28.11.12

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"Wall-crossing and the crepant conjecture"

X projective $\underset{90s}{\leadsto}$ $QH(X) = H(X) \otimes \Lambda_X$ ← Novikov ring (degrees)

Ruan '94 : $f : X \to Y$

- $QH(Y) \to QH(X)$?
- Compute QH of blow-ups
- X orbifold, $\tilde{X} \to X$ resolution which is crepant (i.e. preserves canonical sheaf)

$QH(\tilde{X})$ versus $QH(X)$

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, dashed, "\text{flop}"] \arrow[dr] & & \tilde{X}_2 \arrow[dl] \\ & X & \end{tikzcd}$$

Today: * variation of polarization $L \to X$

⇝ wall-crossing formula for:

- gauged GW-invariants of X
- GW-invariants of $X/\!/G$

* conditions under which GW "does not change".

$G \circlearrowright X$ with $L \downarrow X$, X projective (X affine also)

↑ reductive

$$X^{ss} := \{ x \in X \mid \exists\, s \in \Gamma(L^{\otimes n})^G,\ n ;\ s(x) \neq 0 \}$$

$X/\!\!/_\varsigma := X^{ss}/G$ DM-stack (finite stabilizers)

If $L_\pm \to X$, $\varphi : X/\!\!/_- G \dashrightarrow X/\!\!/_+ G$,

"GIT" birational transformations

Classical story:

$$\begin{tikzcd}
 & H_G(X) \arrow[dl, "K_+"'] \arrow[dr, "K_-"] & \\
H(X/\!\!/_+ G) \arrow[dr, "\tau_+"'] & & H(X/\!\!/_- G) \arrow[dl, "\tau_-"] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Kirwan maps

traces ($\int$)

discrepancy on commutativity from localization

$$\tau_+ K_+(\alpha) - \tau_- K_-(\alpha) = \sum_{[\xi] \in \mathfrak{g}'} \operatorname{Res}_\xi \tau_{\xi,\tau}(\alpha)$$

Thaddeus' master space construction "$\mathbb{P}(L_+ \oplus L_-)/\!\!/ G$"

X^ξ: fixed point locus of ξ

$\mathbb{C}^\times_\xi \subset G_\xi$

$X^{t,\xi} \subset X^\xi$ wrt $L^t = L_+^{(1+t)/2} \otimes L_-^{(1-t)/2}$

$t \in (-1,1) \cap \mathbb{Q}$

$$\alpha \mapsto \int_{(X^{t,\xi}/\!\!/\tilde{G}_\xi)} \frac{i^*(\alpha)}{\mathrm{Eu}_{\mathbb{C}^\times_\xi}(\nu_{X^{\xi,t}})} \qquad \tilde{G}_\xi = G_\xi / \mathbb{C}^\times_\xi$$

EXAMPLE: Apply for

scrolls $\quad \mathbb{F}_2 \longrightarrow \mathbb{P}(1,1,2)$

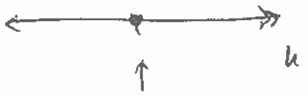
$\mathbb{F}_3 \longrightarrow \mathbb{P}(1,1,1,3)$ $\quad$ "crepant" means sum of weights is $= 0$

EX: $\quad X = \mathbb{C}^{k+1} \circlearrowleft G = \mathbb{C}^*$

acting by $\quad (\underbrace{1,1,\dots,1}_{k \text{ times}},-k)$

- $\mathbb{C}^k/\mathbb{Z}_k$ $\qquad \int Eul(\mathbb{P}^{k-1}) = k$
- $\mathcal{O}(-k) \longrightarrow \mathbb{P}^{k-1}$

ξ = equivariant parameter



$$c_k^G(\mathbb{C}^{k+1}) = (1+\xi)^k(-k\xi+1)$$

$$\mathrm{Res}_\xi \frac{(1+\xi)^k(1-k\xi)}{\xi^k(-k\xi)} = \mathrm{Res}_\xi\left(\left(\frac{1}{k}-k\right)\frac{1}{\xi}+\cdots\right) = \frac{1}{k}-k$$

$$Eul(\mathbb{C}^k/\mathbb{Z}_k) = \frac{1}{k}$$

THM: "Same formula holds in quantum cohomology", redefining diagram and maps

- $H_G(X) \longrightarrow QH_G(X)$
- $H(X/\!/_\pm G) \longrightarrow QH(X/\!/_\pm G)$

- $\mathbb{Q} \rightsquigarrow \Lambda^X_G = \Big\{ \sum\limits_{d \in H_2^G(X,\mathbb{Q})} a_d q^d \;:\; \forall_e,\ \# \{ a_d \mid \langle c_1^G(L), d \rangle \le e \} < \infty \Big\}$

- $K \rightsquigarrow$ quantized Kirwan map

$$\tau : QH(X/\!\!/G) \longrightarrow \Lambda^X_G$$

graph potentials

$$\alpha \longmapsto \sum_{n,d} \frac{q^d}{n!} \int_{\overline{M}_n(\mathbb{P}, X/\!\!/G)} ev(\alpha, \dots, \alpha)$$

$$\overline{M}_n(\mathbb{P}, X/\!\!/G) \swarrow \overline{M}_n(\mathbb{P})$$

parametrized maps

To prove the claim: we first prove it for the gGW potentials

$$\underbrace{\tau^G_{X,-}(\alpha) - \tau^G_{X,+}(\alpha)}_{\text{gauged potentials}} = \sum \mathrm{Res}_\xi \, \tau^G_{\xi,t}(\alpha)$$

$$\begin{tikzcd} QH_G(X) \arrow[rr, "\tau_X^G"] \arrow[dr, "K_X^G"'] & & \Lambda \\ & QH(X/\!\!/G) \arrow[ur] & \end{tikzcd}$$

