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"Gravitating vortices and instantons"

(+ L. Álvarez-Cónsul + M. García-Fernández)

§1. Abelian vortices on compact Riemann surfaces

X = compact Riemann surface

ω = Kähler form

$$\begin{array}{l} L \\ \downarrow \\ X \end{array} \quad \text{holomorphic line bundle} \qquad \varphi \in H^0(X, L)$$

Looking for a Hermitian metric h on L:

$$i \Lambda F_h + |\varphi|_h^2 = \tau \qquad \tau \in \mathbb{R}$$

THM: $\exists\, h$ satisfying vortex equation $\iff$ $\deg L < \tau$
$\varphi \neq 0$

Bradlow, G-P, ...

(If $\varphi = 0$, $\deg L = \tau$.)

§2. Hermitian-Yang-Mills equations

(M, ω_M) compact Kähler manifold

$E \to M$ holomorphic vector bundle

Equation for a Hermitian metric h on E:

$$i \Lambda F_h = \lambda I \qquad \lambda = \frac{\deg_{\omega_M} E}{\operatorname{rk} E} =: \mu(E) \text{ slope}$$

i.e. projective flatness

Recall: E stable $\overset{\text{def}}{\Longleftrightarrow}$ For every coherent subsheaf $E' \subset E$
$$\mu(E') < \mu(E)$$

Existence of a HYM metric on $E \to M$ $\Longleftrightarrow$ polystability of E

$\hookrightarrow$ means: direct sum of stable (sub) bundles of the same slope

Relation of HYM with vortices:

$$(L, \varphi) \rightsquigarrow \quad E \longrightarrow X \times \mathbb{P}^1 \qquad \begin{array}{l} \xrightarrow{p} X \\ \xrightarrow{q} \mathbb{P}^1 \end{array}$$

$\circlearrowleft \mathbb{C}^2$ rk 2 vector bundle

$$0 \longrightarrow p^* L \longrightarrow E \longrightarrow q^* \mathcal{O}(2) \longrightarrow 0$$

$$\left\{ \begin{array}{c} \text{such} \\ \text{extensions} \end{array} \right\} \cong H^1(X \times \mathbb{P}^1, p^* L \otimes q^* \mathcal{O}(-2))$$

$\quad\cong$ Künneth

$$H^0(X, L) \otimes \underbrace{H^1(\mathbb{P}^1, \mathcal{O}(-2))}_{\cong\, \mathbb{C}}$$

$SU(2) \circlearrowright X \times \mathbb{P}^1$

$\quad \llcorner$ trivially on X, $\mathbb{P}^1 = SU(2)/U(1)$

E is an $SU(2)$-invariant holomorphic vector bundle.

Consider the $SU(2)$-invariant Kähler metric on $X \times \mathbb{P}^1$

$$\omega_\tau = p^* \omega_X + q^* \omega_{\mathbb{P}^1} \quad \text{where } \omega_{\mathbb{P}^1} = \frac{4}{\tau} \omega_{\mathbb{P}^1}^{FS} \text{ Fubini-Study}$$

(L, φ) admits a solution to the vortex equations $\Longleftrightarrow$ E admits an $SU(2)$-invariant solution to the HYM equation

To prove the existence theorem on p. $\boxed{1}$, one can e.g. show:

The vector bundle E defined by (L,φ) is stable wrt ω_τ $\iff$ $\deg L < \tau$

§3. Nonabelian vortices on compact Riemann surfaces

Replace L by a vector bundle of any rank, V, $\varphi \in H^0(X,V)$

$$0 \to p^*V \to E \to q^*\mathcal{O}(2) \to 0$$

stability of E wrt ω_τ $\iff$ Bradlow stability for (V,φ)

"Interesting question": compare metrics of moduli of vortices with pulled-back metrics on moduli of instantons.

§4. Triples

$E_1, E_2 \to X$ holomorphic vector bundles

$\varphi : E_2 \to E_1$

$$0 \to p^*E_1 \to E \to p^*E_2 \otimes q^*\mathcal{O}(2) \to 0$$

Coupled vortex equations

variables $\begin{cases} h_1 & \text{metric on } E_1 \to X \\ h_2 & \text{" " } E_2 \to X \end{cases}$ $\qquad \tau_1, \tau_2 \in \mathbb{R}$

$r_1 = \mathrm{rk}\, E_1$

$r_2 = \mathrm{rk}\, E_2$

$$(\ast) \begin{cases} i\Lambda F_{h_1} + \varphi\varphi^* = \tau_1 I_1 \\ i\Lambda F_{h_2} - \varphi^*\varphi = \tau_2 I_2 \end{cases}$$

