=} 0$

 is another knot ...

Need:
- generalization to arbitrary knots (links) K
- generalization to knot homology (introduce another variable t).

$\rightsquigarrow$ vortex counting problem $\quad ch_{q,t}(V_m(K)) =$ "bottom row" of S^m-coloured HOMFLY homology of K

roughly, equivariant characters correspond to regularised volumes of vortex moduli spaces

Both generalizations will be addressed by embedding vortex counting into a larger framework.

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knot K =  R = representation of G

$P_{R,G}(K;q)$ = quantum group invariant of K, coloured by R.

Ex: $G = SU(2)$
$R = V_2$ = fundamental representation $\square$

$P_{G,R}(K;q)$ = Jones polynomial $J(q)$

$$q^2 J(\nwarrow\!\!\!\!\nearrow) - q^{-2} J(\nwarrow\!\!\!\!\nearrow) = (q-q^{-1}) J(\uparrow\uparrow)$$

$$J(\bigcirc) = q + q^{-1} \xrightarrow[q=1]{} 2 = \dim V_2$$

More generally:

$$\begin{tikzcd}
 & ? & \\
\text{HOMFLY-PT polynomial } P(a,q) \arrow[ur] & & \text{Khovanov homology } P(q,t) \arrow[ul] \\
 & \text{quantum group invariants } J(q) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

+ coloured versions

Ex.: $G = SU(N)$ $\quad R = \square$

$$q^N P_{SU(N)}(\times) - q^{-N} P_{SU(N)}(\times) = (q - q^{-1}) P_{SU(N)}(\,)(\,)$$

$$P_{SU(N)}(\bigcirc) = \frac{q^N - q^{-N}}{q - q^{-1}} \xrightarrow{q=1} N$$

$\rightsquigarrow$ "quantum dimension"

Set $q^N = a$ $\curvearrowright$ HOMFLY-PT polynomial

$Ch_q(V_1)$ = "bottom row of HOMFLY polynomial"

Ex: abelian vortex, $V_1 \cong \mathbb{C}$ $\longleftrightarrow$ $ch_q = \frac{1}{1-q} = \text{Bottom}(\bigcirc)$

$$\text{HOMFLY} = \frac{1-a}{1-q}$$

Guess:

$H^*_{U(1)}(V_m) \overset{?}{\simeq}$ bottom of HOMFLY homology

Ex: nonabelian vortex: $\mathcal{V}_1 = \mathbb{C} \times \mathbb{CP}^r$

$G = U(r+1)$

left-action on $\mathrm{Mat}_{(r+1)\times(r+1)}(\mathbb{C})$

$\mathbb{C}$: centre of mass

$\mathbb{CP}^r$: internal structure

$$H^\bullet_{U(1)_q}(\mathcal{V}_1) \xrightarrow[\text{polynomial}]{\text{Poincaré}} \mathrm{Ch}_{q,t}(\mathcal{V}_1) = \frac{\sum_{i=0}^{r}(qt)^{2i}}{1-q}$$

$=$ bottom row of HOMFLY homology of $(2,2r+1)$ torus knot

e.g. $r=1$: 

For multivortices, get colored invariants

Q1: $$\mathrm{Ch}_{q,t}\left(\mathcal{V}_m^{U(r+1)}\right) \overset{?}{=} \frac{q^{-rm}}{(q;q)_m} \sum_{0\le k_r \le \cdots \le k_2 \le k_1 \le m} \begin{bmatrix} m \\ k_1 \end{bmatrix}_q \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} k_{r-1} \\ k_r \end{bmatrix}_q \times$$
$$\times\, q^{(2m+1)(k_1+k_2+\cdots+k_r) - \sum_{i=1}^{r} k_{i-1}k_i}\, t^{2(k_1+k_2+\cdots+k_r)}$$

Claim:

$$\text{link } K \longrightarrow \mathrm{Ch}_{q,t}(\mathcal{V}_m(K))$$

Q2: Explicit vortex solutions to $U(2)\times U(1)$ vortex equations on $D = \mathbb{R}^2$?

coupled! A_2 A_1

(In the case $D = \mathbb{H}^2$, yes! (abelian) $\longrightarrow$ how about nonabelian case?)

II.