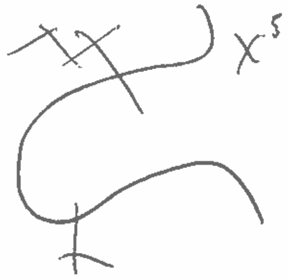
area $\to \infty$

Fixed-point potential:

$$\alpha \longmapsto \sum_{d,n} \frac{q^d}{n!} \int_{\overline{M}_n^{G_\xi}(\xi, L_+, L_-, d)} ev^*(\alpha, \dots, \alpha) \cup \epsilon_+(T(X/\!\!/G))^\wedge$$

$$\overline{M}_n^{G_\xi}(\xi, L_+, L_-, d) \cup M(\xi, d)$$

model this on (stack of) maps $\hat{C} \to X/G_\xi$ principal component map to X^ξ



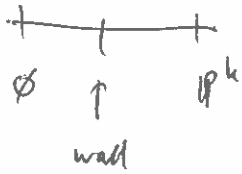
"moving part":

$$\mathrm{Eul}_{\mathbb{C}^\times_\xi}\left(\mathrm{Ind}_E\left(T(X/G)\right)^+\right)$$

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{e} & X \\ f\downarrow & & \\ \mathcal{M} & & \end{array}$$

EXAMPLE: $\mathbb{C}^\times \circlearrowright \mathbb{C}^{k+1} \quad (1, \dots, 1)$

$$c_1^G(X) = (k+1)\xi$$



$\beta =$ pt class in $H^G(\bar{\mathcal{M}}_3(\mathbb{P}^3))$

$w \in H^2(\mathbb{P}^k)$

$$\langle \omega^a, \omega^b, \omega^c \rangle_{\substack{0,1\\ d=1}} = \mathrm{Res}_\xi \int_{\mathcal{M}^{G_3}(\mathbb{P})} \frac{ev^*\xi^a \cup ev^*\xi^b \cup ev^*\xi^c \cup f(\beta)}{e(\mathrm{Ind}(X/G)^+)}$$

$$\mathrm{Ind}(\,)|_0 = H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1) \times_{\mathbb{C}^\times} \mathbb{C}^{k+1}) \cong \mathbb{C}^{2(k+1)}$$

$$= \mathrm{Res} \int_{\bar{\mathcal{M}}_3(\mathbb{P})} \frac{\xi^{a+b+c}\,\beta}{\xi^{2(k+1)}} = \begin{cases} 1, & a+b+c = 2k+1 \\ 0, & \text{otherwise} \end{cases}$$

$$\omega^a * \omega^b = \omega^{a+b-k-1} q \qquad a+b \le 2k$$

THM: Suppose $X^{\xi,t}$ connected

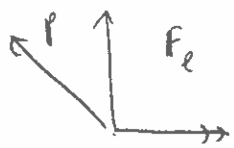
normal bundle $\hookrightarrow$ $\nu_{X^\xi/X} = \bigoplus \nu_{\mu_i}$ $\qquad \mu_i =$ weights

$\uparrow \mathbb{C}^\times_\xi$

DEF.: $X/\!\!/_+ G \dashrightarrow X/\!\!/_- G$ crepant if $\sum \mu_i = 0$

THM: (G-Woodward) $\tau_+ K_+ \underset{\text{a.e.}}{=} \tau_- K_-$ if φ crepant.

EX:



$$\begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad \mathbb{Z}\oplus\mathbb{Z}$$

PROOF: action of $\mathbb{Z} \ni u$ on $\mathcal{M}(\zeta, d)$:

$Q \in \mathrm{Pic}(C) \qquad \delta \in H^2$

$$Q^r : (P, u) \longmapsto (P \times_{\mathbb{C}^*} Q, u)$$

changing degree!

$$d \longmapsto d + r\delta \qquad (P \times_{\mathbb{C}^*} Q)(X^\xi) \supset P(X^\xi)$$

Difference in the contributions to the fixed point potentials:

$$\mathrm{Eul}_{\mathbb{C}^*}\Big(\bigoplus_{i=1}^m L_{\mu_i}^{\oplus r\mu_i}\Big)$$

$$= \prod_{i=1}^m \big(\xi\mu_i + c_1(V_i)\big)^{\mu_i r} = \prod_i \big(\xi + c_1(V_i)/\mu_i\big)^{\mu_i r} \Big(\prod_i \mu_i^{\mu_i r}\Big)$$

$$= \xi^{r\mu} + \xi^{r\mu - 1}\Big(r\sum_{i=1}^m c_1(V_i)\Big) + \dots \quad \text{polynomial in } r$$

$$\mu = \sum_i \mu_i = 0$$

Adding over all degrees:

$$\sum \underbrace{f(r)}_{\substack{\uparrow \\ \text{polynomial}}} q^{\delta r} \left(\prod \mu_i^{\mu_i}\right)^r q^d \underset{a.e.}{=} 0$$

$$\not\uparrow$$

$$\sum_{n \in \mathbb{Z}} q^n = \delta|_{q=1}$$