$r_1\tau_1 + r_2\tau_2 = \deg E_1 + \deg E_2$ $\qquad\qquad \alpha := \tau_1 - \tau_2$

$\exists$ solutions of $(\star)$ $\iff$ $\exists$ HYM metric on E wrt $\omega_\alpha := \alpha p^*\omega + \omega_{\mathbb{P}^1}^{FS}$ which is $SU(2)$-invariant

Bradlow & G-P: Stability for triples

(E_1, E_2, φ) α-stable $\iff$ E stable wrt ω_α

((© Nick: $\underset{U(2)\text{-bundle}}{\varphi : E_2} \longrightarrow \underset{\text{line bundle}}{E_1}$ "standard model"))

\+ Álvarez-Cónsul '00:

$$E_n \longrightarrow E_{n-1} \longrightarrow \cdots \longrightarrow E_1$$

many parameters

⌈ NB: rk 2 Higgs bundles

Hitchin — localization to vortex moduli space allows to compute Poincaré polynomial (uses Macdonald's result on Poincaré polynomial of $\mathrm{Sym}^k\Sigma$).

Gothen — similar result for rank 3, but needed moduli spaces of triples. ⌋

Quiver bundles: $\bullet - \circ - \circ - \circ \cdots \bullet$

dimensional reduction on $X \times G/P$.

§ 5. Coupled Kähler–Yang–Mills equations

+ M. García-Fernández

$$\begin{array}{ll} E & \text{hol. vector bundle} \\ \downarrow & \\ M & \text{compact complex manifold} \end{array}$$

Equations for: · g Kähler metric on M

· h Hermitian metric on E

$$(*) \begin{cases} i\Lambda F_h = \lambda I_E \\ S_g - \alpha \Lambda^2 \operatorname{Tr}(F_h \wedge F_h) = c \end{cases}$$

$\alpha > 0$

λ, c determined by the topology

Gauge group: $1 \to \underbrace{\mathcal{G}}_{\text{"usual" gauge group}} \to \tilde{\mathcal{G}} \to \underbrace{\mathcal{H}}_{\text{Hamiltonian symplectomorphisms of } (M,\omega)} \to 1$

joint w/ L. Álvarez-Cónsul + G-P

Mario García-Fernández's thesis discusses stability conditions for these equations. Can you understand them in "simple cases"?

X = compact Riemann surface

$L \to X$ holomorphic line bundle, $\varphi \in H^0(X, L)$

On $M = X \times \mathbb{P}^1$ have $SU(2)$-invariant bundles described by extensions $0 \to p^*L \to E \to q^*\mathcal{O}(2) \to 0$.

(Calabi-Yang-Mills-Higgs)

$SU(2)$- invariant solution to CYMH on $E \to X \times \mathbb{P}^1$

$$\Updownarrow$$

solving for g = metric on X

h = metric on L

satisfying: (**)

$$\begin{cases} i\Lambda F_h + |\varphi|^2 - \tau = 0 \\ S_g + \alpha(\Delta + \tau)(|\varphi|^2 - \tau) = c \end{cases}$$

call these "equations for gravitating vortices"

τ, α real parameters $\qquad \alpha = 2\pi G$ ← Newton's gravitational constant

c determined by topology

$c = 0$: cosmic strings on $\mathbb{R}^{1,1} \times X$

Comtet & Gibbons } '88
Linet

Spruck - Yang '95

$$c = \frac{2\pi}{\mathrm{Vol}(X)} \left(\chi(X) - \alpha\tau c_1(L) \right)$$

$c = 0 \Leftrightarrow X = \mathbb{P}^1$ zero cosmological constant

$c < 0 \Leftrightarrow$ higher genus

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THM (Y. Yang) : Let $D = \sum n_i p_i$ in $\mathbb{P}^1$ s.t. $\chi(\mathbb{P}^1) = 2\pi G\tau c_1(L)$. Then $(\mathbb{P}^1, L, \varphi)$ admits a solution to (**) if $n_j < \frac{c_1(L)}{2}$ for all j, and also have a solution if $D = np + n\sigma(p)$ antipodal and $n = \frac{c_1(L)}{2}$.

cf. stability for $SL_2\mathbb{C}$-action on $\mathrm{Sym}^N(\mathbb{P}^1)$

Higher genus: work in progress. $N = c_1(L) = \sum_{i=1} n_i$

Moduli space: maybe GIT quotient $SL_2\mathbb{C} \circlearrowright \mathbb{P}^N = \mathrm{Sym}^N(\mathbb{P}^1)$

NB: CYMH functional available for equations (*)