Vortices on $D = \mathbb{R}^2$ $\longleftrightarrow$ knots in $\mathbb{R}^3$ How / Why ?

Right setup is: $D \times \mathbb{R}^3 \rightsquigarrow$?

$$\begin{array}{ccc} & D \times \mathbb{R}^3 & \\ \swarrow & & \searrow \\ \mathrm{ch}_{q,t}(V) \sim Z_{\text{vortex}} & \overset{!}{=} & Z_{\text{knot}} \end{array}$$

Knot K $\longrightarrow$ homology theory $\mathcal{H}(K)$

$\quad\quad = $ space of supersymmetric configurations (BPS states) $\mathcal{H}_{BPS}(K)$

10-dimensional string theory on $\mathbb{R}^4 \times X$

X: Calabi-Yau 3fold

choice of X yields a certain gauge theory on $\mathbb{R}^4$

concretely: $\mathcal{O}(-1) \times \mathcal{O}(-1) \to \mathbb{CP}^1$

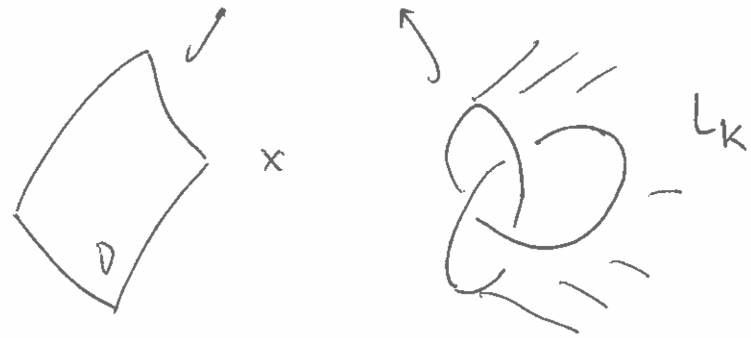
Incorporating knot K:

$K \longrightarrow$ Lagrangian submanifold $L_K \subset X$

$\hookrightarrow$ implemented by Taubes '01, ...

Introduce a defect $\rightsquigarrow$ "brane" supported on $\underset{\overset{\shortparallel}{\mathbb{R}^2}}{D} \times L_K$

$$\mathbb{R}^4 \times X$$



Counting: localized $\times$ bordered Riemann surface $(\Sigma, \partial\Sigma) \hookrightarrow (X, L_K)$

cf. "enumerative counting" Donaldson – Thomas theory

Different vantage points:

- from (X, L_K):

  $\shortparallel$ $Z_{enumer}(X, L_K)$ G – Schwartz – Vafa ORS ⋮ large industry

- from $\mathbb{R}^4 \supset D$ $\rightarrow$ domain, but also divisor where there is some ramification data

  $\shortparallel$ $Z_{ram\ inst}(\mathbb{R}^4, D)$ today! $U(1)_q \times U_t(1)$ not much explored

- from "brane" observer, i.e. $D \times L_K$

  $$Z_{sd}(D \times L_K) \underset{\substack{\uparrow \\ \text{Witten}}}{=} P(\mathcal{H}_{Knot}) \text{ HOMFLY}$$

$$Z_{enumer} = Z_{ram\ inst} = Z_{sd}$$

$K \subset S^3 \quad \underset{\text{Taubes}}{\rightsquigarrow} \quad L_K \subset X = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$

$$\begin{array}{c} \mathcal{O}(-1)\oplus\mathcal{O}(-1) \\ \downarrow \\ \mathbb{CP}^1 \end{array}$$

Steps: 1. Construct 2-dim Lagrangian $L^{(2)} \subset \mathbb{C}^2$ with

$$\partial L^{(2)} = K \subset S^3 = \partial(\mathbb{C}^2)$$

2. $$\mathbb{C}^2 \otimes \mathcal{O}(-1) = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

3. $$\begin{array}{ccc} L_K = L^{(2)} & \hookrightarrow & \mathcal{O}(-1)\oplus\mathcal{O}(-1) \\ \downarrow & & \downarrow \\ S^1 & \hookrightarrow & \mathbb{CP}^1 \end{array}$$

$$\partial X = S^2 \times S^3$$

$$\partial L_K = S^1 \times K$$

cf. arXiv: 1006.0977 Dimofte + G + Hollands